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FRACTIONAL INTEGRAL INEQUALITIES OF HADAMARD-TYPE FOR m-CONVEX FUNCTIONS VIA CAPUTO k-FRACTIONAL DERIVATIVES

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Abstract. In this paper, certain Hadamard and Hadamard-type inequalities are proved for m-convex functions via Caputo k-fractional derivatives. These results have some relationship with the Hadamard-type inequalities and related inequalities via Caputo fractional derivatives.

1. Introduction

Fractional calculus is one of those misnormers which are essence of mathematics. Fractional order systems i.e, the systems which contain fractional derivatives and integrals, have been studied by many in engineering and science areas [27].
Recently scientists and engineers rediscovered fractional calculus and apply it in number of fields namely control engineering, electromagnetism and signal processing. This study has motivated by the number of physical and engineering processes which are described by fractional differential equations [23].

Inequalities are everywhere in nature, in our lives, in almost all sciences. Particularly, in mathematical analysis, in optimization theory and problems their importance is remarkable. In fractional calculus point of view a rich theory of fractional inequalities have been produced in recent decades. Actually fractional inequalities are useful in the theory of fractional calculus, for example in the subject of differential equations, where they take a necessary part in establishing different kinds of problems in terms of fractional differential equations and their solutions.

We are interested to find some fractional integral inequalities using Caputo k-fractional derivatives. In the following we give some preliminaries and basic definitions for which we define convex functions, m-convex functions, Hadamard inequality, Caputo fractional derivatives and then Caputo k-fractional derivatives.

Definition 1.1. [3] A function $f : [a, b] \to \mathbb{R}$ is convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

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holds where for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. If reverse of the above inequality holds, then f is said to be a concave function.

In [28] Toader define the concept of m-convexity, an intermediate between usual convexity and star shape function.

Definition 1.2. A function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0, 1]$, if we have

$$f\left(tx + m(1-t)y\right) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

For $m = 1$, we get the definition of convex functions defined on $[0, b]$. For $m = 0$, we get the concept of starshaped functions on $[0, b]$ which is defined as: A function $f : [0, b] \to \mathbb{R}$ is called *starshaped* if

$$f(tx) \leq tf(x) \text{ for all } t \in [0, b] \text{ and } x \in [0, b].$$

Denote by $K_m(b)$ the set of the m-convex functions on $[0, b]$ for which $f(0) < 0$, then we have

$$K_1(b) \subset K_m(b) \subset K_0(b),$$

whenever $m \in (0, 1)$. Notably the class $K_1(b)$ are only convex functions $f : [0, b] \to \mathbb{R}$ for which $f(0) \leq 0$ (see [8]).

Example 1.3. [21] The function $f : [0, \infty) \to \mathbb{R}$, given by

$$f(x) = \frac{1}{12}\left(x^4 - 5x^3 + 9x^2 - 5x\right)$$

is $\frac{16}{17}-$convex function but it is not convex function.

Convex functions are equally defined with the Hadamard inequality stated in the following.

Theorem 1.4. *If $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$ with $a < b$, then the following inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

It is well known as the Hadamard inequality which got the attention of many mathematicians and is studied extensively (see, [1–12, 15, 18, 20, 24–26, 28]).
In the following we define Caputo fractional derivatives [19].

Definition 1.5. Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$, the space of functions having nth derivatives absolutely continuous. The right-sided and left-sided Caputo fractional derivatives of order α are defined as follows:

$$({}^C D^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt, x > a \tag{1}$$

and

$$({}^C D^{\alpha}_{b-} f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt, x < b. \tag{2}$$

If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative $({}^C D^n_{a+} f)(x)$ coincides with $f^{(n)}(x)$ whereas $({}^C D^n_{b-} f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular we have

$$({}^C D^0_{a+} f)(x) = ({}^C D^0_{b-} f)(x) = f(x) \tag{3}$$

where $n = 1$ and $\alpha = 0$.

In the following we define Caputo k-fractional derivatives.

Definition 1.6. [12] Let $\alpha > 0, k \geq 1$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$. The Caputo k-fractional derivatives of order α are defined as follows:

$${}^C D^{\alpha,k}_{a+} f(x) = \frac{1}{k\Gamma_k(n - \frac{\alpha}{k})} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}} dt, x > a \tag{4}$$

and

$${}^C D^{\alpha,k}_{b-} f(x) = \frac{(-1)^n}{k\Gamma_k(n - \frac{\alpha}{k})} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}} dt, x < b \tag{5}$$

where $\Gamma_k(\alpha)$ is the k-Gamma function defined as

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{\frac{-t^k}{k}} dt,$$

also

$$\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha).$$

If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo k-fractional derivative $({}^C D^{\alpha,1}_{a+} f)(x)$ coincides with $f^{(n)}(x)$, whereas $({}^C D^{\alpha,1}_{b-} f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$.

