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APPROXIMATE CONTROLLABILITY OF FRACTIONAL INTEGRO-DIFFERENTIAL EVOLUTION EQUATIONS WITH NONLOCAL AND NON-INSTANTANEOUS IMPULSIVE CONDITIONS

ARSHI MERAJ, DWIJENDRA N. PANDEY

Abstract. In this article, we will discuss the existence of mild solutions and approximate controllability for a class of fractional semilinear integro-differential equations with nonlocal and impulsive conditions for which the impulses are not instantaneous. The results are obtained by using semigroup theory, Kuratowski measure of noncmpactness and ρ-set contractive fixed point theorem, without imposing the condition of Lipschitz continuity on nonlinear term as well as the condition of compactness on impulsive functions and nonlocal function. At the end, an example is presented to illustrate the obtained results.

1. Introduction

In the recent years, many researchers paid attention to study the differential equations with instantaneous impulses, which have been used to described abrupt changes such as shocks, harvesting and natural disasters. Particularly, the theory of instantaneous impulsive equations has wide applications in control, mechanics, electrical engineering, biological and medical fields.

It seems that models with instantaneous impulses could not explain certain dynamics of evolution process in pharmacotherapy. For example, one considers the hemodynamic equilibrium of a person, the introduction of the drugs in bloodstream and the consequent absorbtion for the body are gradual and continuous process, we can interpret the above situations as an impulsive action which starts abruptly and stays active for a finite time interval. Hernández and O'Regan [11] and Pierri et al. [23], initially studied Cauchy problems of first order evolution equations with non-instataneous impulses. Kumar et al. [14] established the existence and uniqueness of mild solutions for non-instantaneous impulsive fractional differential equations. Chen et al. [6] invetigated the existence of mild solutions for first order semi-linear evolution equations with non-instantaneous impulses using noncompact semigroup. Kumar et al. [15] derived a set of sufficient conditions for the existence

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and uniqueness of mild solutions to fractional integro-differential equations with non-instantaneous impulses. Nonlocal initial value problem was first studied by Byszewski. In [4], Byszewski established the existence and uniqueness of solutions for a semilinear nonlocal Cauchy problem. In many cases, the nonlocal condition has better effect rather than the classical initial condition.

Controllability is one of the most important issue in mathematical control theory and engineering. The problem of controllability for various kinds of differential, integro-differential equations and impulsive differential equations are studied. In case of controllability, the literature on abstract impulsive differential equations consists basically problems for which the impulses are abrupt and instantaneous. In [5], Balasubramaniam et al. derived sufficient conditions for approximate controllability of impulsive fractional integro-differential equations with nonlocal conditions, by assuming the compactness of impulsive and nonlocal functions in a Hilbert space. Zhang et al. [30] discussed the approximate controllability of fractional impulsive integro-differential equations in a Hilbert space with the help of Krasnoselskii fixed point theorem and comapact analytic semigroup theory. Dong et al. [8] studied approximate controllability of semilinear fractional evolution equations with nonlocal conditions via approximate technique.

The purpose of this article is to establish sufficient conditions for the approximate controllability of a certain class of abstract fractional evolution equations of the form :

$$\begin{aligned} {}^cD^q x(t) &= Ax(t) + f\left(t, x(t), \int_0^t h(t,s,x(s))ds\right) + Bu(t), \quad t \in \cup_{k=0}^m (s_k, t_{k+1}], \\ x(t) &= \gamma_k(t, x(t)), \quad t \in \cup_{k=1}^m (t_k, s_k], \\ x(0) + g(x) &= x_0, \end{aligned} \tag{1}$$

where ${}^cD^q$ is the Caputo fractional derivative of order q, $0 < q < 1$, $J = [0, b]$, $b > 0$ is a constant, the state variable x takes values in a separable reflexive Banach space X, $A : D(A) \subset X \to X$ is closed linear operator that generates a C_0 semigroup $T(t)(t \geq 0)$ on X, the control function $u \in L^2(J, U)$ where U is a Banach space, $B : U \to X$ is a bounded linear operator, $0 < t_1 < t_2 < \ldots < t_m < t_{m+1} := b$, $s_0 := 0$, $s_k \in (t_k, t_{k+1})$ for each $k = 1, 2, \ldots, m$ and $f : J \times X \times X \to X$, $g : PC(J, X) \to X$ are given functions satisfying certain assumptions, $\gamma_k : (t_k, s_k] \times X \to X$ is non-instataneous impulsive function for all $k = 1, 2, \ldots, m$, $h : D \times X \to X$ is continuous function where $D := \{(t, s) : 0 \leq s < t \leq b\}$ and $x_0 \in X$.

To the best of our knowledge, there is no work yet reported on approximate controllability of fractional integro-differential equation with nonlocal and non-instantaneous impulsive conditions. Therefore inspired by this fact, we consider the system (1) to study the approximate controllability with the help of Kuratowski measure of noncompactness and ρ-set contractive fixed point theorem without assuming the compactness condition on impulsive and nonlocal functions.

The rest of the paper is organized as following. In section 2, we will recall some basic definitions, notations, theorems and will introduce the expression of mild solutions for the system (1). In section 3, we will discuss the existence of mild solutions for the system (1) under the feedback control $u_\lambda(t, x)$ defined in (9). In section 4, we will show that the control system (1) is approximately controllable on $[0, b]$. Finally, in section 5, we will present an example to illustrate our results.

2. Preliminaries

Now, we recall some basic theory which is required for our main results. Let X be a separable reflexive Banach space with norm $\|\cdot\|$, and $C(J,X)$ be a Banach space of all continuous functions from J into X endowed with supremum norm $\|x\|_C = \sup_{t\in J}\|x(t)\|$. Consider the space $PC(J,X) = \{x : J \to X : x$ is continuous at $t \neq t_k$, $x(t_{k-}) = x(t_k)$ and $x(t_{k+})$ exists for all $k = 1, 2, \dots .m\}$, which is a Banach space with supremum norm $\|x\|_{PC} = \sup_{t\in J}\|x(t)\|$. For each finite constant $r > 0$, let $\Omega_r = \{x \in PC(J,X) : \|x(t)\| \leq r, t \in J\}$, we use θ to denote the zero function in $PC(J,X)$. Let $L^p(J,X)(1 \leq p < \infty)$ be the Banach space of all X-valued Bochner integrable functions defined on J with norm $\|x\|_{L^p(J,X)} = (\int_0^b \|x(t)\|^p dt)^{\frac{1}{p}}$. Let $M = \sup_{t\in J}\|T(t)\|_{\mathcal{L}(X)}$, where $\mathcal{L}(X)$ stands for the Banach space of all linear and bounded operators on X, note that $M \geq 1$. We denote $Gx(t) := \int_0^t h(t,s,x(s))ds$.

