Journal of Fractional Calculus and Applications
Vol. 10(1) Jan. 2019, pp. 228-235.
ISSN: 2090-5858.
http://math-frac.oreg/Journals/JFCA/

SOME APPLICATIONS OF FRACTIONAL CALCULUS IN TECHNOLOGICAL DEVELOPMENT

SUCHANA MISHRA, LAKSHMI NARAYAN MISHRA*, RABINRA KISHORE MISHRA, SRIKANTA PATNAIK

Abstract. The present paper is an effort to explore potential utilities and possible physical implications of the mathematical machinery of fractional derivative and fractional integrals in electrical and electronics technologies, Starting with some of the common definitions, properties of fractional differintegrals, various applications of Fractional calculus in electrical and electronics technology are briefly reviewed in this paper. Applications like Fractional cross product and Fractional curl, Fractional calculus in Antenna Radiation Engineering, Electronic Circuit Analysis, Fractional calculus in Control engineering, Fractional Calculus in Electonic System designing are review and presented.

1. Introduction

The theory of Fractional Order Calculus (FOC) is 400 years old, but researchers could able to use it in the last two decades on account of available of computational recourses. The major constraint of fractional calculus is the rigorous mathematics that has come into the literature is almost difficult for the engineers to understand. Some of the researchers have shown interest in the fractional calculus in their respective fields, but the intensity of research is very small as compared to the latent potential of FOC. Fractional calculus was introduced on September 30, 1695. On that day, Leibniz wrote a letter to L'Hpital, raising the possibility of generalizing the meaning of derivatives from integer order to non-integer order derivatives. L'Hpital wanted to know the result for the derivative of order $n = 1/2$. Leibniz replied that "one day, useful consequences will be drawn" and, in fact, his vision became a reality.However, the study of non-integer order derivatives did not appear in the literature until 1819, when Lacroix presented a definition of fractional derivative based on the usual expression for the nth derivative of the power function (Lacroix 1819). Within years the fractional calculus became a very attractive subject to mathematicians, and many different forms of fractional (i.e., non-integer) differential operators were introduced: the Grunwald-Letnikow, Riemann-Liouville,

---

2010 *Mathematics Subject Classification.* 26A33.
Key words and phrases. Fraction Order calculus, Fractional cross product .
Submitted July 22, 2018. Revised Aug. 1, 2018
*Corresponding author.

Hadamard, Caputo, Riesz (Hilfer2000; Kilbas 2006; Podlubny1999; Samko 1993) and the more recent notions of Cresson (2007), Katugampola (2011), Klimek (2005), Kilbas and Saigo (2004) or variable order fractional operators introduced by Samko and Ross (1993).

In 2010, an interesting perspective to the subject, unifying all mentioned notions of fractional derivatives and integrals, was introduced in Agrawal (2010) and later studied in Bourdin (2014), Klimek and Lupa (2013), Odzijewicz(2012). Precisely, authors considered general operators, which by choosing special kernels, reduce to the standard fractional operators. However, other nonstandard kernels can also be considered as particular cases[1].Furthermore, researchers in some fields of applied science [2-7] and in engineering including signal processing, controls and many other fields such as biological science and neuroscience have used some aspects of fractional derivatives and integrals in their work.some of the results are also found in various books or related review article. Very recently an article is published by H G Sun et al. related to application of fractional calculus in real world problem [2].

2. Preliminaries and Definitions

Some of the common definitions and properties of fractional differ-integrals are briefly presented below. A well-known definition of fractional integrals by Riemann-Liouville known as the Riemann-Liouville integral is written as

$$ {}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_a^x (x-u)^{-\alpha-1} f(u)du, \ for\ \alpha < 0\ and\ x > a. \tag{1}$$

where ${}_aD_x^{\alpha}$ denotes the (fractional)α th order integration of the function $f(x)$, with the lower limit of integration being$a$, and $\Gamma(.)$ is the Gamma function.This definition is a generalization of Cauchy's repeated-integration formula. For $\alpha = -n$ the above definition turns to the below equation:

$$ {}_aD_x^{\alpha} f(x) = \frac{1}{(n-1)!} \int_a^x (x-u)^{n-1} f(u)du, \ for\ x > a. \tag{2}$$