In particular we have

$$({}^C D^{0,1}_{a+} f)(x) = ({}^C D^{0,1}_{b-} f)(x) = f(x) \tag{6}$$

where $n, k = 1$ and $\alpha = 0$.
For $k = 1$, Caputo k-fractional derivatives give the definition of Caputo fractional derivatives.

In this paper we are motivated to give a version of the Hadamard inequality and some related inequalities for m-convex functions via Caputo k-fractional derivatives. We study the bounds of differences of this inequality and also we connect our results with some already known inequalities for convex functions.
In the whole paper $C^n[a, b]$ denotes the space of n-times differentiable functions such that $f^{(n)}$ are continuous on $[a, b]$.

2. Hadamard inequality for m-convex functions via Caputo k-fractional derivatives and related inequalities

Here we give the Hadamard inequality for m-convex functions via Caputo k-fractional derivatives and related inequalities. Also in particular we give results for convex functions as special cases.

Theorem 2.1. *Let $f:[0,\infty)\to\mathbb{R}$ be a positive function such that $f\in C^n[0,\infty)$. If $f^{(n)}$ is m-convex on $[0,\infty)$ and $0\leq a\leq mb$, then following inequalities for Caputo k-fractional derivatives hold*

$$\begin{aligned} & f^{(n)}\left(\frac{a+mb}{2}\right) \\ & \leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^C D^{\alpha,k}_{a^+}f)(mb)+(-1)^n m^{n-\frac{\alpha}{k}+1}({}^C D^{\alpha,k}_{b-}f)\left(\frac{a}{m}\right)\right] \\ & \leq \frac{n-\frac{\alpha}{k}}{2(n-\frac{\alpha}{k}+1)}\left[f^{(n)}(a)-m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right]+\frac{m}{2}\left[f^{(n)}(b)+mf^{(n)}\left(\frac{a}{m^2}\right)\right]. \end{aligned} \tag{7}$$

Proof. Since $f^{(n)}$ is m-convex function, so for $x,y\in[a,b]$ we have

$$f^{(n)}\left(\frac{x+my}{2}\right)\leq\frac{f^{(n)}(x)+mf^{(n)}(y)}{2}.$$

Taking $x=ta+m(1-t)b$, $y=tb+\frac{1}{m}(1-t)a$ and $t\in[0,1]$, then integrating over $[0,1]$ the resulting inequality, after multiplying with $t^{n-\frac{\alpha}{k}-1}$, we get

$$\begin{aligned} & \frac{2}{n-\frac{\alpha}{k}}f^{(n)}\left(\frac{a+mb}{2}\right) \\ & \leq \int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(ta+m(1-t)b\right)dt+m\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(tb+\frac{1}{m}(1-t)a\right)dt \end{aligned}$$

By change of variables we get

$$\begin{aligned} & \frac{2}{n-\frac{\alpha}{k}}f^{(n)}\left(\frac{a+mb}{2}\right) \\ & \leq \int_{mb}^{a}\left(\frac{mb-u}{mb-a}\right)^{n-\frac{\alpha}{k}-1}\frac{f^{(n)}(u)du}{a-mb}+m^2\int_{\frac{a}{m}}^{b}\left(\frac{v-\frac{a}{m}}{b-\frac{a}{m}}\right)^{n-\frac{\alpha}{k}-1}\frac{f^{(n)}(v)dv}{mb-a} \\ & = k\Gamma_k\left(n-\frac{\alpha}{k}\right)\frac{1}{(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^C D^{\alpha,k}_{a^+}f)(mb)+(-1)^n m^{n-\frac{\alpha}{k}+1}({}^C D^{\alpha,k}_{b-}f)\left(\frac{a}{m}\right)\right], \end{aligned}$$

which gives the first inequality in (7). In order to prove the second inequality of (7) we proceed as follows:
Since $f^{(n)}$ is m-convex, so for $t\in[0,mb]$ we have

$$\begin{aligned} & f^{(n)}(ta+m(1-t)b)+mf^{(n)}\left(tb+\frac{(1-t)}{m}a\right) \\ & \leq tf^{(n)}(a)+m(1-t)f^{(n)}(b)+mtf^{(n)}(b)+m^2(1-t)f^{(n)}\left(\frac{a}{m^2}\right). \end{aligned} \tag{8}$$

