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TREATISE ON FRACTIONAL ORDER DYNAMICAL EQUATIONS INVOLVING RANDOM VARIABLE

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Abstract. In this paper, fractional order dynamical equation involving random walk is discussed. The solution of a dynamical system is obtained using ψ-Hilfer fractional derivative. Firstly, the solutions of the fractional dynamical system with random initial conditions are obtained. Further, the random impulsive effect is taken into account to verify the system. The sufficient conditions are obtained using the standard fixed point method. The stability check is made sure by Ulam-Hyers stability method.

1. Introduction

Arbitrary (non-integer) order differential equations arise in many engineering and scientific order as the mathematical modelling of systems and progression in the fields of biology, physics and so on, and they gained much value and consideration, due to their relevance in many other fields. During the last two decades, fractional calculus has increasingly attracted the attention of researchers of many different fields, (see [5, 11, 12, 13, 14, 19]) and the references therein.
Here we study the idea of fractional derivative on time scale $\mathbb{T}$. Interesting in applications, it is the possibility to deal with more complex time domains. One extreme case, covered by the theory of time scales and surprisingly relevant also for the process of signals, appears when one fix the time scale to be the Cantor set. For further information about the theoretical and potential applications of time scales, (see [1, 2, 3, 4]).
The knowledge about the parameters of a dynamic equation is of probabilistic nature; modelling of such systems is called random differential equations (RDEs) or stochastic differential equations. The analyses of FDEs with random variable are studied in, [9, 15, 20].
The Ulam-Hyers(U-H) and Ulam-Hyers-Rassias(U-H-R) types of stability of functional differential equation are discussed vastly in recent days. The stability properties of dynamical equations have attracted many mathematicians. Particularly, the U-H-R stability was briefly studied in, [8, 10, 19, 21].

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More recently, a new fractional derivative has established by Sousa and Oliveira [17] so-called ψ-Hilfer fractional derivative (HFD), which unifies several fractional derivatives, that is, by generalizing those kernels of function can be seen in [17]. Dynamical behaviour of FDEs involving ψ-HFD is discussed in [7, 17, 18].
Motivated by all the above works here we study dynamic equation on, time scales with ψ-HFD with random variable ω and random initial condition is given by

$$\begin{cases} {}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{u}(\tau,\omega) = \mathfrak{g}(\tau,\mathfrak{u}(\tau,\omega),\omega), \quad \tau \in J \subseteq \mathbb{T}, \\ {}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(\tau,\omega)|_{\tau=0} = \mathfrak{u}_0(\omega), \end{cases} \tag{1}$$

where (Ω, F, p) is a complete probability space, $\omega \in \Omega$, ${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}$ is the ψ-HFD defined on $\mathbb{T}$, $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\mathfrak{I}^{1-\gamma;\psi}$ is ψ-fractional internal of order $1-\gamma(\gamma = \alpha+\beta-\alpha\beta)$. Let $\mathbb{T}$ be a time scale, that is nonempty subset of Banach space. The function $\mathfrak{g} : J := [0,b] \times R \times \Omega \to R$ is a right-dense continuous function. Here, the Eq. (1) satisfies the random integral equation of the form

$$\mathfrak{u}(t,\omega) = \frac{\mathfrak{u}_0(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s. \tag{2}$$

Next, we discuss the existence, uniqueness and stability of solutions of RDEs with jump conditions involving ψ-HFD of the form

$$\begin{cases} {}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{u}(\tau,\omega) = \mathfrak{g}(\tau,\mathfrak{u}(\tau,\omega),\omega), \quad \tau \in J \subseteq \mathbb{T}, \\ \Delta{}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(t,\vartheta)|_{\tau=\tau_k} = \mathfrak{u}_{\tau_k}(\omega), \\ {}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(\tau,\omega)|_{\tau=0} = \mathfrak{u}_0(\omega) \end{cases} \tag{3}$$

where $\mathfrak{u}_k(\omega) : J \times \Omega \to R$ is continuous for all $k = 1,2,...,m$, and $0 = \tau_0 < \tau_1 < ... < \tau_m < \tau_{m+1} = b$, $\Delta{}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(\tau,\omega)|_{\tau=\tau_k} ={}^{\mathbb{T}} \mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}_{(\tau_k^+)}(\omega) -{}^{\mathbb{T}} \mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}_{(\tau_k^-)}(\omega)$, ${}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}_{(\tau_k^+)}(\omega) = \lim_{h\to 0+}\mathfrak{u}_{(\tau_k+h)}(\omega)$ and ${}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}_{(\tau_k^-)}(\omega) = \lim_{h\to 0-}\mathfrak{u}_{(\tau_k+h)}(\omega)$ represent the right and left limits of $\mathfrak{u}(\tau,\omega)$ at $\tau = \tau_k$. The equivalent integral equation of the Eq. (3) is given by