Lemma 2.1.([9]) If h satisfies a uniform Hölder continuity with exponent $\beta \in (0,1]$, then the unique solution of the Cauchy problem

$$\begin{aligned} {}^cD^q x(t) &= Ax(t) + h(t), \quad t \in J, \\ x(0) &= x_0 \in X, \end{aligned} \tag{2}$$

is given by

$$x(t) = U(t)x_0 + \int_0^t (t-s)^{q-1}V(t-s)h(s)ds, \tag{3}$$

where

$$\begin{aligned} U(t) &= \int_0^\infty \zeta_q(\theta)T(t^q\theta)d\theta\,,\; V(t) = q\int_0^\infty \theta\zeta_q(\theta)T(t^q\theta)d\theta, \\ \zeta_q(\theta) &= \frac{1}{q}\theta^{-1-\frac{1}{q}}\rho_q(\theta^{\frac{-1}{q}}), \\ \rho_q(\theta) &= \frac{1}{\pi}\sum_{n=0}^\infty(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}sin(n\pi q),\, \theta \in (0,\infty), \end{aligned} \tag{4}$$

$\zeta_q(\theta)$ is a probability density function defined on $(0,\infty)$.

Remark 2.2. $\zeta_q(\theta) \geq 0$, $\theta \in (0,\infty)$, $\int_0^\infty \zeta_q(\theta)d\theta = 1$, $\int_0^\infty \theta\zeta_q(\theta)d\theta = \frac{1}{\Gamma(1+q)}$.

Lemma 2.3.([29]) The operators U and V have the following properties :

(i) $U(t)$ and $V(t)$ are strongly continuous for $t \geq 0$.
(ii) $U(t)$ and $V(t)$ are linear and bounded operators for any fixed $t \geq 0$ and satisfying $\|U(t)x\| \leq M\|x\|$, $\|V(t)x\| \leq \frac{M}{\Gamma(q)}\|x\|$ for any $x \in X$.
(iii) If $T(t)(t > 0)$ is a compact semigroup, then $U(t)$ and $V(t)$ are compact operators on X for $t > 0$.

Definition 2.4.([14]) A function $x \in PC(J,X)$ is said to be a mild solution of the problem (1) if for any $u \in L^2(J,U)$, x satisfies $x(0) = x_0 - g(x)$, $x(t) = \gamma_k(t,u(t))$

for all $t \in \cup_{k=1}^{m}(t_k, s_k]$, and

$$x(t) = \begin{cases} U(t)(x_0 - g(x)) + \int_0^t (t-s)^{q-1} V(t-s)[f(s, x(s), Gx(s)) \\ \quad +Bu(s)]ds, \quad t \in (0, t_1]; \\ U(t-s_k)\gamma_k(s_k, x(s_k)) + \int_{s_k}^t (t-s)^{q-1} V(t-s)[f(s, x(s), Gx(s)) \\ \quad +Bu(s)]ds, \quad t \in \cup_{k=1}^{m}(s_k, t_{k+1}]. \end{cases}$$

Let $x_b(x_0, u)$ be the state value of (1) at terminal time b corresponding to the control u and the initial value x_0. Introduce the set $\mathcal{R}(b, x_0) = \{x_b(x_0, u) : u \in L^2(J, U)\}$, which is called the reachable set for the system (1) at terminal time b, it's closure in X is denoted by $\overline{\mathcal{R}(b, x_0)}$.

Definition 2.5.([22]) The system (1) is said to be approximately controllable on J, if $\overline{\mathcal{R}(b, x_0)} = X$, that is, given any $\epsilon > 0$, it is possible to steer from the point x_0 to within a distance ϵ from all points in the state space X at time b.

Consider the following linear fractional control system

$$\begin{aligned} {}^cD^q x(t) &= Ax(t) + Bu(t), \quad t \in J, \\ x(0) &= x_0. \end{aligned} \tag{5}$$

Now, we introduce the controllability and resolvent operators associated with (5) as :

$$\Gamma_0^b = \int_0^b (b-s)^{q-1} V(b-s) BB^* V^*(b-s) ds, \tag{6}$$

$$R(\lambda, \Gamma_0^b) = (\lambda I + \Gamma_0^b)^{-1}, \ \lambda > 0. \tag{7}$$

respectively, where B^* and $V^*(t)$ denote the adjoint of B and $V(t)$ respectively. It is easy to see that Γ_0^b is a linear bounded operator. Let us consider the following basic hypothesis :

(H0) $\lambda R(\lambda, \Gamma_0^b) \to 0$ as $\lambda \to 0^+$ in the strong operator topology.

Theorem 2.6.([20]) Let Z be a separable reflexive Banach space and let Z^* stands for it's dual space. Assume that $\Gamma : Z^* \to Z$ is a symmetric map, then the following are equivalent :

(i) $\Gamma : Z^* \to Z$ is positive, that is $\langle z^*, \Gamma_0^b z^* \rangle > 0$ for all nonzero $z^* \in Z^*$..

(ii) For all $z \in Z$, $\lambda(\lambda I + \Gamma \mathfrak{J})^{-1}(z)$ strongly converges to zero as $\lambda \to 0^+$. Here $\mathfrak{J}$ is the duality map from $Z \to Z^*$.

Lemma 2.7.([22]) The linear fractional control system (5) is approximately controllable on J if and only if $(H0)$ holds.

Proof. The system (5) is approximately controllable on J if and only if $\langle x, \Gamma_0^b x \rangle > 0$, for all nonzero $x \in X$ (see Theorem 4.1.7 of [7]). Hence the lemma is straightforward consequence of Theorem 2.6. □

Remark 2.8. Notice that the system (5) is approximately controllable on J if and only if $\langle x, \Gamma_0^b x \rangle = \int_0^b (b-s)^{q-1} \|B^* V^*(b-s)x\|^2 ds > 0$, for all nonzero $x \in X$, which is further equivalent to $B^* V^*(b-s)x = 0$, $0 \le s < b \Longrightarrow x = 0$.

