THE MODIFIED FRACTIONAL POWER SERIES FOR SOLVING A CLASS OF FRACTIONAL STURM-LIOUVILLE EIGENVALUE PROBLEMS

M.I. SYAM, M. ALQURAN, H.M. JARADAT, S. AL-SHARA'

ABSTRACT. This article is devoted to both theoretical and numerical studies of eigenvalues of regular fractional 2α-order Sturm-Liouville problem where $\frac{1}{2} < \alpha \leq 1$. In this paper, we implement the modified fractional power series (MFPS) method to approximate the eigenvalues. To find the eigenvalues, we force the approximate solution produced by the MMPS method satisfies the boundary condition at $x = 1$. The fractional derivative is described in the Caputo sense. Numerical results demonstrate the accuracy of the present algorithm. In addition, we prove the existence of the eigenfunctions of the proposed problem. The convergence of the approximate eigenfunctions produced by the MFPS to the exact eigenfunctions is proven.

1. INTRODUCTION

Fractional differential equations (FDEs) appear as generalizations to existing models with integer derivative and they also present new models for some physical problems [1]. In recent years, great interests were devoted to the analytical and numerical treatments of fractional differential equations. In general, fractional differential equations don't have exact solutions in closed form, and therefore, numerical methods such as, the variational iteration [2], the homotopy analysis method [3], and the Adomian decomposition method [4, 5, 6], have been implemented for several types of fractional differential equations. Also, the maximum principle and the method of lower and upper solutions have been extended to deal with FDEs and obtain analytical and numerical results [7]. The Tau method, the Pseudo-spectral method, and the wavelet method based on the legendre polynomials have been implemented for several types of FDEs [8].

The Sturm-Liouville eigenvalue problem has played an important role in modeling many physical problems [9]. The theory of the problem is well developed and many results have been obtained concerning the eigenvalues and corresponding eigenfunctions. Since finding analytical solutions for this problem is a difficult

---

task, many numerical algorithms have been investigated to find approximate solutions.

The fractional Sturm-Liouville eigenvalue problem was studied earlier [10] and [11]. In [10], the existence of a solution to such boundary value problem was established. In [11], the aforementioned relation between eigenvalues and zeros of Mittag-Leffler function was shown. The Adomian decomposition method was established for estimating fractional second order eigenvalues [12, 13]. The Homotopy Analysis method has been used to numerically approximate the eigenvalues of the fractional Sturm-Liouville Problems [14]. In [15], fractional differential transform method used to approximate the eigenvalues of Sturm–Liouville problems of fractional order. Fourier series was used in [16], the method of Haar wavelet operational matrix was used in [17] and [18]. In [19]-[22], extended some spectral properties of fractional Sturm-Liouville problem. Variational Methods and Inverse Laplace transform method applied in [23] and [24], respectively. Recently P. Antunes and R. Ferreira constructed numerical schemes using radial basis functions [25], B. Jin et al used Galerkin finite element method to solve fractional eigenvalue problems [26].

In this paper, we develop a numerical technique for approximating the eigenvalues of the following regular fractional Sturm-Liouville problem of the form

$$D^\alpha[P(x)D^\alpha u(x)] + \lambda Q(x)u(x) = \mu(x)u(x),\ 0 \le x \le 1, \frac{1}{2} < \alpha \le 1 \qquad (1)$$

subject to

$$a_0 u(0) + a_1\ D^\alpha u(0) \ = \ 0,\ a_0^2 + a_1^2 > 0, \qquad (2)$$
$$a_2 u(1) + a_3\ D^\alpha u(1) \ = \ 0,\ a_2^2 + a_3^2 > 0, \qquad (3)$$

where a_0, a_1, a_2, a_3 are constants, $p(x), q(x),\ r(x)$ are continuous functions with $p(x), q(x) > 0$ for all $x \in [0, 1]$, and D^α is the Caputo fractional derivative.

This paper is organized as follows. In section 2, we present some preliminaries which we will use in this paper. A description of the modified fractional power series (MFPS) method for discretization of the fractional 2α-order Sturm-Liouville problem (1.1)-(1.2) is presented in section 3. Convergence analysis is presented in Section 4. Several numerical examples and conclusions are discussed in Section 5. Conclusions and closing remarks are given in Section 6.

2. Preliminaries

In this section, we review some preliminaries definitions and theorems which we use in this paper. First, we write the definition and some preliminary results of the Caputo fractional derivatives, as well as, the definition of the Riemann-Liouville fractional and their properties.

Definition 1   A real function $f(t), t > 0$,is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu,\ m \in \mathbb{N}$.

Definition 2 The left Riemann-Liouville fractional integral of order $\delta \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined by

$$I^\delta f(t) = \begin{cases} \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s) ds, & \delta > 0, \\ f(t), & \delta = 0. \end{cases} \tag{4}$$

Definition 3 For $\delta > 0$, $m - 1 < \delta < m$, $m \in \mathbb{N}, t > 0$, and $f \in C_{-1}^m$, the left Caputo fractional derivative is defined by

$$D^\delta f(t) = \begin{cases} \frac{1}{\Gamma(m-\delta)} \int_0^t (t-s)^{m-1-\delta} f^{(m)}(s) ds, & \delta > 0, \\ f'(t), & \delta = 0, \end{cases} \tag{5}$$

where Γ is the well-known Gamma function.