The above equation, according to Cauchy's repeated-integration formula, is equivalent to,

$$ {}_aD_x^{\alpha} f(x) = \frac{1}{(n-1)!} \int_a^x (x-u)^{n-1} f(u)du \ = \int_a^x dx_{n-1} \int_a^{x_{n-1}} dx_{n-2} ... \int_a^{x_1} f(x_0)dx_0. \tag{3}$$

For fractional derivatives with a¿ 0, the above Riemann-Liouville fractional-integration definition can still be applied, if used in conjunction with the following additional step:

$$ {}_aD_x^{\alpha} f(x) \ = \ \frac{d^m}{dx^m}\, {}_aD_x^{\alpha-m} f(x), \ for\ a > 0, \tag{4}$$

where m is chosen so that $(\alpha - m) < 0$, and thus the Riemann-Liouville integration can be applied for $aD_x^{\alpha-m} f(x)$ .Then, $\frac{d^m}{dx^m}$ is the ordinary mth order differential operator [8-10]. There is another option for computing fractional derivatives. That is the Caputo fractional derivative. It was introduced by M. Caputo in his 1967. In contrast to the Riemann Liouville fractional derivative, when solving differential equations using Caputo's definition, it is not necessary to define the fractional order initial conditions. Caputo's definition is illustrated as follows.

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^n(u)}{(x-a)^{\alpha+1-n}} f(u)du \tag{5} $$

There are several other equivalent definitions for fractional derivatives and integrals, which can be found in various references[8-15].

3. Application of Fractional Calculus In Electronics

3.1. **(a)Fractional cross product.** Cross product operation can be fractionized by geometrical discourse and then derived in a simple way the expressions for fractional curl. This gives a simple understanding of the process, and is very helpful in various practical cases. The new method can be verified against the classical method of fractionizing the linear operator by rules of operator algebra. The formula derived will be useful for several applications and can be applied in various vector fields with the geometrical explanation(see figure-1) are thus helpful in understanding the utility of fractional cross product and fractional curl[16].
The fractional cross product obtained

$$ (\hat{z}_k \times)^\alpha \hat{x}_k = [\cos(\frac{\alpha\pi}{2})]\hat{x}_k + [\sin(\frac{\alpha\pi}{2})]\hat{y}_k \tag{6} $$

$$ (\hat{z}_k \times)^\alpha \hat{y}_k = [-\sin(\frac{\alpha\pi}{2})]\hat{x}_k + [\cos(\frac{\alpha\pi}{2})]\hat{y}_k \tag{7} $$

$$ (\hat{z}_k \times)^\alpha \hat{z}_k = 0 \tag{8} $$

3.2. **(b)Fractional Calculus and Electronic Circuit Analysis in Radiation Engineering.** Electro magnetics is the basis of wireless communication and deals with calculus of integer-order. It is of interest for the researchers to see how fractional calculus can be explored in this field to get the physical significance of such non-integer based differential or integral operators. Researchers have shown that these mathematical operators can be interesting and useful mathematical tools in electromagnetic/wireless theory [17-20]. The major work includes the novel concept of fractional multi-poles in electromagnetism, electrostatic fractional image methods for perfectly conducting wedges and cones, and the mathematical link between the electrostatic image methods for the conducting sphere and the dielectric sphere.

It is an established fact that the scalar Helmholtz equation the canonical solutions are identified as plane, cylindrical, and spherical waves for the one, two and three-dimensional cases respectively. The corresponding sources being one, two and three-dimensional Dirac delta functions. Researchers expected an intermediate wave between two canonical cases. That means thinking the transition to be continuous instead of distinct. This type of solution cannot be expected from ordinary calculus, which is the positive integer based. It has been shown that fractional integration, differentiations, which are mathematical tools studied in the field of fractional calculus, can be utilized to find the "intermediate" sources. The waves that satisfy the conventional scalar Helmholtz equation [17-19] can help the-electromagnetic/wireless researchers. A fractional parameter v which is used as a determining factor for calculating the intermediate values, and takes the fractional value between zero and unity. The solution is such that when $v = 0$ represents the case of the cylindrical wave propagation and $v = 1$ denotes the plane-wave propagation.

Electrical circuits with fractance: Resistors and capacitors are described by an integer- order models in Classical electrical circuits. Electrical element with fractional-order impedance [21-23] is called fractance. Basically, there are two kinds of fractance: (i) tree fractance and (ii) chain fractance. The electrode-electrolyte interface is an example of a fractional-order process. Value of the fractional coefficient η is closely associated with the smoothness of the interface, as the surface is infinitely smoothed then $\eta \to 1$. The current field in the transmission line of infinite length is expressed in terms of the fractional derivative of order 1/2 of the potential $\varphi(0, t)$ is also a fractional based model.