Next, we introduce the Kuratowski measure of noncompactness $\alpha(\cdot)$ defined on each bounded subset of a Banach space X. For more details, we refer [2, 10]. The following results are useful to prove our main results.

Lemma 2.9.([2]) Let X be a Banach space and $S \subset C(J, X)$. For $t \in J$, the set $S(t) = \{x(t) : x \in S\}$. If S is bounded and equicontinuous in $C(J, X)$, then $\alpha(S(t))$ is continuous on J and $\alpha(S) = \sup_{t \in J} \alpha(S(t))$.

Lemma 2.10.([10]) If X be a Banach space and $D = \{x_n\}_{n=1}^{\infty} \subset PC(J, X)$ be a bounded sequence, then $\alpha(D(t))$ is Lebesgue integrable on J and

$$\alpha\left(\left\{\int_0^t x_n(s)ds\right\}_{n=1}^{\infty}\right) \leq 2\int_0^t \alpha(\{x_n(s)\}_{n=1}^{\infty} ds.$$

Lemma 2.11.([3]) Let X be a Banach space and S is a bounded subset of X, then there exists a countable set $D = \{x_n\}_{n=1}^{\infty} \subset S$ such that $\alpha(S) \leq 2\alpha(D)$.

Lemma 2.12.([2]) Let X and E be Banach spaces and $C, D \subset X$ be bounded subsets, then the following properties are satisfied :

(i) D is precompact if and only if $\alpha(D) = 0$.

(ii) $\alpha(C + D) \leq \alpha(C) + \alpha(D)$.

(iii) $Q : D(Q) \subset E \to X$ is Lipschitz continuous with Lipschitz constant L, then $\alpha(Q(V)) \leq L\alpha(V)$ for any bounded subset $V \subset D(Q)$.

Definition 2.13.([6]) Let X be a Banach space and S be a nonempty subset of X. A continuous map $Q : S \to X$ is called ρ-set contractive if there exists a constant $\rho \in [0, 1)$ such that

$$\alpha(Q(\Omega)) \leq \rho\alpha(\Omega), \text{ for every bounded set } \Omega \subset S.$$

Theorem 2.14.([6]) Let X be a Banach space, $\Omega \subset X$ be a closed bounded and convex subset. Suppose that $Q : \Omega \to \Omega$ is a ρ-set contractive map, then Q has atleast one fixed point in Ω.

3. Existence of Mild Solutions

In this section, we prove the existence of mild solutions to the system (1), with the help of following basic assumptions :

(H1) $T(t)(t > 0)$ is a compact semigroup.

(H2) The function $f(t, \cdot, \cdot) : X \times X \to X$ is continuous for each $t \in J$ and the function $f(\cdot, x, y) : J \to X$ is Lebesgue measurable for all $(x, y) \in X \times X$.

(H3) There exist a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, a constant $q_1 \in (0, q)$ and a function $\phi \in L^{\frac{1}{q_1}}(J, \mathbb{R}^+)$ such that

$$\|f(t, x, y)\| \leq \phi(t)\psi(\|x\|), \quad \forall\, x\,,\, y \in X\,;\, t \in J.$$

(H4) $g : PC(J, X) \to X$ is continuous and there exists a constant $\alpha > 0$ such that

$$\|g(x) - g(y)\| \leq \alpha\|x - y\|, \quad \forall\, x\,,\, y \in PC(J, X).$$

(H5) $\gamma_k : [t_k, s_k] \times X \to X$ is continuous and there exists a constant $K_{\gamma_k} > 0$, $k = 1, 2, \ldots, m$, such that

$$\|\gamma_k(t, x) - \gamma_k(t, y)\| \leq K_{\gamma_k}\|x - y\|, \quad \forall\, x\,,\, y \in X\,;\, t \in [t_k, s_k].$$

For our convenience, we use the following notations:

$$\begin{aligned} K &= \max_{k=1,2,\dots,m} K_{\gamma_k}, \quad K_1 = \max\{K, \alpha\}, \quad M_B := \|B\|, \\ \overline{M} &= \frac{b^q M^2 (M_B)^2}{q\lambda(\Gamma(q))^2}, \quad q_2 = \frac{q-1}{1-q_1} \in (-1,0), \quad M_1 = \psi(R)\|\phi\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)}, \\ M_b &= \frac{MM_1}{\Gamma(q)(1+q_2)^{1-q_1}} b^{(1+q_2)(1-q_1)}. \end{aligned} \tag{8}$$

For an arbitrary function $x \in PC(J,X)$, considering the form of a mild solution as defined in Definition 2.4, as well as the controllability and resolvent operator in (6), (7), we choose the feedback control function associated with the nonlinear system (1) as following :

$$u(t) = u_\lambda(t,x) = B^*V^*(b-t)R(\lambda, \Gamma_0^b)p(x), \tag{9}$$

where

$$p(x) = \begin{cases} x_b - U(b)(x_0 - g(x)) - \int_0^b (b-s)^{q-1} V(b-s) f(s, x(s), Gx(s))ds, \\ \qquad \text{for } t \in (0, t_1], \\ x_b - U(b-s_k)\gamma_k(s_k, x(s_k)) - \int_{s_k}^b (b-s)^{q-1} V(b-s) f(s, x(s), Gx(s))ds, \\ \qquad \text{for } t \in \cup_{k=1}^m (s_k, t_{k+1}]. \end{cases}$$

By using the control function (9), for any $\lambda > 0$ define the operator F_λ on $PC(J,X)$ as following :

$$(F_\lambda x)(t) = (\Phi_\lambda x)(t) + (\Psi_\lambda x)(t), \tag{10}$$

where

$$(\Phi_\lambda x)(t) = \begin{cases} U(t)(x_0 - g(x)), & t \in [0, t_1], \\ \gamma_k(t, x(t)), & t \in \cup_{k=1}^m (t_k, s_k], \\ U(t-s_k)\gamma_k(s_k, x(s_k)), & t \in \cup_{k=1}^m (s_k, t_{k+1}], \end{cases} \tag{11}$$

$$(\Psi_\lambda x)(t) = \begin{cases} \int_{s_k}^t (t-s)^{q-1} V(t-s)[f(s, x(s), Gx(s)) \\ \qquad + Bu_\lambda(s,x)]ds, \ t \in \cup_{k=0}^m (s_k, t_{k+1}] \\ 0, \ \text{otherwise}. \end{cases} \tag{12}$$

Theorem 3.1. Assume that the functions $g(\theta)$ and $\gamma_k(\cdot, \theta)$ are bounded for $k = 1, 2, \dots, m$, and the assumptions $(H1)$-$(H5)$ are satisfied. Then the system (1) has atleast one PC- mild solution provided that

$$\rho := MK_1 < 1. \tag{13}$$

Proof. Obviously, the fractional Cauchy problem (1) with the control (9) has a mild solution if and only if the operator F_λ has a fixed point.