The Caputo derivative defined in (5) is related to the Riemann-Liouville fractional integral, I^δ, of order $\delta \in \mathbb{R}^+$, by $D^\delta f(t) = I^{m-\delta} f^{(m)}(t)$. The Caputo fractional derivative satisfy the following properties for $f \in C_\mu$, $\mu \geq -1$ and $\alpha \geq 0$, see [27].

(1) $D^\alpha I^\alpha f(t) = f(t)$,
(2) $I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}$,
(3) $D^\alpha c = 0$, where c is constant,
(4) $D^\alpha t^\gamma = \left\{ \begin{array}{ll} 0, & \gamma < \alpha, \gamma \in \{0, 1, 2, ...\} \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & \text{otherwise} \end{array} \right\}$,
(5) $D^\alpha \left(\sum_{k=0}^m c_i f_i(t) \right) = \sum_{k=0}^m c_i \ D^\alpha f_i(t)$, where $c_1, c_2, ..., c_m$ are constants.

Second, we write the definition and one of the properties of the fractional power series which are used in this paper. More details can be found in [28]-[37].

Definition 4 A power series expansion of the form

$$\sum_{m=0}^\infty c_m (x - x_0)^{m\alpha} = c_0 + c_1 (x - x_0)^\alpha + c_2 (x - x_0)^{2\alpha} + ...$$

where $0 \leq m - 1 < \alpha \leq m$, is called fractional power series FPS about $x = x_0$. Suppose that f has a fractional FPS representation at $x = x_0$ of the form

$$g(x) = \sum_{m=0}^\infty c_m (x - x_0)^{m\alpha}, \quad x_0 \leq x < x_0 + \beta.$$

If $D^{m\alpha} g(x)$, $m = 0, 1, 2, ..$ are continuous on $\mathbb{R}$, then $c_m = \frac{D^{m\alpha} g(x_0)}{\Gamma(1+m\alpha)}$.

3. The MFPS method for a class of fractional second-order Sturm-Liouville Problems

In this section, we discuss the numerical solution of the following class of fractional 2α-order Sturm-Liouville Problems using MFPS:

$$D^\alpha [P(x) D^\alpha u(x)] + \lambda Q(x) u(x) = \mu(x) u(x), \ 0 < x < 1, \frac{1}{2} < \alpha \leq 1 \tag{6}$$

subject to

$$a_0 u(0) + a_1 \, D^\alpha u(0) \quad = \quad 0, \; a_0^2 + a_1^2 > 0, \tag{7}$$
$$a_2 u(1) + a_3 \, D^\alpha u(1) \quad = \quad 0, \; a_2^2 + a_3^2 > 0, \tag{8}$$

where a_0, a_1, a_2, a_3 are constants, $P(x), Q(x),$ and $\mu(x)$ are continuous with $P(x), \mu(x) > 0$ for all $x \in [0,1]$. Assume that $u(0) = \theta$ and $D^\alpha u(0) = \eta$. We find the values of θ and η from the boundary conditions (3.2)-(3.3) later. Using the MRPS method, the solution problem (3.1)-(3.3) can be written in the fractional power series form as

$$u(x) = \sum_{n=0}^{\infty} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{9}$$

To obtain the approximate values of f_n in Eq. (9), we write the k-th truncated series $u_k(x)$ in the form

$$u_k(x) = \sum_{n=0}^{k} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{10}$$

Since $u(0) = f_0 = \theta$ and $D^\alpha u(0) = f_1 = \eta$, we rewrite (10) as

$$u_k(x) = \theta + \eta \frac{x^\alpha}{\Gamma(1+\alpha)} + \sum_{n=2}^{k} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}, \quad k = 2, 3, ... \tag{11}$$

where $u_1(x) = \theta + \eta \frac{x^\alpha}{\Gamma(1+\alpha)}$ is considered to be the 1st MFPS approximate solution of $u(x)$. Let $P_k(x) = \sum_{n=0}^{k} p_n x^{n\alpha}$, $Q_k(x) = \sum_{n=0}^{k} q_n x^{n\alpha}$, and $\mu_k(x) = \sum_{n=0}^{k} m_n x^{n\alpha}$. Then,

$$P_k(x) D^\alpha u_k(x) = \left(\sum_{n=0}^{k} p_n x^{n\alpha} \right) \left(\sum_{n=0}^{k-1} f_{n+1} \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right)$$
$$= \sum_{n=0}^{k-1} \left(\sum_{i=0}^{n} \frac{f_{i+1}}{\Gamma(1+i\alpha)} p_{n-i} \right) x^{n\alpha} + h.o.t_1,$$

$$Q_k(x) u_k(x) = \left(\sum_{n=0}^{k} q_n x^{n\alpha} \right) \left(\sum_{n=0}^{k} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right)$$
$$= \sum_{n=0}^{k} \left(\sum_{i=0}^{n} \frac{f_i}{\Gamma(1+i\alpha)} q_{n-i} \right) x^{n\alpha} + h.o.t_2,$$

and

$$\mu_k(x) u_k(x) = \left(\sum_{n=0}^{k} m_n x^{n\alpha} \right) \left(\sum_{n=0}^{k} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right)$$
$$= \sum_{n=0}^{k} \left(\sum_{i=0}^{n} \frac{f_i}{\Gamma(1+i\alpha)} m_{n-i} \right) x^{n\alpha} + h.o.t_3,$$

where $h.o.t_1$ means a linear combination of $\{x^k, x^{k+1}, ..., x^{2k-1}\}$ and $h.o.t_2$ and $h.o.t_3$ means linear combination of $\{x^{k+1}, x^{k+2}, ..., x^{2k}\}$. Let $Res_k(u(x))$ be the k-th residual function which is defined by

$$Res_k(u(x)) = D^\alpha[P_k(x) D^\alpha u_k(x)] + \lambda Q_k(x) u_k(x) - \mu_k(x) u_k(x).$$

