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FRACTIONAL NONLINEAR EVOLUTION EQUATIONS WITH SECTORIAL LINEAR OPERATORS

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ABSTRACT. We study the existence and uniqueness of a local mild solution for a class of nonlinear evolution equations involving the Caputo fractional time derivative of order α $(0 < \alpha < 1)$ and a sectorial linear operator A in the linear part. We put on the nonlinear part some conditions involving the fractional power of A. By applying Banach Fixed Point Theorem, a unique local mild solution with smoothing effects, estimates, and a behavior at time t close to 0 is obtained. An example associated with anomalous diffusion with chemotaxis, as an application of our result, is given.

1. INTRODUCTION

Consider the fractional chemotaxis-diffusion system

$$\begin{aligned} &D_t^\alpha u = \Delta u - \nabla \cdot u \nabla v, \quad \text{in } \Omega \times (0, \infty), \\ &D_t^\alpha v = \Delta v - v + u, \quad \text{in } \Omega \times (0, \infty), \\ &\frac{\partial v}{\partial n} = \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega \times (0, \infty), \\ &u(\cdot, 0) = u_0,\ v(\cdot, 0) = v_0, \quad \text{in } \Omega \end{aligned} \tag{1.1}$$

where $0 < \alpha < 1$, D_t^α is the Caputo fractional derivative of order α, and $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^2 boundary. The first equation of the system (1.1) which is called the fractional chemotaxis-diffusion equation was derived by Langlands and Henry in [1]. When $\alpha = 1$, the system (1.1) is well known by the Keller-Segel chemotaxis (KS) model. In this model (KS), u and v stand for the concentration of amoebae and acrasin, respectively, where acrasin is a chemoattractant produced by the amoebae. The model (KS) describes the space and time evolution of the concentration of diffusing amoebae that is chemotactically attracted by diffusing acrasin (see [2]). In general, the model (KS) can be used to explain the space and time evolution of the concentration of a diffusing species that is chemotactically attracted by a diffusing chemoattractant. Meanwhile, when $0 < \alpha < 1$, the

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system (1.1) describes the process in which the attracted species and attracting chemoattractant diffuse anomalously.

In this paper, we study the existence and uniqueness of a local mild solution to the fractional abstract Cauchy problem associated with the system (1.1), that is

$$\begin{aligned} &D_t^\alpha u = Au + f(u),\ t > 0,\ 0 < \alpha < 1, \\ &u(0) = u_0, \end{aligned} \tag{1.2}$$

where X is a Banach space, D_t^α is the Caputo fractional derivative of order α, $A : D(A) \subseteq X \to X$ is a sectorial linear operator, $u_0 \in X$, and $f : X \to X$ satisfies some nonlinear conditions. Guswanto and Suzuki in [3] also studied this problem with f and u_0 satisfying the conditions :

(f1) $f(0) = 0$,

(f2) there exist $C_0 > 0$, $\vartheta > 1$, and $0 < \beta < 1$ such that

$$\|f(u) - f(v)\| \le C_0(\|A^\beta u\| + \|A^\beta v\|)^{\vartheta-1}\|A^\beta u - A^\beta v\|,$$

for all $u, v \in D(A^\beta)$,

(f3) $u_0 \in D(A^\nu)$ for some $0 < \nu < 1$.

Meanwhile, here, we use another conditions on f and u_0 as stated below :

(F1) $f(0) = 0$,

(F2) there exist $C_0 > 0$ and $0 < \beta < 1$ such that

$$\begin{aligned} \|f(u) - f(v)\| \le C_0 \big[&(\|u\| + \|v\|)\|A^\beta u - A^\beta v\| \\ &+(\|A^\beta u\| + \|A^\beta v\|)\|u - v\|\big], \end{aligned}$$

for all $u, v \in D(A^\beta)$,

(F3) $u_0 \in D(A)$.

The conditions (F1)-(F3) are the case of Yagi [4]. Von Wahl, in [5], used these conditions to study the Navier-Stokes equations. As in [3, 4, 5], we apply Banach Fixed Point Theorem to construct a local mild solution to the problem (1.2) by employing the properties of the solution operators generated by A and the fractional power of A. In this paper, we obtain the existence and uniqueness of a local mild solution with smoothing effects, estimates, and a behavior at time t close to 0.

This paper is composed of four sections. In section 2, we introduce briefly the fractional integration and differentiation of Caputo operator. In this section, we also provides some properties of analytic solution operators for fractional evolution equations including some estimates involving the fractional power of sectorial operators. In the next section, our main result is shown. Finally, in the last section, an application of our main result to investigate the solution to the system (1.1) describing anomalous Diffusion problem with chemotaxis is given.

2. Preliminaries

2.1. **Fractional Time Derivative.** Let $0 < \alpha < 1$, $a \ge 0$ and $I = (a, T)$ for some $T > 0$. The *fractional integral* of order α is defined by

$$J_{a,t}^\alpha f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \quad f \in L^1(I),\ t > a. \tag{2.1}$$

We set $J_{a,t}^0 f(t) = f(t)$. The fractional integral operator (2.1) obeys the semigroup property

$$J_{a,t}^\alpha J_{a,t}^\beta = J_{a,t}^{\alpha+\beta}, \quad 0 \le \alpha,\ \beta < 1. \tag{2.2}$$

Caputo fractional derivative of order α is defined by

$$D_{a,t}^{\alpha}f(t) = D_t \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}(f(s)-f(0))ds,\ t>a, \tag{2.3}$$

if $f \in L^1(I),\ t^{-\alpha} * f \in W^{1,1}(I)$, or

$$D_{a,t}^{\alpha}f(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}D_s f(s)ds,\ t>a, \tag{2.4}$$

if $f \in W^{1,1}(I)$ where $*$ denotes the convolution of functions

$$(f*g)(t) = \int_a^t f(t-\tau)g(\tau)d\tau, \quad t>a,$$

and $W^{1,1}(I)$ is the set of all functions $u \in L^1(I)$ such that the distributional derivative of u exists and belongs to $L^1(I)$. The operator $D_{a,t}^{\alpha}$ is a left inverse of $J_{a,t}^{\alpha}$, that is

$$D_{a,t}^{\alpha}J_{a,t}^{\alpha}f(t) = f(t), \quad t>a, \tag{2.5}$$

but it is not a right inverse, that is

$$J_{a,t}^{\alpha}D_{a,t}^{\alpha}f(t) = f(t) - f(a), \quad t>a. \tag{2.6}$$

For $a=0$, we set $J_{a,t}^{\alpha} = J_t^{\alpha}$ and $D_{a,t}^{\alpha} = D_t^{\alpha}$. We refer to Kilbas et al. [6] or Podlubny [7] for more details concerning the fractional integral and derivative.