Now integrating over $[0,1]$ after multiplying above inequality with $\frac{(n-\frac{\alpha}{k})}{2}t^{n-\frac{\alpha}{k}-1}$, we get

$$\begin{aligned}
&\frac{n-\frac{\alpha}{k}}{2}\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(ta+m(1-t)b\right)dt\\
&+\frac{(n-\frac{\alpha}{k})}{2}m\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(tb+\frac{(1-t)}{m}a\right)dt\\
&\le \frac{n-\frac{\alpha}{k}}{2}\left[f^{(n)}(a)-m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right]\int_0^1 t^{n-\frac{\alpha}{k}}dt+\frac{m(n-\frac{\alpha}{k})}{2}\times\\
&\left[f^{(n)}(b)+mf^{(n)}\left(\frac{a}{m^2}\right)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}dt\\
&=\frac{n-\frac{\alpha}{k}}{2(n-\frac{\alpha}{k}+1)}\left[f^{(n)}(a)-m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right]+\frac{m}{2}\left[f^{(n)}(b)+mf^{(n)}\left(\frac{a}{m^2}\right)\right].
\end{aligned}$$

From which we get the second inequality in (7). □

Remark 2.2. (i) If we put $k=1$ in Theorem 2.1, then we get [22, Theorem 2.1].
(ii) If we put $m=1$ in Theorem 2.1, then we get [13, Theorem 2.1].
(iii) If we put $m=1$ along with $k=1$ in Theorem 2.1, then we get [16, Theorem 4].

For the next result we need the following lemma.

Lemma 2.3. *Let $f:[a,mb]\to\mathbb{R}$ be a function such that $f\in C^{n+1}[a,mb]$ with $a<mb$. Then the following equality for Caputo k-fractional derivatives holds*

$$\begin{aligned}
&\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}[({}^C D^{\alpha,k}_{a^+}f)(mb)+(-1)^n({}^C D^{\alpha,k}_{mb-}f)(a)] \quad (9)\\
&=\frac{mb-a}{2}\int_0^1[(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}]f^{(n+1)}(ta+m(1-t)b)dt.
\end{aligned}$$

Proof. Consider right hand side of equality (9), we have

$$\begin{aligned}
&\frac{mb-a}{2}\int_0^1[(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}]f^{(n+1)}(ta+m(1-t)b)dt \quad (10)\\
&=\frac{mb-a}{2}\int_0^1(1-t)^{n-\frac{\alpha}{k}}f^{(n+1)}(ta+m(1-t)b)dt\\
&-\frac{mb-a}{2}\int_0^1 t^{n-\frac{\alpha}{k}}f^{(n+1)}(ta+m(1-t)b)dt.
\end{aligned}$$

Now

$$\begin{aligned}
&\frac{mb-a}{2}\int_0^1(1-t)^{n-\frac{\alpha}{k}}f^{(n+1)}(ta+m(1-t)b)dt\\
&=\frac{f^{(n)}(mb)}{2}+\frac{(n-\frac{\alpha}{k})}{2}\int_{mb}^a\left(\frac{mb-a}{x-a}\right)^{\frac{\alpha}{k}-n+1}\frac{f^{(n)}(x)}{mb-a}dx\\
&=\frac{f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}(-1)^n({}^C D^{\alpha,k}_{mb-}f)(a).
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{mb-a}{2}\int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)}(ta+m(1-t)b)dt \\
&=\frac{f^{(n)}(a)}{2}-\frac{n-\frac{\alpha}{k}}{2}\int_{mb}^{a}\left(\frac{mb-a}{mb-x}\right)^{\frac{\alpha}{k}-n+1}\frac{f(x)}{a-mb}dx \\
&=\frac{f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}({}^C D_{a^+}^{\alpha,k} f)(mb).
\end{aligned}$$

Putting the values of above integrals in (10), we get the required result. □

Remark 2.4. (i) If we put $m=1$ in Lemma 2.3, then we get [13, Lemma 2.3]. (ii) If we put $m=1$ along with $k=1$ in Lemma 2.3, then we get [16, Lemma 3].

Using the above lemma we give the following Hadamard-type inequality for m-convex functions.