$$\begin{aligned} \mathfrak{u}(\tau,\omega) =& \frac{\mathfrak{u}_0(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} + \frac{\sum_{0<t_k<\tau}\mathfrak{u}_{t_k}(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)ds. \end{aligned} \tag{4}$$

The novelty of the paper is given as follows: In Section 2, basic definitions and preliminary are discussed. In section 3, existence, uniqueness and stability of RDEs are discussed. Finally, the dynamical behavior of impulsive RDEs is obtained in Section 4.

2. Preliminaries

Definition 2.1. *Let $C(J)$ be continuous function endowed with the norm*

$$\|\mathfrak{u}\|_C = \max\left\{|\mathfrak{u}(\tau,\omega)| : \tau \in J\right\}.$$

We denote the $C_{1-\gamma,\psi}(J)$ as follows

$$C_{1-\gamma,\psi}(J) := \left\{\mathfrak{g}(\tau,\omega) : J\times\Omega \to R|\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}\mathfrak{g}(\tau,\omega) \in C(J)\right\}, 0 \leq \gamma < 1$$

where $C_{1-\gamma,\psi}(J)$ is the weighted space of the continuous functions $\mathfrak{g}$ on the finite interval J.

Obviously, $C_{1-\gamma,\psi}(J)$ is the Banach space with the norm

$$\|\mathfrak{g}\|_{C_{1-\gamma,\psi}} = \left\| (\psi(\tau) - \psi(0))^{1-\gamma} \mathfrak{g}(t,\omega) \right\|_{C}.$$

Definition 2.2. *Let the space $PC(J)$ be a piecewise continuous space from J into R with the norm*

$$\|\mathfrak{u}\|_{PC} = \max \{ |\mathfrak{u}(\tau,\omega)| : \tau \in J \}.$$

The weighted space $PC_{1-\gamma,\psi}(J)$ of functions $\mathfrak{g}$ on J is defined by

$$PC_{1-\gamma,\psi}(J) = \left\{ \mathfrak{g} : J \times \Omega \to R : (\psi(\tau) - \psi(0))^{1-\gamma} \mathfrak{g}(\tau,\omega) \in PC(J) \right\}, 0 \leq \gamma < 1,$$

with the norm

$$\|\mathfrak{g}\|_{PC_{1-\gamma,\psi}} = \left\| (\psi(\tau) - \psi(0))^{1-\gamma} \mathfrak{g}(\tau,\omega) \right\|_{PC} = \max_{\tau \in J} \left| (\psi(\tau) - \psi(0))^{1-\gamma} \mathfrak{g} \right|.$$

Definition 2.3. *Let time scale be $\mathbb{T}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf \{ s \in \mathbb{T} : s > t \}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup \{ s \in \mathbb{T} : s < t \}$.*

Proposition 2.4. *Suppose $\mathbb{T}$ is a time scale and $[a,b] \subset \mathbb{T}$, $\mathfrak{g}$ is increasing continuous function on $[a,b]$. If the extension of $\mathfrak{g}$ is given in the following form:*

$$F(s) = \begin{cases} \mathfrak{g}(s); & s \in \mathbb{T} \\ \mathfrak{g}(\tau); & s \in (\tau, \sigma(\tau)) \notin \mathbb{T}. \end{cases}$$

Then we have

$$\int_a^b \mathfrak{g}(\tau) \Delta \tau \leq \int_a^b F(\tau) d\tau.$$

Definition 2.5. *Let $\mathbb{T}$ be a time scale, $J \in \mathbb{T}$. The left-sided R-L fractional integral of order $\alpha \in R^+$ of function $\mathfrak{f}(\tau)$ is defined by*

$$\left({}^{\mathbb{T}}\mathfrak{J}^{\alpha} \mathfrak{g} \right)(\tau) = \int_0^{\tau} \psi'(s) \frac{(\psi(\tau) - \psi(s))^{\alpha - 1}}{\Gamma(\alpha)} \mathfrak{g}(s) \Delta s, \quad (t > 0),$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.6. *Suppose that $\mathbb{T}$ is a time scale, $[0,b]$ is an interval of $\mathbb{T}$. The left-sided R-L fractional derivative of order $\alpha \in [n-1,n)$, $n \in \mathbb{Z}^+$ of function $f(t)$ is defined by*