First let us observe that, for $x \in \Omega_R$ $(R > 0)$ with the help of Hölder inequality and (H3), we obtain

$$\begin{aligned} \int_0^t \|(t-s)^{q-1} f(s, x(s), Gx(s))\| ds &\le \left(\int_0^t (t-s)^{q_2} ds \right)^{1-q_1} \psi(R)\|\phi\|_{L^{\frac{1}{q_1}}(J,\mathbb{R}^+)} \\ &\le \frac{M_1}{(1+q_2)^{1-q_1}} b^{(1+q_2)(1-q_1)}. \end{aligned} \tag{14}$$

The proof of this theorem is long and technical. Therefore it is convenient to divide it into several steps.

Step 1: For any $\lambda > 0$, there exists a constant $R = R(\lambda) > 0$ such that $F_\lambda(\Omega_R) \subset$

Ω_R. Let $x \in \Omega_r$ for any positive constant r. If $t \in [0, t_1]$, then by using (9)and (14), we have

$$\begin{aligned} u_\lambda(t,x) &= B^*V^*(b-t)R(\lambda,\Gamma_0^b)\Big[x_b - U(b)(x_0 - g(x)) \\ &\quad - \int_0^b (b-s)^{q-1}V(b-s)f(s,x(s),Gx(s))ds\Big] \\ \|u_\lambda(t,x)\| &\leq \frac{MM_B}{\lambda\Gamma(q)}\Big[\|x_b\| + M(\|x_0\| + \alpha\|x-\theta\| + \|g(\theta)\|) + M_b\Big] \\ &\leq \frac{MM_B}{\lambda\Gamma(q)}\Big[\|x_b\| + M(\alpha r + \|x_0\| + \|g(\theta)\|) + M_b\Big], \qquad (15) \end{aligned}$$

and from (10), (15), we obtain

$$\begin{aligned} (F_\lambda x)(t) &= U(t)(x_0 - g(x)) + \int_0^t (t-s)^{q-1}V(t-s)f(s,x(s),Gx(s))ds \\ &\quad + \int_0^t (t-s)^{q-1}V(t-s)Bu_\lambda(s,x)ds \\ \|(F_\lambda x)(t)\| &\leq M(\alpha r + \|x_0\| + \|g(\theta)\|) + M_b + \int_0^t (t-s)^{q-1}\|V(t-s)\|\|Bu_\lambda(s,x)\|ds \\ &\leq M(\alpha r + \|x_0\| + \|g(\theta)\|) + M_b \\ &\quad + \frac{b^q M^2 (M_B)^2}{q\lambda(\Gamma(q))^2}\Big[\|x_b\| + M(\alpha r + \|x_0\| + \|g(\theta)\|) + M_b\Big]. \qquad (16) \end{aligned}$$

If $t \in (t_k, s_k]$; $k = 1, 2, \ldots, m$, then by (10) and (H5), we obtain

$$\begin{aligned} \|(F_\lambda x)(t)\| &= \|\gamma_k(t, x(t))\| \\ &\leq K_{\gamma_k}\|x(t)\| + \|\gamma_k(t,\theta)\| \\ &\leq Kr + \beta \leq M(Kr + \beta), \qquad (17) \end{aligned}$$

where $\beta = \max_{k=1,2,\ldots,m}\{\sup_{t\in J}\|\gamma_k(t,\theta)\|\}$. If $t \in (s_k, t_{k+1}]$; $k = 1, 2, \ldots, m$ then (9), (10)and (14) yield the following estimations

$$\begin{aligned} \|u_\lambda(t,x)\| &\leq \frac{MM_B}{\lambda\Gamma(q)}\Big[\|x_b\| + M(Kr+\beta) + M_b\Big], \qquad (18) \\ \|(F_\lambda x)(t)\| &\leq M(Kr+\beta) + M_b \\ &\quad + \frac{b^q M^2 (M_B)^2}{q\lambda(\Gamma(q))^2}\Big[\|x_b\| + M(Kr+\beta) + M_b\Big]. \qquad (19) \end{aligned}$$

Combining (16), (17) and (19), we obtain

$$\begin{aligned} \|(F_\lambda x)(t)\| &\leq M_b + M(Kr+\beta) + M(\alpha r + \|x_0\| + \|g(\theta)\|) + \overline{M}M(Kr+\beta) \\ &\quad + \overline{M}\Big[\|x_b\| + M(\alpha r + \|x_0\| + \|g(\theta)\|) + M_b\Big]. \qquad (20) \end{aligned}$$

Then we get that for large enough $R > 0$, $F_\lambda(\Omega_R) \subset \Omega_R$ holds.
Step 2: We show that $\Phi_\lambda : \Omega_R \to \Omega_R$ is Lipschitz continuous. Let $x, y \in \Omega_R$, for $t \in [0, t_1]$ using (11) and $(H4)$ we have

$$\|(\Phi_\lambda x)(t) - (\Phi_\lambda y)(t)\| \leq M\|g(x) - g(y)\| \leq M\alpha\|x - y\|, \qquad (21)$$

for $t \in (t_k, s_k]$, $k = 1, 2, \ldots, m$, by (11) and the assumption $(H5)$, we obtain

$$\|(\Phi_\lambda x)(t) - (\Phi_\lambda y)(t)\| \le K_{\gamma_k}\|x(t) - y(t)\| \le MK\|x - y\|, \tag{22}$$

for $t \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, using $(H5)$, we have

$$\begin{aligned}\|(\Phi_\lambda x)(t) - (\Phi_\lambda y)(t)\| &\le M\|\gamma_k(s_k, x(s_k)) - \gamma_k(s_k, y(s_k))\| \\ &\le MK\|x - y\|. \end{aligned}\tag{23}$$