For simplicity, we write the k-th residual function as

$$
\begin{aligned}
Res_k(u(x)) &= D^\alpha \left[\sum_{n=0}^{k-1} \left(\sum_{i=0}^{n} \frac{f_{i+1}}{\Gamma(1+i\alpha)} p_{n-i} \right) x^{n\alpha} \right] \\
&+ \lambda \sum_{n=0}^{k} \left(\sum_{i=0}^{n} \frac{f_i}{\Gamma(1+i\alpha)} q_{n-i} \right) x^{n\alpha} - \sum_{n=0}^{k} \left(\sum_{i=0}^{n} \frac{f_i}{\Gamma(1+i\alpha)} m_{n-i} \right) x^{n\alpha}
\end{aligned}
$$

or

$$
\begin{aligned}
Res_k(u(x)) &= \sum_{n=0}^{k-2} \left(\sum_{i=0}^{n+1} \frac{f_{i+1}}{\Gamma(1+i\alpha)} p_{n+1-i} \right) \frac{\Gamma(1+(n+1)\alpha)}{\Gamma(1+n\alpha)} x^{n\alpha} \qquad (12) \\
&+ \lambda \sum_{n=0}^{k} \left(\sum_{i=0}^{n} \frac{f_i}{\Gamma(1+i\alpha)} q_{n-i} \right) x^{n\alpha} - \sum_{n=0}^{k} \left(\sum_{i=0}^{n} \frac{f_i}{\Gamma(1+i\alpha)} m_{n-i} \right) x^{n\alpha}.
\end{aligned}
$$

To find the values of the MFPS-coefficients f_j, $j \in \{2, 3, 4, .., k\}$, we solve the fractional differential equation

$$
D^{(j-1)\alpha} Res_k(u(0)) = 0.
$$

For $j \in \{0, 1, 2, .., k-2\}$,

$$
\begin{aligned}
0 &= D^{(j-1)\alpha} Res_k(u(0)) = \left(\sum_{i=0}^{j+1} \frac{f_{i+1}}{\Gamma(1+i\alpha)} p_{j+1-i} \right) \frac{\Gamma(1+(j+1)\alpha)}{\Gamma(1+j\alpha)} \frac{\Gamma(1+j\alpha)}{\Gamma(1+(j-1)\alpha)} \\
&+ \lambda \left(\sum_{i=0}^{j} \frac{f_i}{\Gamma(1+i\alpha)} q_{j-i} \right) \frac{\Gamma(1+j\alpha)}{\Gamma(1+(j-1)\alpha)} - \left(\sum_{i=0}^{j} \frac{f_i}{\Gamma(1+i\alpha)} m_{j-i} \right) \frac{\Gamma(1+j\alpha)}{\Gamma(1+(j-1)\alpha)}
\end{aligned}
$$

or

$$
\frac{\Gamma(1+(j+1)\alpha)}{\Gamma(1+j\alpha)} \sum_{i=0}^{j+1} \frac{f_{i+1}}{\Gamma(1+i\alpha)} p_{j+1-i} + \lambda \sum_{i=0}^{j} \frac{f_i}{\Gamma(1+i\alpha)} q_{j-i} - \sum_{i=0}^{j} \frac{f_i}{\Gamma(1+i\alpha)} m_{j-i} = 0
$$

Thus,

$$
f_{j+2} = \frac{\Gamma(1+j\alpha) \sum_{i=0}^{j} \frac{f_i}{\Gamma(1+i\alpha)} m_{j-i} - \lambda\Gamma(1+j\alpha) \sum_{i=0}^{j} \frac{f_i}{\Gamma(1+i\alpha)} q_{j-i} - \Gamma(1+(j+1)\alpha) \sum_{i=0}^{j} \frac{f_{i+1}}{\Gamma(1+i\alpha)} p_{j+1-i}}{p_0}.
$$

Simple calculations imply that

$$
\begin{aligned}
f_0 &= \theta, \\
f_1 &= \eta, \\
f_2 &= \frac{m_0 - \lambda q_0}{p_0} \theta - \frac{\Gamma(1+\alpha) p_1}{p_0} \eta, \\
f_3 &= \left(\frac{(m_0 - \lambda q_1)\Gamma(1+\alpha)}{p_0} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} \frac{m_0 - \lambda q_0}{p_0} p_1 \right) \theta - \frac{\Gamma(1+2\alpha) p_2 + m_0 - \lambda q_0 + \Gamma(1+\alpha) p_1^2}{p_0} \eta, \\
&\vdots
\end{aligned}
$$