Theorem 2.5. *Let $f:[a,mb]\to\mathbb{R}$ be a function such that $f\in C^{n+1}[a,mb]$ with $0\le a<mb$. If $\left|f^{n+1}\right|$ is m-convex on $[a,mb]$, then the following inequality for Caputo k-fractional derivatives holds*

$$\begin{aligned}
&\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^C D_{a^+}^{\alpha,k} f)(mb)+(-1)^n({}^C D_{mb-}^{\alpha,k} f)(a)\right]\right| \\
&\le\frac{mb-a}{2(n-\frac{\alpha}{k}+1)}\left(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right)[f^{(n+1)}(a)+mf^{(n+1)}(b)].
\end{aligned}$$

Proof. Using Lemma 2.3 and m-convexity of $\left|f^{(n+1)}\right|$, we have

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}[({}^C D_{a^+}^{\alpha,k} f)(mb)+(-1)^n({}^C D_{mb-}^{\alpha,k} f)(a)]\right| \tag{11}$$

$$\begin{aligned}
&\le\frac{mb-a}{2}\int_0^1\left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right|\left|f^{(n+1)}(ta+m(1-t)b)\right|dt \\
&\le\frac{mb-a}{2}\int_0^1\left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right|\left[t\left|f^{(n+1)}(a)\right|+m(1-t)\left|f^{(n+1)}(b)\right|\right]dt \\
&=\frac{mb-a}{2}\left(\int_0^{\frac{1}{2}}\left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right|\left[t\left|f^{(n+1)}(a)\right|+m(1-t)\left|f^{(n+1)}(b)\right|\right]dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\left|t^{n-\frac{\alpha}{k}}-(1-t)^{n-\frac{\alpha}{k}}\right|\left[t\left|f^{(n+1)}(a)\right|+m(1-t)\left|f^{(n+1)}(b)\right|\right]dt\right)
\end{aligned}$$

Consider

$$\begin{aligned}
&\left|f^{(n+1)}(a)\right|\left[\int_0^{\frac{1}{2}} t(1-t)^{n-\frac{\alpha}{k}}dt-\int_0^{\frac{1}{2}} t^{n-\frac{\alpha}{k}+1}dt\right]\\
&+m\left|f^{(n+1)}(b)\right|\left[\int_0^{\frac{1}{2}}(1-t)^{n-\frac{\alpha}{k}+1}dt-\int_0^{\frac{1}{2}}(1-t)t^{n-\frac{\alpha}{k}}dt\right]\\
&=\left|f^{(n+1)}(a)\right|\left[\frac{1}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)}-\frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right]\\
&+m\left|f^{(n+1)}(b)\right|\left[\frac{1}{n-\frac{\alpha}{k}+2}-\frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right].
\end{aligned}$$

Similarly

$$\begin{aligned}
&\left|f^{(n+1)}(a)\right|\left[\int_{\frac{1}{2}}^1 t^{n-\frac{\alpha}{k}+1}dt-\int_{\frac{1}{2}}^1 t(1-t)^{n-\frac{\alpha}{k}}dt\right]\\
&+m\left|f^{(n+1)}(b)\right|\left[\int_{\frac{1}{2}}^1(1-t)t^{n-\frac{\alpha}{k}}dt-\int_{\frac{1}{2}}^1(1-t)^{n-\frac{\alpha}{k}+1}dt\right]\\
&=\left|f^{(n+1)}(a)\right|\left[\frac{1}{n-\frac{\alpha}{k}+2}-\frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right]\\
&+m\left|f^{(n+1)}(b)\right|\left[\frac{1}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)}-\frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right].
\end{aligned}$$

Putting above values in (11), we have

$$\begin{aligned}
&\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left[D^{\alpha,k}_{a^+}f(mb)+(-1)^nD^{\alpha,k}_{mb-}f(a)\right]\right|\\
&\leq\frac{mb-a}{2}\left(\left|f^{(n+1)}(a)\right|\left[\frac{1}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)}-\frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right.\right.\\
&\left.+\frac{1}{n-\frac{\alpha}{k}+2}-\frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right]+m\left|f^{(n+1)}(b)\right|\left[\frac{1}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)}\right.\\
&\left.\left.-\frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}+\frac{1}{n-\frac{\alpha}{k}+2}-\frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right]\right)\\
&=\frac{mb-a}{2}\left(\left|f^{(n+1)}(a)\right|\left[\frac{1}{n-\frac{\alpha}{k}+1}-\frac{(\frac{1}{2})^{n-\frac{\alpha}{k}}}{n-\frac{\alpha}{k}+1}\right]\right.\\
&\left.+m\left|f^{(n+1)}(b)\right|\left[\frac{1}{n-\frac{\alpha}{k}+1}-\frac{(\frac{1}{2})^{n-\frac{\alpha}{k}}}{n-\frac{\alpha}{k}+1}\right]\right)\\
&=\frac{mb-a}{2(n-\frac{\alpha}{k}+1)}\left(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right)\left[\left|f^{(n+1)}(a)\right|+m\left|f^{(n+1)}(b)\right|\right],
\end{aligned}$$

which is the required result. □

Remark 2.6. (i) If we put $k=1$ in Theorem 2.5, then we get [22, Theorem 2.6]. (ii) If we put $m=1$ in Theorem 2.5, then we get [13, Theorem 2.4].