The tree fractance and chain fractance consist not only of resistors and capacitor's properties, but also exhibit electrical properties with non-integer-order impedance. One can think of the generalized voltage divider.

In [24] the authors have used 'Fractors'[25]. That is fractional order capacitive and inductive elements with a resistor to make a simple RLC type input tuning network into a large circular loop antenna and simulated where the result clearly shows the radiation pattern of the antenna is more strengthened when the fractional order components are used instead of integer order components. For a particular, this is observed in Figure -2.

Figure-2 shows the effect of changing α or the order of the fractional capacitance. The significant effect is that the amplitude of the radiation lobe for$\alpha = 0.9$ (red color) is 3 times larger than the usual integer order (blue color) radiation amplitude. Fundamentally this is due to the ability of a fractional order element.

3.3. (c)Fractional Calculus in Control Engineering.

It is the challenge of the researchers to find new and effective methods for the time-domain analysis of fractional-order dynamical systems [26-27] for tackling problems of control theory. As a new generalization of the classical PID -controller namely the $PI^{\lambda}D^{\mu}$ -controller has been developed by researchers. The idea of $PI^{\lambda}D^{\mu}$ -controller,involving fractional-order integrator and fractional-order differentiator, is a more efficient control of fractional-order dynamical systems. If $\lambda = 1$ and $= 0$, the equation converts to a PI controller. Likewise, if $= 0$ and $= 1$ the equation yields the PD controller. It has been successfully used the fractional-order controller to develop the so-called CRONE-controller (Commande Robusted'Ordre Non Entrier controller) which is an interesting example of application of fractional derivatives in control theory. The prime advantage of the $PI^{\lambda}D^{\mu}$ -controller compared to the conventional classical PID -controller has a better performance record.

3.4. (d)Fractional Calculus in Electronic System Designing.

The response of a fractional order system [28-40] by using analogical circuits with fractional order behavior can be done in three methods briefly presented.

i. Component by component implementation: The approximation of the transfer function is done by the recursive circuit. The gain between V_o and V_1 in Laplace transform is the continuous fraction approximation to the original system [28-29].

$$\frac{V_0}{V} = 1 + \cfrac{w_n}{s + \cfrac{w_{n-1}}{1 + \cfrac{w_{n-2}}{s + \cfrac{w_{n-3}}{..}}}} \tag{9}$$

where

$$w_{n-2j} = \frac{1}{R_j C_j} \; and \; w_{n-2j+1} = \frac{1}{R_{j+1} + C_j} \tag{10}$$

The method presented has two principal disadvantages: first one is the frequency band of work is limited and the second one, this is an approximation. Therefore, it requires a lot of low tolerance components, depending on the accuracy required by the designer.

ii. Field Programmable Analog Array $(FPAA)$: The designer implements the circuit component by component into a $FPAA$. which enables changing of the dynamical behavior of the fractional order system with a few simple modifications and each element has custom tolerance.

iii. Fractional order impedance component: It is a capacitor with fractional order behavior introduced. In general, it consists in a capacitor of parallel plates, where one of them presents a fractal dimension. Each branch could be modeled as a low pass resistor/capacitor (RC) circuit filter, and it is linked to the principal branch

4. Conclusion.

The present paper is a general effort in recent years to explore potential utilities and possible physical implications of the mathematical machinery of fractional derivatives and fractional integrals in electronics. New algorithm in terms of accuracy and speed needs to be developed. Soft computing technique with fractional calculus may be a suitable combination to solve some problems in the future.

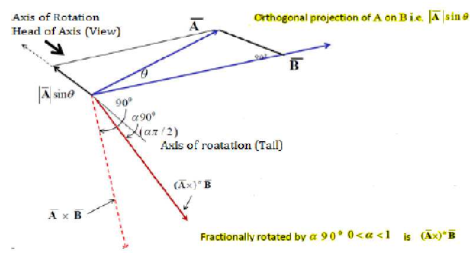

FIGURE 1. cross product

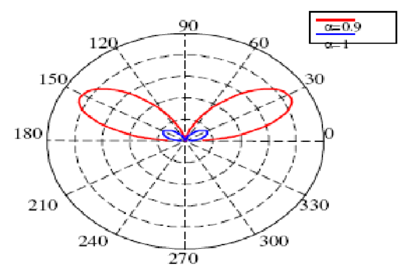

FIGURE 2. antenna

REFERENCES

[1] A.B. Malinowska, T. Odzijewicz, D.F.M. Torres, Advanced Methods in the Fractional Calculusof Variations, SpringerBriefs in Applied Sciences and Technology, 2015,DOI 10.1007/978-3-319-14756-7-2.