Hence,

$$u_k(x) \;\; = \;\; \theta + \eta \frac{x^\alpha}{\Gamma(1+\alpha)} \tag{13}$$

$$+ \sum_{n=2}^{k} \left(\frac{ \begin{array}{c} \Gamma(1+(n-2)\alpha) \sum_{i=0}^{n-2} \frac{f_i}{\Gamma(1+i\alpha)} m_{n-2-i} \\ -\lambda\Gamma(1+(n-2)\alpha) \sum_{i=0}^{n-2} \frac{f_i}{\Gamma(1+i\alpha)} q_{n-2-i} \\ -\Gamma(1+(n-1)\alpha) \sum_{i=0}^{n-2} \frac{f_{i+1}}{\Gamma(1+i\alpha)} p_{n-1-i} \end{array} }{p_0} \right) \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \tag{14}$$

for $k = 1, 2, 3, \ldots$. Simple calculations imply that

$$u_1(x) \;\; = \;\; \theta + \eta \frac{x^\alpha}{\Gamma(1+\alpha)},$$

$$u_2(x) \;\; = \;\; \theta \left(1 + \frac{m_0 - \lambda q_0}{p_0 \Gamma(1+2\alpha)} x^{2\alpha} \right) + \eta \left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{\Gamma(1+\alpha)p_1}{p_0\Gamma(1+2\alpha)} x^{2\alpha} \right),$$

$$u_3(x) \;\; = \;\; \theta \left(1 + \frac{m_0 - \lambda q_0}{p_0 \Gamma(1+2\alpha)} x^{2\alpha} + \left(\frac{(m_0 - \lambda q_1)\Gamma(1+\alpha)}{p_0\Gamma(1+3\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} \frac{m_0 - \lambda q_0}{p_0\Gamma(1+3\alpha)} p_1 \right) \right)$$
$$+ \eta \left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{\Gamma(1+\alpha)p_1}{p_0\Gamma(1+2\alpha)} x^{2\alpha} - \frac{\Gamma(1+2\alpha)p_2 + m_0 - \lambda q_0 + \Gamma(1+\alpha)p_1^2}{p_0\Gamma(1+3\alpha)} x^{3\alpha} \right),$$

$$\vdots$$

It is easy to see that

$$u_k(x) = \theta \, h_1(x,\lambda) + \eta \, h_2(x,\lambda).$$

Using the the first boundary conditions, we get

$$0 \;\; = \;\; a_0 u(0) + a_1 \, D^\alpha u(0)$$
$$= \;\; a_0 \, \theta + a_1 \, \eta.$$

Since $a_0^2 + a_1^2 > 0$ and $\theta \, a_0 + \eta \, a_1 = 0$, we have the following two cases.

- If $a_0 = 0$, $\eta = 0$ and $a_1 \neq 0$. Thus,

$$u_k(x) = \theta \, h_1(x,\lambda).$$

Using the second boundary condition, we get

$$0 \;\; = \;\; a_2 u(1) + a_3 \, D^\alpha u(1)$$
$$= \;\; \theta \left(a_2 h_1(1,\lambda) + a_3 \, D^\alpha h_1(1,\lambda) \right)$$

or

$$a_2 h_1(1,\lambda) + a_3 \, D^\alpha h_1(1,\lambda) = 0. \tag{15}$$

- If $a_0 \neq 0$, $\theta = -\frac{a_1 \, \eta}{a_0}$. Thus,

$$u_k(x) = \eta \left(-\frac{a_1}{a_0} \, h_1(x,\lambda) + \, h_2(x,\lambda) \right).$$

Using the second boundary condition, we get

$$0 \;\; = \;\; a_2 u(1) + a_3 \, D^\alpha u(1)$$
$$= \;\; \eta \left(a_2 \left(-\frac{a_1}{a_0} \, h_1(1,\lambda) + \, h_2(1,\lambda) \right) + a_3 \, D^\alpha \left(-\frac{a_1}{a_0} \, h_1(1,\lambda) + \, h_2(1,\lambda) \right) \right)$$

or

$$a_2 \left(-\frac{a_1}{a_0} \ h_1(1,\lambda) + \ h_2(1,\lambda) \right) + a_3 \ D^\alpha \left(-\frac{a_1}{a_0} \ h_1(1,\lambda) + \ h_2(1,\lambda) \right) = 0. \quad (16)$$

To find the eigenvalues, we solve either Eqs. (3.10) or (3.11).

4. Convergence analysis

In this section, we study the convergence of the series (3.5) of the eigenfunction of problem (3.1)-(3.3).

Theorem 3.1: Suppose that $\sum_{n=0}^{\infty} f_n \ x^{\alpha n}$ converges to $u(x)$ on $(0,1)$ where $0 < \alpha < 1$. Then, $\sum_{n=0}^{\infty} f_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} x^{\alpha(n-1)}$ converges to $D^\alpha u(x)$ on $(0,1)$.