2.2. **Analytic Solution Operators.** In this section, we provide briefly some results concerning solution operators for the fractional Cauchy problem

$$\begin{aligned} D_t^{\alpha}u(t) &= Au(t) + f(t),\ t>0, \\ u(0) &= u_0. \end{aligned} \tag{2.7}$$

For more details, we refer to Guswanto [8].

Henceforth, we assume that the linear operator $A : D(A) \subseteq X \to X$ satisfies the properties that there is a constant $\theta \in (\pi/2, \pi)$ such that

$$\rho(A) \supset S_\theta = \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \theta\}, \tag{2.8}$$

$$\|R(\lambda;A)\| \le \frac{M}{|\lambda|},\ \lambda \in S_\theta, \tag{2.9}$$

where $R(\lambda;A) = (\lambda - A)^{-1}$ and $\rho(A)$ are the resolvent operator and resolvent set of A, respectively. We call A a sectorial operator.

Proposition 2.1. *Every sectorial operator is closed*

Proof. We suppose A is a sectorial operator. To prove A is closed, we must show that if $\{x_n\}_{n\in\mathbb{N}} \subseteq D(A)$, $x_n \to x \in X$, and $Ax_n \to y \in X$, as $n \to \infty$, then $x \in D(A)$ and $Ax = y$.

Note that the resolvent set $\rho(A)$ of A contains all $\lambda \in \mathbb{C}$ such that $\lambda - A : D(A) \to X$ is bijective and the resolvent operator $R(\lambda;A)$ of A is bounded. Thus, since, for any $\lambda \in \rho(A)$, $\lambda - A$ is bijective, we have if $z_n = (\lambda - A)x_x$ then $x_n = R(\lambda;A)z_n$ for $n \in \mathbb{N}$. Observe that $z_n \to \lambda x - y$, as $n \to \infty$. Consequently, by the boundedness (which is equivalent to the conitinuity) of $R(\lambda;A)$, we get $x = R(\lambda;A)(\lambda x - y)$ implying $(\lambda - A)x = \lambda x - y$. We obtain $x \in D(A)$ and $Ax = y$. □

Definition 2.1. *For $r>0$ and $\pi/2 < \omega < \theta$,*

$$\Gamma_{r,\omega} = \{\lambda \in \mathbb{C} : |\arg(\lambda)| = \omega, |\lambda| \ge r\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda)| \le \omega, |\lambda| = r\}.$$

The linear operator A generates solution operators for the problem (2.7), those are

$$S_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha; A)d\lambda, \quad t>0, \tag{2.10}$$

$$P_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t}R(\lambda^\alpha; A)d\lambda, \quad t>0, \tag{2.11}$$

where $\Gamma_{r,\omega}$ is oriented counterclockwise. By Cauchy's theorem, the integral form (2.10) and (2.11) are independent of $r>0$ and $\omega \in (\pi/2, \theta)$.

Let $B(X;Y)$ be the set of all bounded linear operators $T : X \to Y$ where X and Y are Banach spaces. If $X = Y$ then $B(X;X) := B(X)$. Also, let $BC((0,T];X)$ be the set of all bounded and continuous functions $w : (0,T] \to X$.

The properties of the families $\{S_\alpha(t)\}_{t>0}$ and $\{P_\alpha(t)\}_{t>0}$ are given in the following theorems.

Theorem 2.1. *Let A be a sectorial operator and $S_\alpha(t)$ be the operator defined by (2.10). Then the following statements hold.*

(i) $S_\alpha(t) \in B(X)$ *and there exists a constant $C_1 = C_1(\alpha) > 0$ such that*

$$\|S_\alpha(t)\| \le C_1, \quad t>0,$$

(ii) $S_\alpha(t) \in B(X;D(A))$ *for $t>0$, and if $x \in D(A)$ then $AS_\alpha(t)x = S_\alpha(t)Ax$. Moreover, there exists a constant $C_2 = C_2(\alpha) > 0$ such that*

$$\|AS_\alpha(t)\| \le C_2 t^{-\alpha}, \quad t>0,$$

(iii) *The function $t \mapsto S_\alpha(t)$ belongs to $C^\infty((0,\infty); B(X))$ and it holds that*

$$S_\alpha^{(n)}(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{t\lambda}\lambda^{\alpha+n-1}R(\lambda^\alpha; A)d\lambda,\ n = 1,2,\ldots$$

and there exist constants $M_n = M_n(\alpha) > 0, n = 1,2,\ldots$ such that

$$\|S_\alpha^{(n)}(t)\| \le M_n t^{-n}, \quad t>0,$$

Moreover, it has an analytic continuation $S_\alpha(z)$ to the sector $S_{\theta-\pi/2}$ and, for $z \in S_{\theta-\pi/2}$, $\eta \in (\pi/2,\theta)$, it holds that

$$S_\alpha(z) = \frac{1}{2\pi i}\int_{\Gamma_{r,\eta}} e^{\lambda z}\lambda^{\alpha-1}R(\lambda^\alpha; A)d\lambda.$$

Theorem 2.2. *Let A be a sectorial operator and $P_\alpha(t)$ be the operator defined by (2.11). Then the following statements hold.*

(i) $P_\alpha(t) \in B(X)$ *and there exists a constant $L_1 = L_1(\alpha) > 0$ such that*

$$\|P_\alpha(t)\| \le L_1 t^{\alpha-1}, \quad t>0,$$

(ii) $P_\alpha(t) \in B(X;D(A))$ *for all $t>0$, and if $x \in D(A)$ then $AP_\alpha(t)x = P_\alpha(t)Ax$. Moreover, there exists a constant $L_2 = L_2(\alpha) > 0$ such that*

$$\|AP_\alpha(t)\| \le L_2 t^{-1}, \quad t>0,$$

(iii) *The function $t \mapsto P_\alpha(t)$ belongs to $C^\infty((0,\infty); B(X))$ and it holds that*

$$P_\alpha^{(n)}(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{t\lambda}\lambda^{n}R(\lambda^\alpha; A)d\lambda,\ n = 1,2,\ldots$$

and there exist constants $K_n = K_n(\alpha) > 0, n = 1,2,\ldots$ such that

$$\|P_\alpha^{(n)}(t)\| \le K_n t^{\alpha-n-1}, \quad t>0,$$

Moreover, it has an analytic continuation $P_\alpha(z)$ to the sector $S_{\theta-\pi/2}$ and, for $z \in S_{\theta-\pi/2}$, $\eta \in (\pi/2, \theta)$, it holds that

$$P_\alpha(z) = \frac{1}{2\pi i}\int_{\Gamma_{r,\eta}} e^{\lambda z} R(\lambda^\alpha; A) d\lambda.$$

The following theorem states some identities concerning the operators $S_\alpha(t)$ and $P_\alpha(t)$ including the semigroup-like property.