[2] H.G. Sun, Y Zhang, D Baleanu, W Chen, YQ Chen, A new collection of real world application of fractional calculus in science and engineering, Communications in Nonlinear Science and Numerical Simulation, Apr 2018

[3] M. Ochmann and S. Makarov, "Representation of the Absorption of Nonlinear Waves by Fractional Derivatives," Journal of Acoustical Society of America, 94, 6, December 1993, pp. 3392- 3399.

[4] W. R. Schneider and W. Wyss, "Fractional Diffusion and Wave Equation," Journal of Mathematical Physics, 30, 1, January 1989, pp. 134-144.

[5] M. Caputo and F. Mainardi, "Linear Models of Dissipation in Anelastic Solids," La Rivista del NuovoCimento (dellasocietitaliana di fisica), 1, series 2,2, April-June 1971, pp. 161-198.

[6] C. F. Chen, Y. T. Tsay, and T. T. Wu, "Walsh Operational Matrices for Fractional Calculus and Their Application to Distributed Systems," Journal of the Franklin Institute, 303, 3, March 1977, pp. 267-284.

[7] Z. Altman, D. Renaud, and H. Baudrand, "On the Use of Differential Equations of Nonentire Order to Generate Entire Domain Basis Functions with Edge Singularity," IEEE Transactions on Microwave Theory and Techniques, MTT-42,10, October 1994, pp. 1966-1972.

[8] K.S. Miller and B. Ross. An Introduction to the Fractional Calculus and Fractional Differential Equations.Wiley, 1993.

[9] I. Podlubny. Fractional Differential Equations, Mathematics in Science and Engineering V198, Academic Press, 1999.

[10] P. M. Morse and H. Feshbach. Methods of Theoretical Physics, McGraw-Hill, New York, 2 vols.,1953.

[11] S.G. Samko, A.A. Kilbas, O. I. Marichev. Fractional Integrals and Derivatives-Theory and Applications Gordon and Breach, Linghorne, P.A.,1993.