From (21), (22) and (23), we obtain

$$\|\Phi_\lambda x - \Phi_\lambda y\| \le MK_1\|x - y\|. \tag{24}$$

Step 3: Ψ_λ is continuous in Ω_R. Let $\{x_n\}$ be a sequence in Ω_R such that $\lim_{n\to\infty} x_n = x$ in Ω_R. By the continuity of nonlinear term f with respect to second and third variables, for each $s \in J$, we have

$$\lim_{n\to\infty} f(s, x_n(s), Gx_n(s)) = f(s, x(s), Gx(s)). \tag{25}$$

So, we can conclude that

$$\sup_{s\in J}\|f(s, x_n(s), Gx_n(s)) - f(s, x(s), Gx(s))\| \to 0 \quad \text{as} \quad n \to \infty. \tag{26}$$

For $t \in (s_k, t_{k+1}]$, (H5) and (26) yield the following

$$\begin{aligned}\|p(x_n) - p(x)\| &\le M\|\gamma_k(s_k, x_n(s_k)) - \gamma_k(s_k, x(s_k))\| \\ &\quad + \frac{M}{\Gamma(q)}\int_{s_k}^{b}(b - s)^{q-1}\|f(s, x_n(s), Gx_n(s)) - f(s, x(s), Gx(s))\|ds \\ &\le M\|\gamma_k(s_k, x_n(s_k)) - \gamma_k(s_k, x(s_k))\| \\ &\quad + \frac{Mb^q}{\Gamma(q+1)}\sup_{s\in J}\|f(s, x_n(s), Gx_n(s)) - f(s, x(s), Gx(s))\| \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}\tag{27}$$

Therefore, (9) and (27) imply that

$$\|u_\lambda(s, x_n) - u_\lambda(s, x)\| \to 0 \quad \text{as} \quad n \to \infty, \tag{28}$$

also (12), (26) and (28) yield

$$\begin{aligned}\|(\Psi_\lambda x_n)(t) - (\Psi_\lambda x)(t)\| &\le \frac{M}{\Gamma(q)}\int_{s_k}^{t}(t - s)^{q-1}\|f(s, x_n(s), Gx_n(s)) - f(s, x(s), Gx(s))\|ds \\ &\quad + \frac{M}{\Gamma(q)}\int_{s_k}^{t}(t - s)^{q-1}\|B\|\|u_\lambda(s, x_n) - u_\lambda(s, x)\|ds \\ &\le \frac{Mb^q}{\Gamma(q+1)}\sup_{s\in J}\|f(s, x_n(s), Gx_n(s)) - f(s, x(s), Gx(s))\| \\ &\quad + \frac{b^q MM_B}{\Gamma(q+1)}\sup_{s\in J}\|u_\lambda(s, x_n) - u_\lambda(s, x)\| \\ &\to 0 \quad \text{as} \quad n \to \infty, \end{aligned}\tag{29}$$

which means that Ψ_λ is continuous in Ω_R.

Step 4: $\Psi_\lambda : \Omega_R \to \Omega_R$ is compact. We shall get this result by using Arzela-Ascoli theorem. For this we have to prove that

(i): For any $t \in J$, the set $\{(\Psi_\lambda x)(t) : x \in \Omega_R\}$ is relatively compact in X. For $t \notin (s_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$, obviously the set $\{(\Psi_\lambda x)(t) : x \in \Omega_R\} = \{0\}$ which

is compact in X. Let $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$ be fixed. For any $\varepsilon \in (s_k, t)$ and $\delta > 0$, we define an operator $\Psi_\lambda^{\varepsilon,\delta}$ on Ω_R as following

$$\begin{aligned}(\Psi_\lambda^{\varepsilon,\delta} x)(t) &= q\int_{s_k}^{t-\varepsilon}\int_{\delta}^{\infty}\theta(t-s)^{q-1}\zeta_q(\theta)T((t-s)^q\theta)[f(s,x(s),Gx(s))+Bu_\lambda(s,x)]d\theta ds\\ &= T(\varepsilon^q\delta)q\int_{s_k}^{t-\varepsilon}\int_{\delta}^{\infty}\theta(t-s)^{q-1}\zeta_q(\theta)T((t-s)^q\theta-\varepsilon^q\delta)[f(s,x(s),Gx(s))\\ &\quad +Bu_\lambda(s,x)]d\theta ds\\ &:= T(\varepsilon^q\delta)y(t,\varepsilon).\end{aligned}$$

Since $T(\varepsilon^q\delta)(\varepsilon^q\delta > 0)$ is compact on X and $y(t,\varepsilon)$ is bounded on Ω_R, we obtain that the set $\{(\Psi_\lambda^{\varepsilon,\delta}x)(t) : x \in \Omega_r\}$ is relatively compact in X. On the other hand

$$\begin{aligned}\|(\Psi_\lambda x)(t) - (\Psi_\lambda^{\varepsilon,\delta}x)(t)\| &= q\Big\|\int_{s_k}^{t}\int_{0}^{\delta}\theta(t-s)^{q-1}\zeta_q(\theta)T((t-s)^q\theta)[f(s,x(s),Gx(s))\\ &\quad +Bu_\lambda(s,x)]d\theta ds\\ &\quad +\int_{s_k}^{t}\int_{\delta}^{\infty}\theta(t-s)^{q-1}\zeta_q(\theta)T((t-s)^q\theta)[f(s,x(s),Gx(s))\\ &\quad +Bu_\lambda(s,x)]d\theta ds\\ &\quad -\int_{s_k}^{t-\varepsilon}\int_{\delta}^{\infty}\theta(t-s)^{q-1}\zeta_q(\theta)T((t-s)^q\theta)[f(s,x(s),Gx(s))\\ &\quad +Bu_\lambda(s,x)]d\theta ds\Big\|\\ &= q\Big\|\int_{s_k}^{t}\int_{0}^{\delta}\theta(t-s)^{q-1}\zeta_q(\theta)T((t-s)^q\theta)[f(s,x(s),Gx(s))\\ &\quad +Bu_\lambda(s,x)]d\theta ds\\ &\quad +\int_{t-\epsilon}^{t}\int_{\delta}^{\infty}\theta(t-s)^{q-1}\zeta_q(\theta)T((t-s)^q\theta)[f(s,x(s),Gx(s))\\ &\quad +Bu_\lambda(s,x)]d\theta ds\Big\|\\ &\le q(I_1+I_2),\end{aligned} \tag{30}$$

where

$$\begin{aligned}I_1 &= \|\int_{s_k}^{t}\int_{0}^{\delta}\theta(t-s)^{q-1}\zeta_q(\theta)T((t-s)^q\theta)[f(s,x(s),Gx(s))+Bu_\lambda(s,x)]d\theta ds\|,\\ I_2 &= \|\int_{t-\epsilon}^{t}\int_{\delta}^{\infty}\theta(t-s)^{q-1}\zeta_q(\theta)T((t-s)^q\theta)[f(s,x(s),Gx(s))+Bu_\lambda(s,x)]d\theta ds\|.\end{aligned}$$