(iii) If we put $m = 1$ along with $k = 1$ in Theorem 2.5, then we get [16, Theorem 5].

Theorem 2.7. *Let $f : [a, b] \to \mathbb{R}$ be a positive function such that $f \in C^n[a, b]$, with $0 \le a < b$. Also let $f^{(n)}$ be m-convex function on $[a, b]$. Then the following inequalities for Caputo k-fractional derivatives hold*

$$
\begin{aligned}
&f^{(n)}\left(\frac{a+mb}{2}\right) \le \frac{2^{n-\frac{\alpha}{k}-1}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}} \\
&\left[({}^{C}D^{\alpha,k}_{(\frac{a+mb}{2})+}f)(mb) + (-1)^n m^{n-\frac{\alpha}{k}+1}({}^{C}D^{\alpha,k}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right] \\
&\le \frac{n-\frac{\alpha}{k}}{4(n-\frac{\alpha}{k}+1)}\left[f^{(n)}(a) - m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right] + \frac{m}{2}\left[f^{(n)}(b) + mf^{(n)}\left(\frac{a}{m^2}\right)\right].
\end{aligned}
$$

Proof. Since $f^{(n)}$ is m-convex, so we have

$$
f^{(n)}\left(\frac{x+my}{2}\right) \le \frac{f^{(n)}(x) + mf^{(n)}(y)}{2}.
$$

Taking $x = \frac{t}{2}a + m\frac{(2-t)}{2}b$, $y = \frac{(2-t)}{2m}a + \frac{t}{2}b$ where $t \in [0, 1]$, then integrating over $[0, 1]$ after multiplying resulting inequality with $t^{n-\frac{\alpha}{k}-1}$, we get

$$
\begin{aligned}
\frac{2}{n-\frac{\alpha}{k}}f^{(n)}\left(\frac{a+mb}{2}\right) \le& \int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(\frac{t}{2}a + m\frac{2-t}{2}b\right)dt \\
&+ m\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(\frac{2-t}{2m}a + \frac{t}{2}b\right)dt.
\end{aligned}
$$

By using change of variables we have

$$
\begin{aligned}
&\frac{2}{n-\frac{\alpha}{k}}f^{(n)}\left(\frac{a+mb}{2}\right) \le \int_{mb}^{\frac{a+mb}{2}}\left(\frac{2}{mb-a}(mb-u)\right)^{n-\frac{\alpha}{k}-1}\frac{(-2f^{(n)}(u))du}{mb-a} \\
&+ m^2\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(\frac{2}{b-\frac{a}{m}}(v-\frac{a}{m})\right)^{\alpha-1}\frac{2f^{(n)}(v)dv}{mb-a} \\
&= \frac{2^{n-\frac{\alpha}{k}}k\Gamma_k(n-\frac{\alpha}{k})}{(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^{C}D^{\alpha,k}_{(\frac{a+mb}{2})+}f)(mb) + (-1)^n m^{n-\frac{\alpha}{k}+1}({}^{C}D^{\alpha,k}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right],
\end{aligned}
$$

which is the first required inequality. On the other hand m-convexity of $f^{(n)}$ gives

$$
\begin{aligned}
&f^{(n)}\left(\frac{t}{2}a + m\frac{2-t}{2}b\right) + mf^{(n)}\left(\frac{2-t}{2m}a + \frac{t}{2}b\right) \\
&\le \frac{t}{2}\left[f^{(n)}(a) - m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right] + m\left[f^{(n)}(b) + mf^{(n)}\left(\frac{a}{m^2}\right)\right].
\end{aligned}
$$

Now integrating over $[0, 1]$ after multiplying above inequality with $t^{n-\frac{\alpha}{k}-1}$, we get

$$
\begin{aligned}
&\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(\frac{t}{2}a + m\frac{2-t}{2}b\right)dt + m\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(\frac{2-t}{2m}a + \frac{t}{2}b\right)dt \\
&\le \frac{1}{2}\left(f^{(n)}(a) - m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{n-\frac{\alpha}{k}}dt + m\left(f^{(n)}(b) + mf^{(n)}\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}dt.
\end{aligned}
$$