Proof: For $x \in (0,1)$,

$$
\begin{aligned}
D^\alpha u(x) &= \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} \frac{du(s)}{ds} ds \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} \frac{d}{ds} \left(\sum_{n=0}^{\infty} f_n \ s^{n\alpha} \right) ds \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} f_n \int_0^x (x-s)^{-\alpha} \frac{d}{ds} (s^{n\alpha}) ds \\
&= \sum_{n=0}^{\infty} f_n D^\alpha x^{n\alpha} \\
&= \sum_{n=0}^{\infty} f_n \frac{\Gamma(n\alpha + 1)}{\Gamma((n-1)\alpha + 1)} x^{\alpha(n-1)}.
\end{aligned}
$$

Thus, $\sum_{n=0}^{\infty} f_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} x^{\alpha(n-1)}$ converges to $D^\alpha u(x)$.

Theorem 3.2. Let $P(x) = \sum_{n=0}^{\infty} p_n x^{n\alpha}$, $Q(x) = \sum_{n=0}^{\infty} q_n x^{n\alpha}$, and $\mu(x) = \sum_{n=0}^{\infty} m_n x^{n\alpha}$. Then, the sequence $\{u_k\}$ defined in Eq. (3.5) is convergent to $u(x)$.

Proof: Let

$$
\begin{aligned}
&D^\alpha \left[\left(\sum_{n=0}^{\infty} p_n x^{n\alpha} \right) D^\alpha \left(\sum_{n=0}^{\infty} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right) \right] \\
&+ \lambda \left(\sum_{n=0}^{\infty} q_n x^{n\alpha} \right) \left(\sum_{n=0}^{\infty} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right) - \sum_{n=0}^{\infty} m_n x^{n\alpha} \\
&= \sum_{n=0}^{\infty} \delta_n x^{n\alpha}.
\end{aligned}
$$

Since $P(x) = \sum_{n=0}^{\infty} p_n x^{n\alpha}$, $Q(x) = \sum_{n=0}^{\infty} q_n x^{n\alpha}$, and $\mu(x) = \sum_{n=0}^{\infty} m_n x^{n\alpha}$,

$$\sum_{n=0}^{\infty} \delta_n x^{n\alpha} = 0.$$

Let

$$S_m = \sum_{n=m}^{\infty} \delta_n x^{n\alpha}.$$

Then, the sequence $\{S_m\}$ converges to zero. From Eq. (3.7), we see that if we substitute Eq. (3.5) into Eq (3.1),

$$Res_m(u(x)) = S_m.$$

Thus,

$$\underset{m \longrightarrow \infty}{Lim} Res_m(u(x)) = \underset{m \longrightarrow \infty}{Lim} S_m = 0.$$

Hence, the sequence $\{u_k\}$ defined in Eq. (3.5) is convergent to $u(x)$.

5. Numerical results

In this section, we present three examples to show the efficiency of the proposed method.

Example 1. Consider the following fractional Sturm-Liouville problem

$$D^\alpha[P(x)D^\alpha u(x)] + \lambda Q(x)u(x) = \mu(x)u(x), \ 0 \le x \le 1, \frac{1}{2} < \alpha \le 1$$

subject to

$$u(0) = 0, u(1) = 0$$

where $P(x) = Q(x) = 1$, and $\mu(x) = 0$. Figure 1 shows the graph of the eigenfunctions for $\alpha = 0.75$ and λ_1 and λ_2.
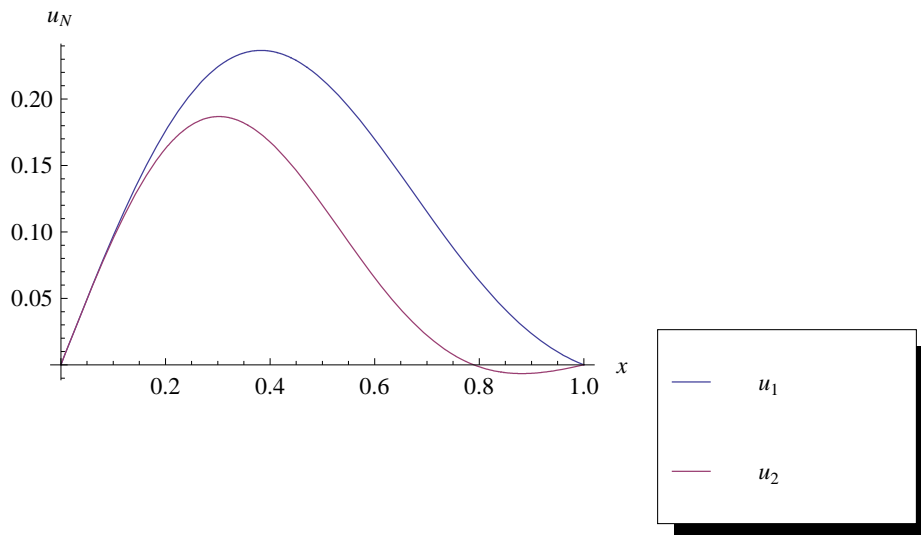

FIGURE 1. Graph of the eigenfunctions for $\alpha = 0.75$ and λ_1 and λ_2.