Now, by (14) and (18) we have

$$\begin{aligned}I_1 &\le M\Big(\int_0^\delta\theta\zeta_q(\theta)d\theta\Big)\Big[\int_{s_k}^{t}(t-s)^{q-1}\|f(s,x(s),Gx(s))\|ds + M_B\|u_\lambda\|\frac{b^q}{q}\Big]\\ &\le M\Big(\int_0^\delta\theta\zeta_q(\theta)d\theta\Big)\Big[\frac{M_1}{(1+q_2)^{1-q_1}}b^{(1+q_2)(1-q_1)}\\ &\quad +\frac{M(M_B)^2}{\lambda\Gamma(q)}[\|x_b\| + M(Kr+\beta)+M_b]\frac{b^q}{q}\Big].\end{aligned} \tag{31}$$

Similarly using Remark 2.2, we can obtain

$$\begin{aligned} I_2 &\leq M\bigg(\int_\delta^\infty \theta\zeta_q(\theta)d\theta\bigg)\bigg[\frac{M_1}{(1+q_2)^{1-q_1}}\epsilon^{(1+q_2)(1-q_1)} + \frac{M(M_B)^2}{\lambda\Gamma(q)}[\|x_b\| \\ &\quad +M(Kr+\beta)+M_b]\frac{\epsilon^q}{q}\bigg] \\ &\leq \frac{M}{\Gamma(q+1)}\bigg[\frac{M_1}{(1+q_2)^{1-q_1}}\epsilon^{(1+q_2)(1-q_1)} + \frac{M(M_B)^2}{\lambda\Gamma(q)}[\|x_b\| \\ &\quad +M(Kr+\beta)+M_b]\frac{\epsilon^q}{q}\bigg]. \end{aligned} \tag{32}$$

Therefore by (30), (31), and (32) we conclude that

$$\|(\Psi_\lambda x)(t) - (\Psi_\lambda^{\varepsilon,\delta} x)(t)\| \to 0 \quad as \quad \varepsilon \to 0, \delta \to 0.$$

This implies that for $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$, the set $\{(\Psi_\lambda x)(t) : x \in \Omega_R\}$ is relatively compact in X.
(ii): The family of functions $\{\Psi_\lambda x : x \in \Omega_R\}$ is bounded and equicontinuous. Boundedness is obvious. For any $x \in \Omega_R$ and $s_k \leq t_1 < t_2 \leq t_{k+1}$ for $k = 0, 1, 2, \ldots, m$, we have

$$\begin{aligned} \|(\Psi_\lambda x)(t_2) - (\Psi_\lambda x)(t_1)\| &\leq \bigg\|\int_{t_1}^{t_2}(t_2-s)^{q-1}V(t_2-s)f(s,x(s),Gx(s))ds\bigg\| \\ &\quad +\bigg\|\int_{t_1}^{t_2}(t_2-s)^{q-1}V(t_2-s)Bu_\lambda(s,x)ds\bigg\| \\ &\quad +\bigg\|\int_{s_k}^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]V(t_2-s)f(s,x(s),Gx(s))ds\bigg\| \\ &\quad +\bigg\|\int_{s_k}^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]V(t_2-s)Bu_\lambda(s,x)ds\bigg\| \\ &\quad +\bigg\|\int_{s_k}^{t_1}(t_1-s)^{q-1}[V(t_2-s)-V(t_1-s)]f(s,x(s),Gx(s))ds\bigg\| \\ &\quad +\bigg\|\int_{s_k}^{t_1}(t_1-s)^{q-1}[V(t_2-s)-V(t_1-s)]Bu_\lambda(s,x)ds\bigg\| \\ &= J_1+J_2+J_3+J_4+J_5+J_6, \end{aligned}$$

Now, we only need to check that J_1, J_2, J_3, J_4, J_5 and J_6 tends to 0 independently of $x \in \Omega_R$ when $t_2 \to t_1$. By (14), we have

$$\begin{aligned} J_1 &\leq \frac{M_1 M}{\Gamma(q)(1+q_2)^{1-q_1}}(t_2-t_1)^{(1+q_2)(1-q_1)} \to 0 \quad as \quad t_2 \to t_1, \\ J_2 &\leq \frac{MM_B}{\Gamma(q+1)}(t_2-t_1)^q\|u_\lambda\| \to 0 \quad as \quad t_2 \to t_1. \end{aligned}$$

By $(H3)$, Lemma 2.3, and Hölder inequality, we get that

$$
\begin{aligned}
J_3 &\le \frac{M}{\Gamma(q)}\Big(\int_{s_k}^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]^{\frac{1}{1-q_1}}ds\Big)^{1-q_1}\psi(R)\|\phi\|_{L^{\frac{1}{q_1}}(J,\mathbb{R})}\\
&\le \frac{M_1M}{\Gamma(q)}\Big(\int_{s_k}^{t_1}[(t_1-s)^{q_2}-(t_2-s)^{q_2}]ds\Big)^{1-q_1}\\
&\le \frac{M_1M}{\Gamma(q)(1+q_2)^{1-q_1}}[(t_1)^{1+q_2}-(t_2)^{1+q_2}+(t_2-t_1)^{1+q_2}]^{1-q_1}\\
&\le \frac{M_1M}{\Gamma(q)(1+q_2)^{1-q_1}}(t_2-t_1)^{(1+q_2)(1-q_1)}\to 0 \quad as \quad t_2\to t_1,
\end{aligned}
$$

and

$$
J_4 \le \frac{MM_B}{\Gamma(q+1)}\Big[(t_2-s_k)^q-(t_1-s_k)^q-(t_2-t_1)^q\Big]\|u_\lambda\|\to 0 \quad as \quad t_2\to t_1.
$$