From which by use of change of variables we get

$$\int_{mb}^{\frac{a+mb}{2}} \left(\frac{2}{mb-a}(mb-u)\right)^{n-\frac{\alpha}{k}-1} \frac{2f^{(n)}(u)du}{a-mb}$$
$$+ m^2 \int_{\frac{a}{m}}^{\frac{a+mb}{2m}} \left(\frac{2}{b-\frac{a}{m}}(v-\frac{a}{m})\right)^{n-\frac{\alpha}{k}-1} \frac{2f^{(n)}(v)dv}{mb-a}$$
$$\leq \frac{1}{2(n-\frac{\alpha}{k}+1)}\left[f^{(n)}(a) - m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right] + \frac{m}{n-\frac{\alpha}{k}}\left[f^{(n)}(b) + mf^{(n)}\left(\frac{a}{m^2}\right)\right].$$

Finally we have

$$\frac{2^{n-\frac{\alpha}{k}-1} k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}} \times$$
$$\left[({}^C D^{\alpha,k}_{(\frac{a+mb}{2})+} f)(mb) + (-1)^n m^{n-\frac{\alpha}{k}+1} ({}^C D^{\alpha,k}_{(\frac{a+mb}{2m})-} f)\left(\frac{a}{m}\right)\right]$$
$$\leq \frac{n-\frac{\alpha}{k}}{4(n-\frac{\alpha}{k}+1)}\left[f^{(n)}(a) - m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right] + \frac{m}{2}\left[f^{(n)}(b) + mf^{(n)}\left(\frac{a}{m^2}\right)\right].$$

Which gives the second inequality of the required inequality. □

Remark 2.8. (i) If we put $k=1$ in Theorem 2.7, then we get [22, Theorem 2.9].
(ii) If we put $m=1$ in Theorem 2.7, then we get [14, Theorem 2].
(iii) If we put $m=1$ along with $k=1$ in Theorem 2.7, then we get [17, Theorem 4].

Following lemma is useful for our next results.

Lemma 2.9. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f\in C^{n+1}[a,b]$, with $a<b$. Then the following equality for Caputo k-fractional derivatives holds*

$$\frac{2^{n-\frac{\alpha}{k}-1} k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^C D^{\alpha,k}_{(\frac{a+mb}{2})+} f)(mb) + (-1)^n m^{n-\frac{\alpha}{k}+1} ({}^C D^{\alpha,k}_{(\frac{a+mb}{2m})-} f)\left(\frac{a}{m}\right)\right]$$
$$-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right) + mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]$$
$$= \frac{mb-a}{4}\left[\int_0^1 t^\alpha f^{(n+1)}\left(\frac{t}{2}a + m\frac{2-t}{2}b\right)dt - \int_0^1 t^\alpha f^{(n+1)}\left(\frac{2-t}{2m}a + \frac{t}{2}b\right)dt\right].$$

Proof. Since

$$\frac{mb-a}{4}\left[\int_0^1 t^\alpha f^{(n+1)}\left(\frac{t}{2}a + m\frac{2-t}{2}b\right)dt\right]$$
$$= \frac{mb-a}{4}\left[-\frac{2}{mb-a}f^{(n)}\left(\frac{a+mb}{2}\right)\right.$$
$$\left.+\frac{2(n-\frac{\alpha}{k})}{a-mb}\int_{mb}^{\frac{a+mb}{2}}\left(\frac{2}{mb-a}(mb-x)\right)^{n-\frac{\alpha}{k}-1}\frac{2}{mb-a}f^{(n)}(x)dx\right]$$
$$= \frac{mb-a}{4}\left[-\frac{2}{mb-a}f^{(n)}\left(\frac{a+mb}{2}\right) + \frac{2^{n-\frac{\alpha}{k}+1}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}+1}}({}^C D^{\alpha,k}_{(\frac{a+mb}{2})+} f)(mb)\right].$$

In the same way,

$$-\frac{mb-a}{4}\left[\int_0^1 t^{\alpha} f^{(n+1)}\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)dt\right]$$
$$=-\frac{mb-a}{4}\left[\frac{2m}{mb-a}f^{(n)}\left(\frac{a+mb}{2m}\right)\right.$$
$$\left.-(-1)^n\frac{2^{n-\frac{\alpha}{k}+1}m^{n-\frac{\alpha}{k}+1}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}+1}}({}^CD^{\alpha,k}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right].$$

Adding above equalities we get the required equality. □

Remark 2.10. (i) If we put $m=1$ in above Lemma, then we get an equality in [14, Lemma 1].
(ii) If we put $m=1$ along with $k=1$ in above Lemma, we get an equality in [17, Lemma 2].