Theorem 2.3. *Let A be a sectorial operator, $S_\alpha(t)$ and $P_\alpha(t)$ be the operators defined by* (2.10) *and* (2.11)*, respectively. Then the following statements hold.*

(i) *For $x \in X$ and $t > 0$,*

$$S_\alpha(t)x = J_t^{1-\alpha} P_\alpha(t)x, \quad D_t S_\alpha(t)x = AP_\alpha(t)x,$$

(ii) *For $x \in D(A)$ and $s, t > 0$,*

$$D_t^\alpha S_\alpha(t)x = AS_\alpha(t)x,$$

$$S_\alpha(t+s)x = S_\alpha(t)S_\alpha(s)x - A\int_0^t \int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} P_\alpha(\tau)P_\alpha(r)x dr d\tau.$$

Next theorem shows us the behavior of the operator $S_\alpha(t)$ at t close to 0^+.

Theorem 2.4. *Let A be a sectorial operator and $S_\alpha(t)$ be the operator defined by* (2.10)*. Then the following statements hold.*

(i) *If $x \in \overline{D(A)}$ then $\lim_{t\to 0^+} S_\alpha(t)x = x$,*

(ii) *For every $x \in D(A)$ and $t > 0$,*

$$\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)x d\tau \in D(A),$$

$$\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} AS_\alpha(\tau)x d\tau = S_\alpha(t)x - x,$$

(iii) *If $x \in D(A)$ and $Ax \in \overline{D(A)}$ then*

$$\lim_{t\mapsto 0^+} \frac{S_\alpha(t)x - x}{t^\alpha} = \frac{1}{\Gamma(\alpha+1)} Ax.$$

The representation of the solution to (2.7) in term of $S_\alpha(t)$ and $P_\alpha(t)$ is given in the following theorem.

Theorem 2.5. *Let $u \in C^1((0,\infty); X) \cap L^1((0,\infty); X)$, $u(t) \in D(A)$ for $t \in [0,\infty)$, $Au \in L^1((0,\infty); X)$, $f \in L^1((0,\infty); D(A))$, and $Af \in L^1((0,\infty); X)$. If u is a solution to the problem* (2.7) *then*

$$u(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s) f(s) ds, \quad t > 0. \tag{2.12}$$

Now, we consider the fractional power of operator A

$$A^{-\beta}x = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} \lambda^{-\beta} R(\lambda; A) x d\lambda, \ x \in X, \tag{2.13}$$

and $A^\beta = (A^{-\beta})^{-1}$, for $\beta > 0$. If $x \in D(A)$, we can denote $A^\beta x$ by

$$A^\beta x = A(A^{\beta-1}x) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} \lambda^{\beta-1} R(\lambda; A) Ax d\lambda, \ 0 < \beta < 1. \tag{2.14}$$

Proposition 2.2. *Let A be a sectorial operator and $\beta > 0$. The fractional power $A^{-\beta}$ of A is a bounded operator on X.*

Proof. Since A is sectorial, there is a constant $\theta \in (\pi/2, \pi)$ such that (2.8) and (2.9) are satisfied. Then, by Definition 2.1 and (2.13), for $\beta > 0$ and $x \in X$, we have

$$\begin{aligned}\|A^{-\beta}x\| &\le \frac{1}{2\pi}\int_{\Gamma r,\omega} \|\lambda^{-\beta}R(\lambda;A)x\||d\lambda| \\ &\le \frac{M}{2\pi}\int_{\Gamma r,\omega} |\lambda|^{-\beta-1}|d\lambda|\|x\| \\ &= \frac{M}{2\pi}\left(2\int_r^\infty R^{-\beta-1}dR + \int_{-\omega}^{\omega} r^{-\beta}d\theta\right)\|x\| \\ &= \frac{M}{\pi r^\beta}\left(\frac{1}{\beta}+\omega\right)\|x\|.\end{aligned}$$

Thus $A^{-\beta}$ is a bounded operator on X for $\beta > 0$. □

By the boundedness of $A^{-\beta}$ for $\beta > 0$, we get the closedness of A^β for $\beta > 0$.

Proposition 2.3. *Let A be a sectorial operator and $\beta > 0$. The fractional power A^β of A is closed.*

Proof. To show that A^β is closed, we must prove that if $\{x_n\}_{n\in\mathbb{N}} \subseteq D(A^\beta)$, $x_n \to x \in X$, and $A^\beta x_n \to y \in X$, as $n \to \infty$, then $x \in D(A^\beta)$ and $A^\beta x = y$. Recall that $A^\beta = (A^{-\beta})^{-1}$. It follows that if $y_n = A^\beta x_n$ then $x_n = A^{-\beta}y_n$ for $n \in \mathbb{N}$. By Proposition 2.2, $A^{-\beta}$ is a bounded or continuous operator on X. Consequently, since $x_n \to x$ and $y_n \to y$, as $n \to \infty$, we have $x = A^{-\beta}y$. It implies $x \in D(A^\beta)$ and $y = A^\beta x$. □

Some estimates involving A^β and the operators families $\{S_\alpha(t)\}_{t>0}$, $\{P_\alpha(t)\}_{t>0}$ generated by the sectorial operator A are provided by the following theorem. These estimates are analogous to those as stated in Theorem 6.13 in [9] for analytic semigroups.