For $t_1=s_k$, it is easy to see that $J_5=0$. For $t_1>s_k$ and $\epsilon>0$ small enough, by $(H3)$ and Lemma 2.3, we obtain

$$
\begin{aligned}
J_5 &\le \|\int_{s_k}^{t_1-\epsilon}(t_1-s)^{q-1}[V(t_2-s)-V(t_1-s)]f(s,x(s),Gx(s))ds\|\\
&\quad +\|\int_{t_1-\epsilon}^{t_1}(t_1-s)^{q-1}[V(t_2-s)-V(t_1-s)]f(s,x(s),Gx(s))ds\|\\
&\le \int_{s_k}^{t_1-\epsilon}\|(t_1-s)^{q-1}f(s,x(s),Gx(s))\|ds\sup_{s\in[s_k,t_1-\epsilon]}\|V(t_2-s)-V(t_1-s)\|\\
&\quad +\frac{2M}{\Gamma(q)}\int_{t_1-\epsilon}^{t_1}\|(t_1-s)^{q-1}f(s,x(s),Gx(s))\|ds\\
&\le \frac{M_1}{(1+q_2)^{1-q_1}}((t_1-s_k)^{1+q_2}-\epsilon^{1+q_2})^{1-q_1}\sup_{s\in[s_k,t_1-\epsilon]}\|V(t_2-s)-V(t_1-s)\|\\
&\quad +\frac{2M_1M}{\Gamma(q)(1+q_2)^{1-q_1}}\epsilon^{(1+q_2)(1-q_1)}\to 0 \quad as \quad t_2\to t_1,\ \epsilon\to 0,
\end{aligned}
$$

similarly

$$
\begin{aligned}
J_6 &\le \frac{M_B}{q}[(t_1-s_k)^q-\epsilon^q]\|u_\lambda\|\sup_{s\in[s_k,t_1-\epsilon]}\|V(t_2-s)-V(t_1-s)\|\\
&\quad +\frac{2MM_B}{\Gamma(q+1)}\epsilon^q\|u_\lambda\|\to 0 \quad as \quad t_2\to t_1,\ \epsilon\to 0.
\end{aligned}
$$

As a result $\|(\Psi_\lambda x)(t_2)-(\Psi_\lambda x)(t_1)\|\to 0$ independently of $x\in\Omega_R$ as $t_2\to t_1$, which means that $\Psi_\lambda:\Omega_R\to\Omega_R$ is equicontinuous. Thus, by Arzela-Ascoli theorem Ψ_λ is compact on Ω_R.

Step 5: *We show that F_λ is ρ-set contractive map. For any bounded set $D\subset\Omega_R$, by Lemma 2.11, we know that there exists a countable set $D_0=\{x_n\}\subset D$ such that*

$$\alpha(\Psi_\lambda(D))\le 2\alpha(\Psi_\lambda(D_0)).$$

Since $\Psi_\lambda(D_0)\subset\Psi_\lambda(\Omega_R)$ is bounded and equicontinuous, by Lemma 2.9 we obtain

$$\alpha(\Psi_\lambda(D_0))=\max_{t\in[s_k,t_{k+1}],k=0,1,2,\dots,m}\alpha(\Psi_\lambda(D_0)(t)).$$

By Lemma 2.12 (i) and Step 4 (i), we have $\alpha(\Psi_\lambda(D_0)(t)) = 0$ *for all* $t \in J$*, therefore* $\alpha(\Psi_\lambda(D)) = 0$*. From (24) and Lemma 2.12 (iii), we know that for any bounded set* $D \subset \Omega_R$

$$\alpha(\Phi_\lambda(D)) \leq MK_1\alpha(D).$$

Thus, by Lemma 2.12 (ii)

$$\begin{aligned} \alpha(F_\lambda(D)) &\leq \alpha(\Phi_\lambda(D)) + \alpha(\Psi_\lambda(D)) \\ &\leq MK_1\alpha(D) = \rho\alpha(D). \end{aligned} \tag{33}$$

Now combining (33) with (13) and Definition 2.13, we conclude that $F_\lambda : \Omega_R \to \Omega_R$ *is a* ρ*-set-contractive map with* $\rho = MK_1$*. Hence from Theorem 2.14, it follows that* F_λ *has atleast one fixed point* $x \in \Omega_R$*, which is a PC-mild solution of (1). Thus the proof of the theorem is completed.*

4. Approximate Controllability

In this section, the approximate controllability of (1) will be discussed.

Theorem 4.1. *Assume that the assumptions of Theorem 3 hold and in addition, hypothesis (H0) is satisfied. Moreover assume that the functions* f, g, γ_k $(k = 1, 2, \ldots, m)$ *are uniformly bounded by positive constants* L_1, L_2 *and* $N_k(k = 1, 2, \ldots, m)$*. Then the semilinear fractional system (1) is approximately controllable on* J*.*

Proof. *Let* x_λ *be a fixed point of* F_λ *in* Ω_R*. Any fixed point of* F_λ *is a mild solution of the problem (1) under the control*

$$u_\lambda(t, x_\lambda) = B^*V^*(b - t)R(\lambda, \Gamma_0^b)p(x_\lambda),$$

and satisfies the equality

$$x_\lambda(b) = x_b - \lambda R(\lambda, \Gamma_0^b)p(x_\lambda), \tag{34}$$

where

$$p(x_\lambda) = \begin{cases} x_b - U(b)(x_0 - g(x_\lambda)) - \int_0^b (b-s)^{q-1}V(b-s)f(s, x_\lambda(s), Gx_\lambda(s))ds, \\ \qquad \text{for } t \in (0, t_1], \\ x_b - U(b - s_k)\gamma_k(s_k, x_\lambda(s_k)) - \int_{s_k}^b (b-s)^{q-1}V(b-s)f(s, x_\lambda(s), Gx_\lambda(s))ds, \\ \qquad \text{for } t \in \cup_{k=1}^m (s_k, t_{k+1}]. \end{cases}$$

According to the compactness of $U(t)(t > 0)$ *and the uniform boundedness of* g*, we see that there exists a subsequence of* $\{U(b)g(x_\lambda) : \lambda > 0\}$*, still denoted by it, converges to some* $x_g \in X$ *as* $\lambda \to 0$*. Similarly there exists a subsequence of* $\{U(b - s_k)\gamma_k(s_k, x_\lambda(s_k)) : \lambda > 0\}$*, still denoted by it, converges to some* $x_{\gamma_k} \in X$ *as* $\lambda \to 0$*. By the assumption that* f *is uniformly bounded, we have*

$$\int_0^b \|f(s, x_\lambda(s), Gx_\lambda(s))\|^2 ds \leq L_1^2 b.$$

Hence the sequence $f(\cdot, x_\lambda(\cdot), Gx_\lambda(\cdot))$ *is bounded in* $L^2(J, X)$*. Then there exists a subsequence of* $\{f(\cdot, x_\lambda(\cdot), Gx_\lambda(\cdot)) : \lambda > 0\}$*, still denoted by it, converges weakly to some* $f(\cdot) \in L^2(J, X)$*. Define*

$$\omega = \begin{cases} x_b - U(b)(x_0) + x_g - \int_0^b (b-s)^{q-1}V(b-s)f(s)ds, & t \in (0, t_1]; \\ x_b - x_{\gamma_k} - \int_{s_k}^b (b-s)^{q-1}V(b-s)f(s)ds, & t \in \cup_{k=1}^m (s_k, t_{k+1}]. \end{cases}$$