With the help of above lemma and power mean's inequality we prove the following Hadamard-type inequality.

Theorem 2.11. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f\in C^{n+1}[a,b]$, with $a<b$. If $|f^{(n+1)}|^q$ is also m-convex on $[a,b]$ for $q>1$, then the following inequality for Caputo k-fractional derivatives holds*

$$\left|\frac{2^{n-\frac{\alpha}{k}-1}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^CD^{\alpha,k}_{(\frac{a+mb}{2})+}f)(mb)+(-1)^n m^{n-\frac{\alpha}{k}+1}({}^CD^{\alpha,k}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\le\frac{mb-a}{4(n-\frac{\alpha}{k}+1)}\left(\frac{1}{2(n-\frac{\alpha}{k}+2)}\right)^{\frac{1}{q}}$$
$$\left[\left(\left(n-\frac{\alpha}{k}+1\right)|f^{(n+1)}(a)|^q+m\left(n-\frac{\alpha}{k}+3\right)|f^{(n+1)}(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(m\left(n-\frac{\alpha}{k}+3\right)|f^{(n+1)}\left(\frac{a}{m^2}\right)|^q+\left(n-\frac{\alpha}{k}+1\right)|f^{(n+1)}(b)|^q\right)^{\frac{1}{q}}\right].$$

Proof. Using Lemma 2.9 we have

$$\left|\frac{2^{n-\frac{\alpha}{k}-1}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^CD^{\alpha,k}_{(\frac{a+mb}{2})+}f)(mb)\right.\right.$$
$$\left.\left.+(-1)^n m^{n-\frac{\alpha}{k}+1}({}^CD^{\alpha,k}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\le\frac{mb-a}{4}\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|dt+\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|dt.$$

Applying power mean's inequality we have

$$\Bigg|\frac{2^{n-\frac{\alpha}{k}-1}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^C D^{\alpha,k}_{(\frac{a+mb}{2})^+}f)(mb)+(-1)^n m^{n-\frac{\alpha}{k}+1}({}^C D^{\alpha,k}_{(\frac{a+mb}{2m})^-}f)\left(\frac{a}{m}\right)\right]$$
$$-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\Bigg|$$
$$\le\frac{mb-a}{4}\left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}\left[\left[\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right].$$

Also from m-convexity of $|f^{(n+1)}|^q$, we have

$$\Bigg|\frac{2^{n-\frac{\alpha}{k}-1}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^C D^{\alpha,k}_{(\frac{a+mb}{2})^+}f)(mb)\;+(-1)^n m^{n-\frac{\alpha}{k}+1}({}^C D^{\alpha,k}_{(\frac{a+mb}{2m})^-}f)\left(\frac{a}{m}\right)\right]$$
$$-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\Bigg|$$
$$\le\frac{mb-a}{4}\left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}\left[\left[\int_0^1 t^{n-\frac{\alpha}{k}}\left(\frac{t}{2}|f^{(n+1)}(a)|^q+m\frac{2-t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 t^{n-\frac{\alpha}{k}}\left(m\frac{2-t}{2}|f^{(n+1)}\left(\frac{a}{m^2}\right)|^q+\frac{t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right]$$
$$=\frac{mb-a}{4(n-\frac{\alpha}{k}+1)}\left(\frac{1}{2(n-\frac{\alpha}{k}+2)}\right)^{\frac{1}{q}}\times$$
$$\left[\left(\left(n-\frac{\alpha}{k}+1\right)|f^{(n+1)}(a)|^q+m\left(n-\frac{\alpha}{k}+3\right)|f^{(n+1)}(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(m\left(n-\frac{\alpha}{k}+3\right)|f^{(n+1)}\left(\frac{a}{m^2}\right)|^q+\left(n-\frac{\alpha}{k}+1\right)|f^{(n+1)}(b)|^q\right)^{\frac{1}{q}}\right],$$

which is the required result. □

Remark 2.12. (i) If we put $k=1$ in above result, then we get [22, Theorem 2.13].
(i) If we put $m=1$ in above result, then we get [14, Theorem 3].
(ii) If we put $m=1$ along with $k=1$ in above result, then we get [17, Theorem 5].