$\alpha = 0.501$	$\alpha = 0.75$	$\alpha = 0.95$	$\alpha = 0.99$
	8.78259056	8.2711826	9.6635259
	14.0844504	58.9901632	38.0440806
		96.6736528	84.9714381
		148.2953502	150.1337217
		199.5714025	233.5986357
		277.10713565	335.0977772
		295.45014962	454.7644066
			590.9308952

Table 1: Eigenvalues for different values of α

For $\alpha = 1$, the exact eigenvalues are well-known and they are given by

$$\lambda_n = n^2\pi^2, \quad n = 1, 2, 3, ...$$

It is worth mentioning that the eigenvalues of the problem in this example approaches to $n^2\pi^2$ when α approaches to 1. We noticed that the eigenvalue problem in Example 1 does not have any eigenvalue for $\alpha = 0.501$. For this reason, we look for the numerical value of α^* such that the eigenvalue problem of this example does not have any eigenvalue for $\frac{1}{2} < \alpha < \alpha^*$. We noticed that $\alpha^* = 0.7355$. Let

$$\delta_{i,j} = \left| \int_0^1 u_i(x) \ u_j(x) \ Q(x)dx \right|.$$

For $\alpha = 0.75$, $\delta_{1,2} = 5.7 * 10^{-16}$. Sample of these values for $\alpha = 0.95$ are given as

$$\delta_{1,2} = 5.7 * 10^{-16}, \ \delta_{4,6} = 2.6 * 10^{-16}, \delta_{1,6} = 8.3 * 10^{-16}.$$

Similarly for $\alpha = 0.99$,

$$\delta_{1,2} = 3.1 * 10^{-16}, \ \delta_{4,6} = 4.2 * 10^{-16}, \delta_{1,7} = 2.0 * 10^{-16}.$$

This means, the orthogonality relation holds. We notice that the eigenvalues satisfy the property

$$\lambda_1 \le \lambda_2 \le ....$$

Example 2. Consider the following fractional Sturm-Liouville problem

$$D^\alpha[P(x)D^\alpha u(x)] + \lambda Q(x)u(x) = \mu(x)u(x), \ 0 \le x \le 1, \frac{1}{2} < \alpha \le 1$$

subject to

$$u(0) = 0, u(1) = 0$$

where $P(x) = 1, Q(x) = 1 + x^\alpha$, and $\mu(x) = 0$. The available results for λ obtained by the present are summarized in Table 2.

$\alpha = 0.501$	$\alpha = 0.75$	$\alpha = 0.95$
3.7449685	4.9059651	5.8271193
5.5935961	9.9542383	21.8630978
25.4751193	14.2468657	100.8687952
151.8458500	25.8797084	234.2256821
	124.4751382	439.2009128
		721.0093446
		984.1247813

Table 2: Eigenvalues for different values of α

Let

$$\delta_{i,j} = \left| \int_0^1 u_i(x) \ u_j(x) \ Q(x) dx \right|.$$

For $\alpha = 0.502$, $\delta_{1,2} = 3.3 * 10^{-16}$ and $\delta_{2,4} = 4.9 * 10^{-16}$. Sample of these values for $\alpha = 0.75$ are given as

$$\delta_{1,2} = 2.2 * 10^{-16}, \ \delta_{4,5} = 4.1 * 10^{-16}, \delta_{1,5} = 6.9 * 10^{-16}.$$

Similarly for $\alpha = 0.95$,

$$\delta_{1,2} = 1.2 * 10^{-16}, \ \delta_{4,6} = 2.1 * 10^{-16}, \delta_{1,7} = 4.6 * 10^{-16}.$$

This means, the orthogonality relation holds. We notice that the eigenvalues satisfy the property

$$\lambda_1 \leq \lambda_2 \leq ....$$

Example 3. Consider the following fractional Sturm-Liouville problem

$$D^\alpha[P(x)D^\alpha u(x)] + \lambda Q(x)u(x) = \mu(x), \ 0 \leq x \leq 1, \frac{1}{2} < \alpha \leq 1$$

subject to

$$u(0) - D^\alpha y(0) = 0, u(1) + D^\alpha y(1) = 0$$

where $P(x) = Q(x) = 1$, and $\mu(x) = 0$. The available results for λ obtained by the present method are summarized in Table 3.

$\alpha = 0.75$	$\alpha = 0.8$	$\alpha = 0.99$
1.5022831	1.5178861	1.6932293
12.051004	10.3426409	13.1985857
14.5035813	21.0815838	41.8043677
	44.2389076	88.7795093
		153.9545688
		237.42247398
		338.92086546
		458.58377853
		596.14001070
		752.04906711
		918.71319420

Table 3: Eigenvalues for different values of α

It worth mention that, there are eigenvalues for all $\frac{1}{2} < \alpha \leq 1$. For example,the first eigenvalue for $\alpha = 0.5001$ is 1.68861. Let

$$\delta_{i,j} = \left| \int_0^1 u_i(x) \ u_j(x) \ Q(x) dx \right|.$$

For $\alpha = 0.75$, $\delta_{1,2} = 3.3 * 10^{-16}$ and $\delta_{2,3} = 4.9 * 10^{-16}$. Sample of these values for $\alpha = 0.8$ are given as

$$\delta_{1,2} = 1.6 * 10^{-16}, \ \delta_{2,4} = 1.9 * 10^{-16}, \delta_{3,4} = 2.8 * 10^{-16}.$$

Similarly for $\alpha = 0.99$,

$$\delta_{1,2} = 3.2 * 10^{-16}, \ \delta_{4,6} = 4.5 * 10^{-16}, \delta_{1,7} = 4.1 * 10^{-16}.$$

This means, the orthogonality relation holds. We notice that the eigenvalues satisfy the property

$$\lambda_1 \leq \lambda_2 \leq ....$$

6. Conclusion

In this paper, we have developed a numerical technique to find the eigenvalues of regular 2α-order fractional Sturm-Liouville problem for $\frac{1}{2} < \alpha \leq 1$. The method of solution is based on MFPS. The numerical results for the examples demonstrate the efficiency and accuracy of the present method. From the three examples which we mentioned in the previous section, we notice that our technique is very efficient for computing the eigenvalues of the fractional second order problems. We end this section by the following remarks.