Theorem 2.6. *For each $0 < \beta < 1$, there exist positive constants $C_1' = C_1'(\alpha,\beta)$, $C_2' = C_2'(\alpha,\beta)$, and $C_3' = C_3'(\alpha,\beta)$ such that for all $x \in X$,*

$$\|A^\beta S_\alpha(t)x\| \le C_1' t^{-\alpha}(t^{-\alpha(\beta-1)}+1)\|x\|, \quad t > 0, \tag{2.15}$$

$$\|A^\beta P_\alpha(t)x\| \le C_2' t^{-\alpha(\beta-1)-1}\|x\|, \quad t > 0. \tag{2.16}$$

Moreover, for all $x \in D(A^\beta)$,

$$\|S_\alpha(t)x - x\| \le C_3' t^{\alpha\beta}\|A^\beta x\|, \quad t > 0. \tag{2.17}$$

Now, let $\xi_\zeta = \alpha(\zeta-1)+1$, for $0 < \zeta < 1$, and $x^+ = \max\{0, x\}$, for $x \in \mathbb{R}$. Thus we have the following results.

Corollary 2.1. *For each $\beta > (2-1/\alpha)^+$ and $x \in X$,*

$$t^{\xi_\beta}\|A^\beta S_\alpha(t)x\| \le 2C_1'\|x\|, \quad 0 < t \le 1, \tag{2.18}$$

$$t^{\xi_\beta}\|A^\beta S_\alpha(t)x\| \le 2C_1' t^{1-\alpha}\|x\|, \quad t > 1, \tag{2.19}$$

$$t^{\xi_\beta}\|A^\beta P_\alpha(t)x\| \le C_2'\|x\|, \quad t > 0, \tag{2.20}$$

and

$$t^{\xi_\beta}\|A^\beta S_\alpha(t)x\| \to 0, \quad \textit{as}\, t \to 0^+. \tag{2.21}$$

Remark 2.1. *If $\beta = 2 - 1/\alpha > 0$, implying $\xi_\beta - \alpha = 0$, the estimates* (2.18), (2.19), *and* (2.20) *still hold for all $x \in X$. Furthermore, using* (2.13), (2.14), *Theorem 2.1(i), Theorem 2.1(ii), and Proposition 2.2, if $x \in D(A)$ then*

$$\begin{aligned}\|A^\beta S_\alpha(t)x\| &= \left\| \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} \lambda^{\beta-1} R(\lambda; A) S_\alpha(t) A x d\lambda \right\| \\ &= \|A^{\beta-1} S_\alpha(t) A x\| \\ &\leq C_1 \|A^{\beta-1}\| \|Ax\|\end{aligned}$$

implying (2.21).

Furthermore, we have the same result as Theorem 2.3(ii) with weaker condition.

Theorem 2.7. *Let $0 < \beta < 1$. Then, for $x \in D(A^\beta)$ and $s, t > 0$,*

$$D_t^\alpha S_\alpha(t)x = AS_\alpha(t)x, \tag{2.22}$$

$$S_\alpha(t+s)x = S_\alpha(t)S_\alpha(s)x - A \int_0^t \int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} P_\alpha(\tau) P_\alpha(r) x dr d\tau. \tag{2.23}$$

3. Main Results

In this section, we show the existence and uniqueness of a mild solution for the problem (1.2) under certain conditions by applying Banach Fixed Point Theorem. Based on Theorem 2.5, we define a mild solution to the problem (1.2) as follows.

Definition 3.1. *A continuous function $u : (0, T] \to X$ is a mild solution to the problem* (1.2) *if it satisfies*

$$u(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s) f(u(s)) ds, \quad 0 < t \leq T.$$

The conditions on f are

(i) $f(0) = 0$,
(ii) there exist $C_0 > 0$ and $0 < \beta < 1$ such that

$$\begin{aligned}\|f(u) - f(v)\| \leq C_0 \big[&(\|u\| + \|v\|) \|A^\beta u - A^\beta v\| \\ &+ (\|A^\beta u\| + \|A^\beta v\|) \|u - v\| \big],\end{aligned} \tag{3.1}$$

for all $u, v \in D(A^\beta)$.

Under the conditions on f above, we obtain the following result.

Theorem 3.1. *Let A be sectorial, $u_0 \in D(A)$, and $1/2 < \alpha < 1$. Then there exists $r_\alpha > 0$ such that if $\|u_0\| < r_\alpha$ then, for some $T > 0$, the problem* (1.2) *has a unique mild solution u satisfying*

$$u \in BC((0,T]; D(A^{2-1/\alpha})), \quad t^\alpha A^{2-1/\alpha} u \in BC((0,T]; X),$$
$$\lim_{t \to 0^+} t^\alpha A^{2-1/\alpha} u(t) = 0$$

with

$$\|u(t)\| \leq M_1 \|u_0\|, \quad \|A^{2-1/\alpha} u(t)\| \leq M_2 t^{-\alpha} \|u_0\|, \quad t \in (0, T],$$

for some $M_i > 0$, $i = 1, 2$.

Proof. Observe that $1/2 < \alpha < 1$ assures that $0 < 2 - 1/\alpha < 1$ and if $\beta = 2 - 1/\alpha$ then $\xi_\beta = \alpha$. We then define the Banach space

$$E_{\alpha,T} = \left\{u : [0,T] \to X : u \in BC((0,T]; D(A^\beta)), t^\alpha A^\beta u \in BC((0,T];X)\right\}$$

equipped with the norm

$$|||u|||_{\alpha,T} = \sup_{0<t\le T} t^\alpha \|A^\beta u(t)\| + \sup_{0<t\le T} \|u(t)\|.$$

We also define a mapping F on $B_{\alpha,T}$ by

$$Fu(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s)f(u(s))ds, \quad 0 < t \le T,$$

where $B_{\alpha,T}$ is the closed subset of $E_{\alpha,T}$ defined by

$$B_{\alpha,T} = \left\{u \in E_{\alpha,T} : \sup_{0<t\le T} t^\alpha \|A^\beta u(t)\| \le K_1, \sup_{0<t\le T} \|u(t) - u_0\| \le K_2\right\}$$

with T, K_1, and K_2 are some positive constants which will be specified later.