[12] A.A. Kilbas, H. H. Srivastava, J. J. Trujillo. Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam (2006).
[13] K. B. Oldham and J. Spanier. The Fractional Calculus, Academic Press, New York, 1974.
[14] R. Gorenflo. Fractional Calculus: Some Numerical Methods, Udine, Sept. 1996.
[15] R. Hilfer. Applications of Fractional Calculus in Physics, World Scientific, 2000.
[16] S Das, IJMC, Vol.20, Issue 3,2013,"Geometrically deriving fractional cross product and fractional curl"
[17] N. Engheta. IEEE Transactions on Antennas and Propagation, Vol. 44, No. 4, April 1996, pp. 554-566.
[18] N. Engheta. IEEE Transactions on Antennas and Propagation, Vol. 44, No. 12, December 1996, pp. 1565-1574.
[19] N. Engheta. Journal of Electromagnetic Waves and Applications, Vol. 9, No. 9, September 1995, pp. 1179-1188.
[20] N. Engheta. Electromagnetics, Microwave and Optical Technology Letters, Vol. 17, No. 2, February 5, 1998, pp. 86-91.
[21] M. Nakagawa and K. Sorimachi. IEICE Trans. Fundamentals E75-A , 1992, no.2, 1814-1819.
[22] A. Oustaloup, F. Levron, F. Nanot, and B. Mathieu. IEEE Transactions on Circuits and Systems I:Fundamental Theory and Applications, 47(1), January2000, 25-40.
[23] D. J. Scmidlin. IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications, 43(7), 1996, 588-591.
[24] Calvin Coopmans, Edmund spencer, hadimalek "radiation and impedance characteristics of a circular loop antenna driven by fractional order electronics" ,ASME2013,Aug 4-7 2013,Portland,USA.
[25] G. Bohannan, 2002. "Analog realization of a fractional control element-revisited". In Proc. of the 41st IEEE Int. Conf. on Decision and Control, Tutorial Workshop, Vol.1, pp. 27-30.
[26] M Axtell, M E Bise. Fractional calculus applications in control systems, Proc of IEEE Nat. Aerospace and Electronics Conf, New York 1990, pp. 563-566.
[27] I. Podlubny, Dorcak, Kostial. J. On fractional derivatives, fractional order dynamic systems and PID^{μ} - controllers, Procc. of the 36th IEEE CDC, San Diego, December 2004, pp 4985-4990.
[28] H.E. Jones, B.A. Shenoi. IEEE Trans. on Circuits and Systems, vol. 17, no. pp. 125-128, 1970.
[29] Bohennan Gary et al. Electrical components with fractional order impedances, Patent US 20060267595, November 2006.
[30] L.N. Mishra, M. Sen, On the concept of existence and local attractivity of solutions for some quadratic Volterra integral equation of fractional order, Applied Mathematics and Computation, Vol. 285, 2016, 174-183. DOI: 10.1016/j.amc.2016.03.002
[31] L.N. Mishra, H.M. Srivastava, M. Sen, Existence results for some nonlinear functional-integral equations in Banach algebra with applications, International Journal of Analysis and Applications, Vol. 11, No. 1, 2016, 1-10.
[32] D.L. Suthar, L.N. Mishra, A.M. Khan, A. Alaria, Fractional integrals for the product of Srivastava's polynomial and (p, q)-extended hypergeometric function, TWMS J. Appl. Engg. Math. (2018), DOI: 10.26837/jaem.446155.
[33] L.N. Mishra, M. Sen, R.N. Mohapatra, On existence theorems for some generalized nonlinear functional-integral equations with applications, Filomat, 31:7, 2017, 2081-2091, DOI 10.2298/FIL1707081N
[34] L.N. Mishra, On existence and behavior of solutions to some nonlinear integral equations with Applications, Ph.D. Thesis (2017), National Institute of Technology, Silchar 788 010, Assam, India.
[35] V.N. Mishra, K. Khatri, L.N. Mishra, Using Linear Operators to Approximate Signals of Lip (α, p), $(p \geq 1)$-Class, Filomat, 27:2, 2013, 353-363, DOI 10.2298/FIL1302353M
[36] L.N. Mishra, R.P. Agarwal, On existence theorems for some nonlinear functional-integral equations, Dynamic Systems and Applications, Vol. 25, 2016, pp. 303-320.
[37] L.N. Mishra, R. P. Agarwal, M. Sen, Solvability and asymptotic behavior for some nonlinear quadratic integral equation involving Erdélyi-Kober fractional integrals on the unbounded interval, Progress in Fractional Differentiation and Applications, Vol. 2, No. 3, 2016, 153-168.
[38] V.N. Mishra, Some Problems on Approximations of Functions in Banach Spaces, Ph.D. Thesis (2007), Indian Institute of Technology, Roorkee 247 667, Uttarakhand, India.

[39] V.N. Mishra, L.N. Mishra, Trigonometric Approximation of Signals (Functions) in $L_p(p \geq 1)$-norm, International Journal of Contemporary Mathematical Sciences, Vol. 7, no. 19, 2012, 909-918.

[40] Mohd. F. Ali, M. Sharma, L.N. Mishra, V.N. Mishra, Dirichlet Average of Generalized Miller-Ross Function and Fractional Derivative, Turkish Journal of Analysis and Number Theory, 2015, Vol. 3, No. 1, pp. 30-32. DOI:10.12691/tjant-3-1-7.

Suchana Mishra
Dayananda Sagar College of Engineering, Bengaluru - 560078, India
E-mail address: suchanamishra@gmail.com

Lakshmi Narayan Mishra
Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University, Vellore 632 014, Tamil Nadu, India
L. 1627 Awadh Puri Colony, Beniganj, Phase IIIrd, Opposite Industrial Training Institute (I.T.I.), Ayodhya main road, Faizabad 224 001, Uttar Pradesh, India
E-mail address: lakshminarayanmishra04@gmail.com, l_n_mishra@yahoo.co.in, (Corresponding author)

Rabinra Kishore Mishra
Department of Electronics Science, Berhampur University, Berhampur - 760007, Odisha, India
E-mail address: prof.r.k.mishra@gmail.com

Srikanta Patnaik
Siksha O Anusandhan University, Bhubaneswar-751030, Odisha, India