- From Examples 1-3, we find that the generated eigenvalues satisfy the following property

$$\lambda_1 < \lambda_2 < \lambda_3 < ... < \lambda_n < ....$$

- From Examples 1-3, the orthogonality property

$$\int_0^1 u_i(x) \ u_j(x) \ Q(x) = 0, \ i \neq j$$

holds.

- Example 1 has eigenvalues when $\alpha > 0.7355$.
- The results in this paper confirm that MFPS is a powerful and efficient method for solving fractional Sturm-Liouville problems in different fields of sciences and engineering.
- MFPS is excellent tool due to rapid convergent.
- The existence and the convergent are proven in Theorems (3.2).
- We do not compare our results with others because we are the first who discuss this class of eigenvalues.

Future work:

- We state the following conjecture for the future work:

Conjecture: The eigenvalue problem in Example 1 does not have any eigenvalue for $\alpha < 0.7355$.

We will work to prove it.

- Generalize the proposed method for higher order fractional Sturm-Liouville problems.

Conflict of Interests: The authors declare that there is no conflict of interests regarding the publication of this article.

References

[1] J. He, Some applications of nonlinear fractional differential equations and their approximations, Bull. Sci. Technol., vol 15 (1999), pp. 86-90.
[2] M. Inc, The approximate and exact solutions of the space-and time-fractional Burger's equations with initial conditions by the variational iteration method, J. Math. Anal. Appl., vol 345(1) (2008), pp. 476-484.
[3] N. H. Sweilam, M.M. Khader, R.F. Al-Bar, Numerical studies for a multi-order fractional differential equation, Phys. Lett. A, vol 371(1-2) (2007), pp. 26-33.
[4] H. Jafari and V. Daftardar-Gejji, Solving linear and nonlinear fractional diffusion and wave equations by Adomian decomposition, Appl. Math. Comput., vol 180(2) (2006), pp. 488-497.
[5] A. Wazwaz, R. Rach, L. Bougoffa, Dual solutions for nonlinear boundary value problems by the Adomian decomposition method, International J. of Numerical Methods for Heat and Fluid Flow, (2016). In Press.

[6] A. Wazwaz, R. Rach, J.S. Duan, A review of the Adomian decomposition method and its applications to fractional differential equations, Commun. Frac. Calc., vol 3 (2) (2012), pp. 73 – 99.

[7] M. Syam, M. Al-Refai, Positive solutions and monotone iterative sequences for a class of higher order boundary value problems, Journal of Fractional Calculus and Applications, vol 4(14) (2013), pp. 1-13.

[8] S. Kazem, S. Abbasbandy, S. Kumar, Fractional-order Legendre functions for solving fractional-order differential equations, Applied Mathematical Modeling, vol 37 (2013), pp. 5498-5510.

[9] M. Syam, H. Siyyam, An Efficient Technique for Finding the Eigenvalues of Sixth-Order Sturm-Liouville problems, Applied Mathematical Sciences, vol 5(49) (2011), pp. 2425 - 2436.

[10] M. M. Djrbashian. A boundary value problem for a Sturm-Liouville type differential operator of fractional order. Izv. Akad. Nauk Armjan. SSR Ser. Mat., vol 5(2) (1970), pp. 71–96.

[11] A. M. Nahusev. The Sturm-Liouville problem for a second order ordinary differential equation with fractional derivatives in the lower terms. Dokl. Akad. Nauk SSSR, vol 234(2) (1997), pp.308–311.

[12] Q. M. Al-Mdallal, An efficient method for solving fractional Sturm-Liouville problems. Chaos, Solitons Fractals, vol 40(2009), pp. 183–189.

[13] Q. M. Al-Mdallal, On the numerical solution of fractional Sturm–Liouville problems, International Journal of Computer Mathematics, vol 87 (2010), pp. 2837-2845.

[14] S. Abbasbandy, A. Shirzadi, Homotopy analysis method for multiple solutions of the fractional Sturm-Liouville problems, Numer. Algorithms, vol 54 (4) (2010), pp. 521-532.

[15] V. S. Ertürk, Computing eigenelements of Sturm-Liouville Problems of fractional order via fractional differential transform method, Math. Comput. Appl., vol 16 (3) (2011), pp. 712-720.

[16] Y. Luchko, Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation, Fract. Calc. Appl. Anal., vol 15(1) (2012), pp. 141–160.

[17] A. Neamaty, R. Darzi, S. Zaree, B. M. zadeh, Haar wavelet operational matrix of fractional order integration and its application for eigenvalues of fractional Sturm– Liouville problem, World Applied Sciences Journal, 16 (12) (2012), pp. 1668–1672.