Step 1. We prove that $E_{\alpha,T}$ is a Banach space. Suppose that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $E_{\alpha,T}$. It means that $|||u_m - u_n|||_{\alpha,T} \to 0$, as $m, n \to \infty$. Consequently, for $0 < t \le T$,

$$\|u_m(t) - u_n(t)\| \le \sup_{0<t\le T} \|u_m(t) - u_n(t)\| \to 0, \text{ as } m, n \to \infty \tag{3.2}$$

and

$$t^\alpha \|A^\beta u_m(t) - A^\beta u_n(t)\| \le \sup_{0<t\le T} t^\alpha \|A^\beta u_m(t) - A^\beta u_n(t)\| \to 0, \text{ as } m, n \to \infty. \tag{3.3}$$

In other words, from (3.2) and (3.3), both $\{u_n(t)\}_{n\in\mathbb{N}}$ and $\{t^\alpha A^\beta u_n(t)\}_{n\in\mathbb{N}}$, for $0 < t \le T$, are Cauchy sequences in X. Since X is a Banach space, there exist $u(t) \in X$ and $v(t) \in X$ such that $u_n(t) \to u(t)$ and $t^\alpha A^\beta u_n(t) \to v(t)$, as $n \to \infty$, for $0 < t \le T$, respectively. Moreover, both $\{u_n(t)\}_{n\in\mathbb{N}}$ and $\{t^\alpha A^\beta u_n(t)\}_{n\in\mathbb{N}}$ converge uniformly in X for $0 < t \le T$. Note that, for $0 < t \le T$,

$$\|u(t)\| = \lim_{n\to\infty} \|u_n(t)\| < \infty$$

and

$$\|v(t)\| = \lim_{n\to\infty} t^\alpha \|A^\beta u_n(t)\| < \infty$$

implying that $u(t), v(t) \in B((0,T];X)$. Next, consider that, by Proposition 2.3, A^β is closed. It implies that $u(t) \in D(A^\beta)$ and $t^\alpha A^\beta u(t) = v(t)$, for $0 < t \le T$. Finally, since, for $0 < t \le T$, both $\{u_n(t)\}_{n\in\mathbb{N}}$ and $\{t^\alpha A^\beta u_n(t)\}_{n\in\mathbb{N}}$ converge uniformly in X, we have $u(t) \in C((0,T]; D(A^\beta))$ and $t^\alpha A^\beta u(t) \in C((0,T];X)$. Thus $u(t) \in BC((0,T]; D(A^\beta))$ and $t^\alpha A^\beta u(t) \in BC((0,T];X)$. We obtain $E_{\alpha,T}$ is a Banach space.

Step 2. We prove the continuity of $A^\beta Fu(t)$ and $Fu(t)$ with respect to $t \in (0,T]$ in the norm $\|\cdot\|$ of X. Observe that, by Theorem 2.1(ii), for each $x \in X$, $S_\alpha(t)x \in D(A)$, $t > 0$. Then by (2.13), (2.14), and Theorem 2.1(ii) again, for $u_0 \in D(A)$ and $0 < \beta < 1$,

$$A^\beta S_\alpha(t)u_0 = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} \lambda^{\beta-1} R(\lambda; A) S_\alpha(t) A u_0 d\lambda = A^{\beta-1} S_\alpha(t) A u_0. \tag{3.4}$$

By Proposition 2.2, for $0 < \beta < 1$, $A^{\beta-1}$ is a bounded operator on X. Next, note that, by Theorem 2.1(iii), $S_\alpha(t)Au_0$ is continuous with respect to $t \in (0,\infty)$.

Consequently, by (3.4) and the boundedness of $A^{\beta-1}$, for $0 < \beta < 1$, we have, for $u_0 \in D(A)$, $A^\beta S_\alpha(t)u_0$ is continuous with respect to $t \in (0,\infty)$. Thus it remains to show the continuity of

$$A^\beta \int_0^t P_\alpha(t-s)f(u(s))ds, \quad 0 < t \leq T.$$

Consider that

$$\begin{aligned} A^\beta \int_0^{t+h} & P_\alpha(t+h-s)f(u(s))ds - A^\beta \int_0^t P_\alpha(t-s)f(u(s))ds \\ &= A^\beta \int_{-h}^t P_\alpha(t-s)f(u(s+h))ds - A^\beta \int_0^t P_\alpha(t-s)f(u(s))ds \\ &= A^\beta \int_0^t P_\alpha(t-s)(f(u(s+h)) - f(u(s)))ds \\ &\quad + A^\beta \int_0^h P_\alpha(t+h-s)f(u(s))ds. \end{aligned}$$

Observe that, for $u \in B_{\alpha,T}$,

$$\begin{aligned} \|f(u(t+h)) - f(u(t))\| \leq 2C_0 \big[& (K_2 + \|u_0\|)\|A^\beta u(t+h) - A^\beta u(t)\| \\ & + K_1 t^{-\alpha}\|u(t+h) - u(t)\|\big] \end{aligned} \tag{3.5}$$

and

$$\|f(u(t))\| \leq 2C_0\|u(t)\|\|A^\beta u(t)\| \leq 2C_0K_1(K_2 + \|u_0\|)t^{-\alpha} \tag{3.6}$$

for $0 < t \leq T$. Then, by (2.16) and (3.5), we have

$$\begin{aligned} \int_0^t & \|A^\beta P_\alpha(t-s)(f(u(s+h)) - f(u(s)))\|ds \\ & \leq 2C_0C_2'(\alpha,\beta)(K_2 + \|u_0\|) \int_0^t (t-s)^{-\alpha}\|A^\beta u(s+h) - A^\beta u(s)\|ds \\ & \quad + 2C_0C_2'(\alpha,\beta)K_1 \int_0^t (t-s)^{-\alpha}s^{-\alpha}\|u(s+h) - u(s)\|ds. \end{aligned}$$

Note that, for $0 < s < t \leq T$,

$$(t-s)^{-\alpha}\|A^\beta u(s+h) - A^\beta u(s)\| \leq 2K_1(t-s)^{-\alpha}s^{-\alpha},$$

$$s \mapsto 2K_1(t-s)^{-\alpha}s^{-\alpha} \in L^1((0,t);H).$$

Next, consider that since $t^\alpha A^\beta u \in BC((0,T];X)$, we have that $t^\alpha A^\beta u(t)$ is bounded and continuous with respect to $t \in (0,T]$ in the norm $\|\cdot\|$ of X. Then $A^\beta u(t) = t^{-\alpha}\left(t^\alpha A^\beta u(t)\right)$ is also continuous with respect to $t \in (0,T]$ in the norm $\|\cdot\|$ of X. Thus we have

$$\|A^\beta u(s+h) - A^\beta u(s)\| \to 0, \text{ as } h \to 0.$$

Hence, by the Dominated Convergence Theorem, we get

$$\int_0^t (t-s)^{-\alpha}\|A^\beta u(s+h) - A^\beta u(s)\|ds \to 0, \text{ as } h \to 0. \tag{3.7}$$

Similarly, we obtain

$$\int_0^t (t-s)^{-\alpha}s^{-\alpha}\|u(s+h) - u(s)\|ds \to 0, \text{ as } h \to 0. \tag{3.8}$$