It follows that for $t \in (0, t_1]$ *and* $t \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, m$,

$$\|p(x_\lambda) - \omega\| \to 0 \quad as \quad \lambda \to 0^+, \tag{35}$$

because of compactness of the operator (see [27])

$$l(\cdot) \to \int_0^{\cdot} (\cdot - s)^{q-1} V(\cdot - s) l(s) ds : L^2(J, X) \to C(J, X).$$

Then, from (34), (35), and (H0), we obtain

$$\begin{aligned} \|x_\lambda(b) - x_b\| &\leq \|\lambda R(\lambda, \Gamma_0^b) p(x_\lambda)\| \\ &\leq \|\lambda R(\lambda, \Gamma_0^b)\omega\| + \|\lambda R(\lambda, \Gamma_0^b)\| \|p(x_\lambda) - \omega\| \\ &\leq \|\lambda R(\lambda, \Gamma_0^b)\omega\| + \|p(x_\lambda) - \omega\| \to 0 \ as \ \lambda \to 0^+. \end{aligned}$$

This proves the approximate controllability of the system (1) on J.

5. Example

As an application, we consider a control system governed by a fractional partial differential equation of the form :

$$\begin{cases} {}^cD^{\frac{1}{2}} x(t,z) = \frac{\partial^2}{\partial z^2} x(t,z) + u(t,z) + \frac{1}{25} \frac{e^{-t}}{1+e^t} \frac{|x(t,z)|}{1+|x(t,z)|} + \int_0^t \frac{1}{50} e^{-s} \frac{|x(s,z)|}{1+|x(s,z)|} ds, \\ \qquad z \in (0,1), \ t \in (0, \frac{1}{3}] \cup (\frac{2}{3}, 1], \\ x(t,0) = x(t,1) = 0, \ \ t \in [0,1], \\ x(t,z) = \frac{e^{-(t-\frac{1}{3})}}{4} \frac{|x(t,z)|}{1+|x(t,z)|}, \ \ z \in (0,1), \ t \in (\frac{1}{3}, \frac{2}{3}], \\ x(0,z) + \displaystyle\sum_{i=1}^{2} \frac{1}{3^i} \frac{x(\frac{1}{i}, z)}{1 + x(\frac{1}{i}, z)} = x_0(z), \ \ z \in [0,1], \end{cases} \tag{36}$$

where $X = U = L^2[0,1]$, $J = [0,1]$, $x_0(z) \in X$. Define $Ax = x''$ with

$$D(A) = \{x \in X : x, x' \ are \ absolutely \ continuous \ and \ x'' \in X, x(0) = x(1) = 0\}.$$

Then

$$Ax = \sum_{n=1}^{\infty} -n^2 < x, e_n > e_n, \ x \in D(A), \tag{37}$$

where $e_n(z) = \sqrt{\frac{2}{\pi}} sin(nz)$, $0 \leq z \leq 1$, $n = 1, 2, \ldots$. It is well known that A generates a compact semigroup $T(t)(t > 0)$, on X and is given by

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} < x, e_n > e_n, \ x \in X, \tag{38}$$

with $\|T(t)\| \leq 1$, for any $t \geq 0$. Let $b = t_2 = 1, t_0 = s_0 = 0, t_1 = \frac{1}{3}, s_1 = \frac{2}{3}$. Put $x(t) = x(t, \cdot)$, that is $x(t)(z) = x(t,z)$, $t, z \in [0,1]$. Let $u(t) = u(t, \cdot)$ is continuous and the bounded linear operator $B : U \to X$ is defined as $Bu(t) = u(t, \cdot)$. Further

$$\begin{aligned} f(t, x(t), Gx(t)) &= \frac{1}{25} \frac{e^{-t}}{1+e^t} \frac{|x(t,\cdot)|}{1+|x(t,\cdot)|} + \int_0^t \frac{1}{50} e^{-s} \frac{|x(s,\cdot)|}{1+|x(s,\cdot)|} ds, \\ Gx(t) &= \int_0^t \frac{1}{50} e^{-s} \frac{|x(s,\cdot)|}{1+|x(s,\cdot)|} ds, \\ \gamma_1(t, x(t)) &= \frac{e^{-(t-\frac{1}{3})}}{4} \frac{|x(t,\cdot)|}{1+|x(t,\cdot)|}, \\ g(x) &= \sum_{i=1}^{2} \frac{1}{3^i} \frac{x(\frac{1}{i}, \cdot)}{1 + x(\frac{1}{i}, \cdot)}. \end{aligned}$$

Then the system (36) can be rewritten into the abstract form of (1) for $m = 1$. *It is easy to verify that the assumptions* $(H1)$-$(H5)$ *and condition* (13) *hold with*

$$q = \frac{1}{2},\ M = 1,\ \phi(t) = \frac{1}{25}\frac{e^{-t}}{1+e^{t}} + \frac{1}{50},\ \psi(r) = r,$$
$$\alpha = \frac{4}{9},\ K_{\gamma_1} = \frac{1}{4},\ \rho = \frac{4}{9} < 1.$$

Also f, g *and* γ_1 *are uniformly bounded with* $L_1 = \frac{3}{50}$, $L_2 = \frac{4}{9}$, $N_1 = \frac{1}{4}$ *respectively. Moreover linear system corresponding to (36) is approximately controllable on* $[0, 1]$, *based on the argument in* [22], *it yields that* $(H6)$ *also holds. Thus by Theorem 4.1, the system (36) is approximately controllable.*

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Arshi Meraj
Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee-247667, Uttarakhand, India.
E-mail address: arshimeraj@gmail.com

Dwijendra N. Pandey
Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee-247667, Uttarakhand, India.
E-mail address: dwij.iitk@gmail.com