Next we use the $H\ddot{o}lder's$ inequality and Minkowski's inequality along with Lemma 2.9 to obtain the following result.

Theorem 2.13. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f\in C^{n+1}[a,b]$, with $a<b$. If $|f^{(n+1)}|^q$ is m-convex on $[a,b]$ for $q>1$, then the following inequality for*

Caputo k-fractional derivatives holds

$$\left|\frac{2^{n-\frac{\alpha}{k}-1}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^CD^{\alpha,k}_{(\frac{a+mb}{2})+}f)(mb)+(-1)^n m^{n-\frac{\alpha}{k}+1}({}^CD^{\alpha,k}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{4}\left(\frac{1}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f^{(n+1)}(a)|^q+3m|f^{(n+1)}(b)|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{3m|f^{(n+1)}\left(\frac{a}{m^2}\right)|^q+|f^{(n+1)}(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{mb-a}{16}\left(\frac{4}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|\right.$$
$$\left.+(3m)^{\frac{1}{q}}\left(|f^{(n+1)}\left(\frac{a}{m^2}\right)|+|f^{(n+1)}(b)|\right)\right],$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.9 we have

$$\left|\frac{2^{n-\frac{\alpha}{k}-1}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^CD^{\alpha,k}_{(\frac{a+mb}{2})+}f)(mb)+(-1)^n m^{n-\frac{\alpha}{k}+1}({}^CD^{\alpha,k}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{4}\left[\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|dt\right.$$
$$\left.+\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|dt\right].$$

Applying *Hölder's* inequality, we get

$$\left|\frac{2^{n-\frac{\alpha}{k}-1}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^CD^{\alpha,k}_{(\frac{a+mb}{2})+}f)(mb)+(-1)^n m^{n-\frac{\alpha}{k}+1}({}^CD^{\alpha,k}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]\right.$$
$$\left.-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{4}\left[\left[\int_0^1 t^{np-\frac{\alpha p}{k}}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f^{(n+1)}\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 t^{np-\frac{\alpha p}{k}}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f^{(n+1)}\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right].$$

Since $|f^{(n+1)}|^q$ is m-convex, so we have

$$\left|\frac{2^{n-\frac{\alpha}{k}-1}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^C D^{\alpha,k}_{(\frac{a+mb}{2})+}f)(mb)\right.\right.$$
$$\left.\left.+(-1)^n m^{n-\frac{\alpha}{k}+1}({}^C D^{\alpha,k}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq \frac{mb-a}{4}\left(\frac{1}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}$$
$$\left[\left[\int_0^1\left(\frac{t}{2}|f^{(n+1)}(a)|^q+m\frac{2-t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1\left(m\frac{2-t}{2}|f^{(n+1)}(\frac{a}{m^2})|^q+\frac{t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right]$$
$$=\frac{mb-a}{4}\left(\frac{1}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}$$
$$\left[\left[\frac{|f^{(n+1)}(a)|^q+3m|f^{(n+1)}(b)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3m|f^{(n+1)}\left(\frac{a}{m^2}\right)|^q+|f^{(n+1)}(b)|^q}{4}\right]^{\frac{1}{q}}\right].$$

For the second inequality we apply Minkowski's inequality which gives

$$\left|\frac{2^{n-\frac{\alpha}{k}-1}k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^C D^{\alpha,k}_{(\frac{a+mb}{2})+}f)(mb)\right.\right.$$
$$\left.+(-1)^n m^{n-\frac{\alpha}{k}+1}({}^C D^{\alpha,k}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]$$
$$\left.-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq \frac{mb-a}{16}\left(\frac{4}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left[\left[|f^{(n+1)}(a)|^q+3m|f^{(n+1)}(b)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[3m|f^{(n+1)}\left(\frac{a}{m^2}\right)|^q+|f^{(n+1)}(b)|^q\right]^{\frac{1}{q}}\right]$$
$$\leq \frac{mb-a}{16}\left(\frac{4}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|\right.$$
$$\left.+(3m)^{\frac{1}{q}}\left(|f^{(n+1)}\left(\frac{a}{m^2}\right)|+|f^{(n+1)}(b)|\right)\right].$$

□

Remark 2.14. (i) If we put $k=1$ in above theorem, then we get [22, Theorem 2.15].
(ii) If we put $m=1$ in above theorem, then we get [14, Theorem 4].
(iii) If we put $m=1$ along with $k=1$ in above theorem, then we get [17, Theorem 6].

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