By (3.7) and (3.8), we have

$$\int_0^t \|A^\beta P_\alpha(t-s)(f(u(s+h))-f(u(s)))\|ds \to 0, \text{ as } h\to 0.$$

Next, observe that, by (2.16) and (3.6),

$$\begin{aligned}
&\int_0^h \|A^\beta P_\alpha(t+h-s)\|\|f(u(s))\|ds \\
&\quad \le 2C_0C_2'(\alpha,\beta)K_1(K_2+\|u_0\|)\int_0^h (t+h-s)^{-\alpha}s^{-\alpha}ds \\
&\quad = 2C_0C_2'(\alpha,\beta)K_1(K_2+\|u_0\|)(t+h)^{1-2\alpha}\int_0^{\frac{h}{t+h}}(1-r)^{-\alpha}r^{-\alpha}dr \\
&\quad = 2C_0C_2'(\alpha,\beta)K_1(K_2+\|u_0\|)(t+h)^{1-2\alpha}\cdot\frac{1}{1-\alpha} \\
&\qquad \cdot\left(\frac{h}{t+h}\right)^{1-\alpha}H\left(1-\alpha,\alpha;2-\alpha;\frac{h}{t+h}\right) \\
&\quad = \frac{2C_0C_2'(\alpha,\beta)K_1(K_2+\|u_0\|)}{1-\alpha}h^{1-\alpha}(t+h)^{-\alpha}H\left(1-\alpha,\alpha;2-\alpha;\frac{h}{t+h}\right)
\end{aligned}$$

where

$$H(a,b;c;x)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\frac{t^{b-1}(1-t)^{c-b-1}}{(1-xt)^a}dt, \quad c-b-a>0,\ |x|\le 1,$$

is Hypergeometric function (see [6]). Thus

$$\int_0^h \|A^\beta P_\alpha(t+h-s)\|\|f(u(s))\|ds \to 0, \text{ as } h\to 0.$$

Hence $A^\beta Fu(t)$ is continuous with respect to $t\in(0,T]$ in X since $A^\beta S_\alpha(t)u_0$ is also continuous with respect to $t\in(0,\infty)$ in X. Using the way which is similar to that used to prove that $A^\beta Fu(t)$ is continuous with respect to $t\in(0,T]$ in X, we can also obtain that $Fu(t)$ is continuous with respect to $t\in(0,T]$ in X.

Step 3. We shall find $K_1>0, K_2>0$, and $T>0$ such that

$$\sup_{0<t\le T} t^\alpha\|A^\beta Fu(t)\|\le K_1, \quad \sup_{0<t\le T}\|Fu(t)-u_0\|\le K_2. \tag{3.9}$$

By Theorem 2.2(i) and (3.6), we have, for $0<t\le T$,

$$\begin{aligned}
\int_0^t \|P_\alpha(t-s)f(u(s))\|ds &\le 2C_0L_1(\alpha)K_1(K_2+\|u_0\|)\int_0^t (t-s)^{\alpha-1}s^{-\alpha}ds \\
&= 2C_0L_1(\alpha)K_1(K_2+\|u_0\|)B(1-\alpha,\alpha)
\end{aligned}$$

where

$$B(\eta,\nu)=\int_0^1 r^{\eta-1}(1-r)^{\nu-1}dr, \quad \eta,\nu>0$$

is Beta function. Hence

$$\|Fu(t)-u_0\|\le\|S_\alpha(t)u_0-u_0\|+2C_0L_1(\alpha)K_1(K_2+\|u_0\|)B(1-\alpha,\alpha). \tag{3.10}$$

Similarly, by (2.16) and (3.6), we get

$$\begin{aligned}\int_0^t \|A^\beta P_\alpha(t-s)f(u(s))\|ds &\leq 2C_0C_2'(\alpha,\beta)K_1(K_2+\|u_0\|)\int_0^t (t-s)^{-\alpha}s^{-\alpha}ds \\ &= 2C_0C_2'(\alpha,\beta)K_1(K_2+\|u_0\|)B(1-\alpha,1-\alpha)t^{1-2\alpha}\end{aligned}$$

implying

$$\begin{aligned}t^\alpha\|A^\beta Fu(t)\| \leq\; & t^\alpha\|A^\beta S_\alpha(t)u_0\| \\ & + 2C_0C_2'(\alpha,\beta)K_1(K_2+\|u_0\|)B(1-\alpha,1-\alpha)t^{1-\alpha}.\end{aligned} \tag{3.11}$$

We shall choose $K_1 = K_2 = K > 0$ such that

$$K/4 - 2C_0L_1(\alpha)K(K+\|u_0\|)B(1-\alpha,\alpha) > 0 \tag{3.12}$$

and

$$K/4 - 2C_0C_2'(\alpha,\beta)K(K+\|u_0\|)B(1-\alpha,1-\alpha) > 0. \tag{3.13}$$

In order to do it, we take first

$$r_\alpha = \frac{1}{8C_0}\min\Big\{\big(L_1(\alpha)B(1-\alpha,\alpha)\big)^{-1}, \big(C_2'(\alpha,\beta)B(1-\alpha,1-\alpha)\big)^{-1}\Big\}.$$

It follows that if $\|u_0\| < r_\alpha$ then

$$a = 1/4 - 2C_0L_1(\alpha)B(1-\alpha,\alpha)\|u_0\| > 0$$

and

$$b = 1/4 - 2C_0C_2'(\alpha,\beta)B(1-\alpha,1-\alpha)\|u_0\| > 0$$

implying that we can find such a K since (3.12) and (3.13) are equivalent to

$$aK - cK^2 > 0$$

and

$$bK - dK^2 > 0,$$

respectively, where

$$c = 2C_0L_1(\alpha)B(1-\alpha,\alpha) > 0$$

and

$$d = 2C_0C_2'(\alpha,\beta)B(1-\alpha,1-\alpha) > 0.$$

Note that, by Theorem 2.4(i),

$$\|S_\alpha(t)u_0 - u_0\| \to 0, \text{as } t \to 0^+ \tag{3.14}$$

and, by Remark 2.1,

$$t^\alpha\|A^\beta S_\alpha(t)u_0\| \to 0, \text{as } t \to 0^+. \tag{3.15}$$

Then, by (3.14) and (3.15), we can choose $T > 0$ such that

$$\sup_{0<t\leq T}\|S_\alpha(t)u_0 - u_0\| \leq K/4 - 2C_0L_1(\alpha)K(K+\|u_0\|)B(1-\alpha,\alpha) \tag{3.16}$$

and

$$\sup_{0<t\leq T} t^\alpha\|A^\beta S_\alpha(t)u_0\| \leq K/4 - 2C_0C_2'(\alpha,\beta)K(K+\|u_0\|)B(1-\alpha,1-\alpha), \tag{3.17}$$

respectively. Applying (3.16) to (3.10), we have

$$\begin{aligned}\sup_{0<t\leq T}\|Fu(t)-u_0\| \leq\; & \sup_{0<t\leq T}\|S_\alpha(t)u_0 - u_0\| \\ & + 2C_0L_1(\alpha)K(K+\|u_0\|)B(1-\alpha,\alpha) \\ \leq\; & K/4.\end{aligned}$$