[18] Z. Shi, Y. Y. Cao, Application of Haar wavelet method to eigenvalue problems of higher order, Appl. Math. Model., vol 36 (9) (2012), pp. 4020–4026.

[19] E. Bas, F. Metin, Spectral properties of fractional Sturm–Liouville problem for diffusion operator, arXiv:1212.4761, (2012).

[20] E. Bas, F. METIN, A Note Basis Properties for Fractional Hydrogen Atom Equation, arXiv:1303.2839v2, (2013).

[21] E. Bas, Fundamental Spectral Theory of Fractional Singular Sturm-Liouville Operator, Journal of Function Spaces and Applications, Article ID 915830, 7 pages, Vol (2013).

[22] M. Zayernouri, G. E. Karniadakis, Fractional Sturm–Liouville eigen-problems: theory and numerical approximation, J. Comput. Phys., vol 252 (2013), pp. 495–517.

[23] M. Klimek, T. Odzijewicz, A. B. Malinowska, Variational Methods for the Fractional Sturm–Liouville Problem, Journal of Mathematical Analysis and Applications, vol 416 (1) (2014), pp. 402–426.

[24] F. D. Saei, S. Abbasi, Z. Mirzayi, Inverse Laplace transform method for multiple solutions of the fractional Sturm-Liouville problems, Computational Methods for Differential Equations, vol 2 (1) (2014), pp. 56-61.

[25] P. Antunes, R. Ferreira, An augmented – RBE method for solving fractional Sturm-Liouville eigenvalues problems, SIAM Journal of Scientific Computing, vol 37(1) (2015), A515-A535.

[26] B. Jin, R. Lazarov, X. Lu, Z. Zhou, A simple finite element method for boundary value problems with a Riemann-Liouville derivative, Journal of Computational and Applied Mathematics, arXiv:1502.07840v1, (2015).

[27] K. B. Oldham, J. Spanier, The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order, Courier Corporation, (2002).

[28] M. Alquran, K. Al-Khaled, J. Chattopadhyay, Analytical solutions of fractional population diffusion model: Residual power series, Nonlinear Studies, 2015, 22(1), 31-39

[29] M. Alquran, Analytical solution of time-fractional two-component evolutionary system of order 2 by residual power series method, Journal of Applied Analysis and Computation, 2015, 5(4), 589-599.

[30] M. Alquran M., K. Al-Khaled, T. Sarda, J. Chattopadhyay, Revisited Fisher's equation in a new outlook: A fractional derivative approach, Physica A: Statistical Mechanics and its Applications, 2015, 438, 81-93.

[31] H.M. Jaradat, S. Al-Shara', Q.K.J. Khan, M. Alquran, K. Al-Khaled, Analytical Solution of Time-Fractional Drinfeld-Sokolov-Wilson System Using Residual Power Series Method, IAENG International Journal of Applied Mathematics, 2016, 46(1), 64-70.

[32] H.M. Jaradat, I. Jaradat, M. Alquran, M.M.M. Jaradat, Z. Mustafa, K. Abohassan, R. Abdelkarim, Approximate solutions to the generalized time-fractional Ito system, Italian journal of pure and applied mathematics, 2017, 37, 699-710.

[33] M. Alquran, K. Al-Khaled, S. Sivasundaram, H.M. Jaradat, Mathematical and numerical study of existence of bifurcations of the generalized fractional Burgers-Huxley equation, Nonlinear Studies, 2017, 24(1), 235-244.

[34] A. Ababneh, M. Alquran, K. Al-Khaled, J. Chattopadhyay, A New and Elegant Approach for Solving nxn-Order Linear Fractional Differential Equations. Mediterr. J. Math., 2017, 14(2), 1-18

[35] M. Alquran, H.M. Jaradat,?M.I. Syam, Analytical solution of the time-fractional Phi-4 equation by using modified residual power series method. *Nonlinear Dynamics*. 90(4) (2017) 2525-2529.

[36] I. Jaradat, M. Al-Dolat, K. Al-Zoubi, M. Alquran, Theory and applications of a more general form for fractional power series expansion. *Chaos, Solitons Fractals*. Vol. 108 (2018) 107–110.

[37] M. Alquran, I. Jaradat, A novel scheme for solving Caputo time-fractional nonlinear equations: theory and application. *Nonlinear Dynamics*. 91(4) (2018) 2389-2395.

M.I. SYAM

DEPARTMENT OF MATHEMATICAL SCIENCES, UNITED ARAB EMIRATES UNIVERSITY , COLLEGE OF SCIENCE, P. O. BOX 17551, AL-AIN, UNITED ARAB EMIRATES
    E-mail address: `m.syam@uaeu.ac.ae`

M. ALQURAN

DEPARTMENT OF MATHEMATICS AND STATISTICS, JORDAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, P.O. BOX: 3030, IRBID, 22110, JORDAN
    E-mail address: `marwan04@just.edu.jo`

H.M. JARADAT

DEPARTMENT OF MATHEMATICS, AL AL-BAYT UNIVERSITY, MAFRAQ, JORDAN
    E-mail address: `husseinjaradat@yahoo.com`

S. AL-SHARA'

DEPARTMENT OF MATHEMATICS, AL AL-BAYT UNIVERSITY, MAFRAQ, JORDAN
    E-mail address: `smath1973@yahoo.com`