Similarly, applying (3.17) to (3.11), we get

$$\begin{aligned}\sup_{0<t\le T} t^\alpha \|A^\beta Fu(t)\| &\le \sup_{0<t\le T} t^\alpha \|A^\beta S_\alpha(t)u_0\| \\ &\quad + 2C_0C_2'(\alpha,\beta)K(K+\|u_0\|)B(1-\alpha,1-\alpha) \\ &\le K/4.\end{aligned}$$

Thus (3.9) is satisfied.

Step 4. We prove F is a contraction in $B_{\alpha,T}$. Observe that

$$\|f(u(t)) - f(v(t))\| \le 2C_0(K_1+K_2+\|u_0\|)t^{-\alpha}|||u-v|||_{\alpha,T}.$$

Therefore, by Theorem 2.2(i),

$$\begin{aligned}\|Fu(t)-Fv(t)\| &\le 2C_0L_1(\alpha)(K_1+K_2+\|u_0\|)\int_0^t (t-s)^{\alpha-1}s^{-\alpha}ds|||u-v|||_{\alpha,T} \\ &\le 2C_0L_1(\alpha)(K_1+K_2+\|u_0\|)B(1-\alpha,\alpha)|||u-v|||_{\alpha,T}\end{aligned}$$

and, by (2.16),

$$\begin{aligned}&t^\alpha\|A^\beta Fu(t) - A^\beta Fv(t)\| \\ &\le 2C_0C_2'(\alpha,\beta)(K_1+K_2+\|u_0\|)t^\alpha \int_0^t (t-s)^{-\alpha}s^{-\alpha}ds|||u-v|||_{\alpha,T} \\ &\le 2C_0C_2'(\alpha,\beta)(K_1+K_2+\|u_0\|)B(1-\alpha,1-\alpha)t^{1-\alpha}|||u-v|||_{\alpha,T}.\end{aligned}$$

Then

$$|||Fu-Fv|||_{\alpha,T} \le K'|||u-v|||_{\alpha,T}$$

where

$$K' = 2C_0(K_1+K_2+\|u_0\|)\left[L_1(\alpha)B(1-\alpha,\alpha) + C_2'(\alpha,\beta)B(1-\alpha,1-\alpha)\right].$$

Note that, with $K>0$ specified as in (3.12) and (3.13), we have

$$C_0L_1(\alpha)(K+\|u_0\|)B(1-\alpha,\alpha) < 1/8 \tag{3.18}$$

and

$$C_0C_2'(\alpha,\beta)(K+\|u_0\|)B(1-\alpha,1-\alpha) < 1/8 \tag{3.19}$$

implying $0 < K' < 1$. It means that F is a contraction mapping from $B_{\alpha,T}$ into itself.

Then, by Banach Fixed Point Theorem, we obtain a unique $u \in B_{\alpha,T}$ which is a mild solution to the problem (1.2). Furthermore, consider that, based on (3.11),

$$\begin{aligned}t^\alpha\|A^\beta u(t)\| &\le t^\alpha\|A^\beta S_\alpha(t)u_0\| \\ &\quad + 2C_0C_2'(\alpha,\beta)K(K+\|u_0\|)B(1-\alpha,1-\alpha)t^{1-\alpha}.\end{aligned} \tag{3.20}$$

Since $u_0 \in D(A)$, we have

$$t^\alpha\|A^\beta S_\alpha(t)u_0\| \to 0, \text{ as } t\to 0^+$$

by Remark 2.1. It implies that

$$\lim_{t\to 0^+} t^\alpha\|A^\beta u(t)\| = 0$$

by (3.20). Moreover, we can find that there exist $M_i > 0$, $i=1,2$, such that

$$\|u(t)\| \le M_1\|u_0\|, \quad \|A^\beta u(t)\| \le M_2 t^{-\alpha}\|u_0\|, \quad t\in(0,T].$$

□

4. Applications

We consider again the fractional chemotaxis-diffusion system (1.1)

$$\begin{aligned}
&D_t^\alpha u = \Delta u - \nabla\cdot u\nabla v, \quad \text{in } \Omega\times(0,\infty),\\
&D_t^\alpha v = \Delta v - v + u, \quad \text{in } \Omega\times(0,\infty),\\
&\frac{\partial v}{\partial n} = \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega\times(0,\infty),\\
&u(\cdot,0) = u_0,\ v(\cdot,0) = v_0, \quad \text{in } \Omega
\end{aligned}$$

with $4/5 < \alpha < 1$ and $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^2 boundary. We define the Banach space

$$X = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u \in L^2(\Omega), v \in H_N^2(\Omega) \right\}$$

where

$$H_N^2(\Omega) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}$$

equipped with the norm

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| = \left(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{H^2(\Omega)} \right)^{\frac{1}{2}}, \quad \begin{pmatrix} u \\ v \end{pmatrix} \in X.$$

The abstract formulation of the problem (1.1) is

$$\begin{aligned}
&D_t^\alpha U = AU + F(U), \quad t > 0,\\
&U(0) = U_0
\end{aligned}$$

in $X = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u \in L^2(\Omega), v \in H_N^2(\Omega) \right\}$ where

$$A = \begin{pmatrix} A_1 & 0 \\ I & A_2 \end{pmatrix},\ F(U) = -\begin{pmatrix} \nabla\cdot u\nabla v \\ 0 \end{pmatrix},\ U = \begin{pmatrix} u \\ v \end{pmatrix},\ U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

with $A_1 = \Delta$, $A_2 = \Delta - I$, and

$$D(A) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u \in H_N^2(\Omega), v \in \mathcal{H}^4_{N^2}(\Omega) \right\}$$

where

$$\mathcal{H}^4_{N^2}(\Omega) = \left\{ v \in H_N^2(\Omega) : \Delta v \in H_N^2(\Omega) \right\}.$$

The operator $A_i, i = 1,2$ is dissipative and self adjoint implying that $A_i, i = 1,2$ is sectorial in $H_N^2(\Omega)$. Moreover, for any $\lambda \in S_\theta$ with $\theta \in (\pi/2, \pi)$, we get

$$\begin{aligned}
(\lambda - A)^{-1} &= \begin{pmatrix} (\lambda - A_1)^{-1} & 0 \\ (\lambda - A_2)^{-1}(\lambda - A_1)^{-1} & (\lambda - A_2)^{-1} \end{pmatrix}\\
&= \begin{pmatrix} (\lambda - A_1)^{-1} & 0 \\ A_2^{-1}A_2(\lambda - A_2)^{-1}(\lambda - A_1)^{-1} & (\lambda - A_2)^{-1} \end{pmatrix}\\
&= \begin{pmatrix} (\lambda - A_1)^{-1} & 0 \\ A_2^{-1}[\lambda(\lambda - A_2)^{-1} - I](\lambda - A_1)^{-1} & (\lambda - A_2)^{-1} \end{pmatrix}.
\end{aligned}$$

Thus, there exists $M > 0$ such that $\|(\lambda - A)^{-1}\| \le M/|\lambda|$ for all $\lambda \in S_\theta$.

For $3/4 < \beta \le 1$, we define

$$H_N^{2\beta}(\Omega) = \left\{ u \in H^{2\beta}(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

Since $\nabla \cdot u\nabla v = \nabla u \cdot \nabla v + u\Delta v$, we have, for $1/2 < \beta \le 1$,

$$\|\nabla \cdot u\nabla v\|_{L^2(\Omega)} \le C\|u\|_{H^{2\beta}(\Omega)}\|v\|_{H^2(\Omega)}, \quad u \in H^{2\beta}(\Omega), v \in H^2(\Omega), \tag{4.1}$$

for some constant $C > 0$. Next, for $3/4 < \beta \le 1$, $D(A_1^\beta) = D(A_2^\beta) = H_N^{2\beta}(\Omega)$ and

$$\|u\|_{H^{2\beta}(\Omega)} \le R_i\|A_i^\beta u\|_{L^2(\Omega)}, \quad u \in H_N^{2\beta}(\Omega),\ i = 1, 2, \tag{4.2}$$

for some constants $R_i > 0$, $i = 1, 2$. Furthermore, we obtain that, for $3/4 < \beta \le 1$,

$$D(A^\beta) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u \in H_N^{2\beta}(\Omega), v \in \mathcal{H}_{N^2}^{2(\beta+1)}(\Omega) \right\} \tag{4.3}$$

where

$$\mathcal{H}_{N^2}^{2(\beta+1)}(\Omega) = \left\{ v \in H_N^2(\Omega) : \Delta v \in H_N^{2\beta}(\Omega) \right\}$$

(see [4]).

Thus, using (4.1), (4.2), (4.3), and the inequalities

$$(a+b)^p \le 2^{p-1}(a^p + b^p), \quad a, b \ge 0, \quad p \ge 1,$$
$$a^2 + b^2 \le (a+b)^2, \quad a, b \ge 0,$$

for $3/4 < \beta \le 1$, F satisfies

$$\begin{aligned}
&\|F(U) - F(V)\|^2 \\
&\quad = \left\| \begin{pmatrix} \nabla \cdot (u_1 - u_2)\nabla v_1 - \nabla \cdot u_2 \nabla(v_2 - v_1) \\ 0 \end{pmatrix} \right\|^2 \\
&\quad = \|\nabla \cdot (u_1 - u_2)\nabla v_1 - \nabla \cdot u_2 \nabla(v_2 - v_1)\|_{L^2(\Omega)}^2 \\
&\quad \le C^2 \left[\|u_1 - u_2\|_{H^{2\beta}(\Omega)}\|v_1\|_{H^2(\Omega)} + \|u_2\|_{H^{2\beta}(\Omega)}\|v_1 - v_2\|_{H^2(\Omega)} \right]^2 \\
&\quad \le C^2 R_1^2 \left[\|A_1^\beta(u_1 - u_2)\|_{L^2(\Omega)}\|v_1\|_{H^2(\Omega)} + \|A_1^\beta u_2\|_{L^2(\Omega)}\|v_1 - v_2\|_{H^2(\Omega)} \right]^2 \\
&\quad \le 2C^2 R_1^2 \left[\|A_1^\beta u_1 - A_1^\beta u_2\|_{L^2(\Omega)}^2 \|v_1\|_{H^2(\Omega)}^2 + \|A_1^\beta u_2\|_{L^2(\Omega)}^2 \|v_1 - v_2\|_{H^2(\Omega)}^2 \right] \\
&\quad \le 2C^2 C' \left[\|A^\beta U - A^\beta V\|^2 (\|U\| + \|V\|)^2 + \left(\|A^\beta U\| + \|A^\beta V\|\right)^2 \|U - V\|^2 \right] \\
&\quad \le 2C^2 C' \left[\|A^\beta U - A^\beta V\| (\|U\| + \|V\|) + \left(\|A^\beta U\| + \|A^\beta V\|\right) \|U - V\| \right]^2
\end{aligned}$$

for some constant $C' > 0$ with

$$U = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \in D(A^\beta), \quad V = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \in D(A^\beta).$$

It follows that, for some constant $C_0 > 0$,

$$\|F(U) - F(V)\| \le C_0 \left[(\|U\| + \|V\|) \|A^\beta U - A^\beta V\| + \left(\|A^\beta U\| + \|A^\beta V\|\right) \|U - V\| \right].$$

By Theorem 3.1, for $\beta = 2 - 1/\alpha$ with $4/5 < \alpha < 1$ and $U_0 \in D(A)$ with $\|U_0\| < r_\alpha$, we conclude that, for some $T > 0$, the problem (1.1) has a unique mild solution U satisfying

$$U \in BC((0,T] : D(A^\beta)),\ t^\alpha A^\beta U \in BC((0,T] : X),$$
$$\lim_{t \to 0^+} t^\alpha A^\beta U(t) = 0$$

with

$$\|U(t)\| \le M_1\|U_0\|, \quad \|A^\beta U(t)\| \le M_2 t^{-\alpha}\|U_0\|, \quad t \in (0,T]$$

for some $M_i > 0$, $i = 1, 2$.

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