$$\left({}^{\mathbb{T}}\Delta^{\alpha} \mathfrak{g} \right)(t) = \left(\frac{1}{\psi'(\tau)} \frac{d}{d\tau} \right)^n \int_0^{\tau} \psi'(s) \frac{(\psi(\tau) - \psi(s))^{n - \alpha - 1}}{\Gamma(n - \alpha)} \mathfrak{g}(s) \Delta s, \quad (\tau > 0).$$

Definition 2.7. [11] *The left-sided ψ-HFD of function $\mathfrak{f}(\tau)$ is defined by*

$${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi} \mathfrak{g}(\tau) = \left({}^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi} \ {}^{\mathbb{T}}\Delta ({}^{\mathbb{T}}\mathfrak{J}^{(1-\beta)(1-\alpha);\psi} \mathfrak{g}) \right)(\tau),$$

where ${}^{\mathbb{T}}\Delta := \frac{d}{d\tau}$.

Remark 2.8. (1) *The operator ${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}$ also can be written as*

$${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi} = {}^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi} \ {}^{\mathbb{T}}\Delta {}^{\mathbb{T}}\mathfrak{J}^{(1-\beta)(1-\alpha);\psi} = {}^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi} \ {}^{\mathbb{T}}\Delta^{\gamma;\psi}, \ \gamma = \alpha + \beta - \alpha\beta.$$

(2) *Let $\beta = 0$, the left-sided R-L derivative can be presented as ${}^{\mathbb{T}}\Delta^{\alpha} := {}^{\mathbb{T}}\Delta^{\alpha,0}$.*
(3) *Let $\beta = 0$, left-sided Caputo fractional derivative can be presented as ${}^{\mathbb{T}}_{c}\Delta^{\alpha} := {}^{\mathbb{T}}\mathfrak{J}^{1-\alpha} \ {}^{\mathbb{T}}\Delta$.*

Next, we review some lemmas which will be used to establish our existence results.

Lemma 2.9. *If $\alpha > 0$ and $\beta > 0$, there exists*

$$\left[{}^{\mathbb{T}}\mathfrak{J}^{\alpha}\left(\psi(s)-\psi(0)\right)^{\beta-1}\right](\tau)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\psi(\tau)-\psi(0)\right)^{\beta+\alpha-1}$$

Lemma 2.10. *Let $\alpha \geq 0$, $\beta \geq 0$ and $\mathfrak{g} \in L^1(J)$. Then*

$${}^{\mathbb{T}}\mathfrak{J}^{\alpha}\ {}^{\mathbb{T}}\mathfrak{J}^{\beta}\mathfrak{g}(\tau) \overset{a.e}{=} {}^{\mathbb{T}}\mathfrak{J}^{\alpha+\beta}\mathfrak{g}(\tau).$$

Lemma 2.11. *Let $0 < \alpha < 1$, $0 \leq \gamma < 1$. If $\mathfrak{g} \in C_\gamma(J)$ and ${}^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\mathfrak{g} \in C^1_\gamma(J)$, then*

$${}^{\mathbb{T}}\mathfrak{J}^{\alpha}\ {}^{\mathbb{T}}\Delta^{\alpha}\mathfrak{g}(\tau)=\mathfrak{g}(\tau)-\frac{\left({}^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\mathfrak{g}\right)(0)}{\Gamma(\alpha)}\left(\psi(\tau)-\psi(0)\right)^{\alpha-1}.$$

Lemma 2.12. *Suppose that $\alpha > 0$, $a(\tau,\omega)$ is a nonnegative function locally integrable on $0 \leq \tau < b$ (some $b \leq \infty$), and let $g(\tau,\omega)$ be a nonnegative, nondecreasing continuous function defined on $0 \leq \tau < b$, such that $g(\tau,\omega) \leq K$ for some constant K. Further, let $\mathfrak{u}(\tau,\omega)$ be a nonnegative locally integrable on $a \leq t < b$ function with*

$$|\mathfrak{u}(\tau,\omega)| \leq a(\tau,\omega)+g(\tau,\omega)\int_0^\tau \psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{u}(s,\omega)\Delta s,$$

with some $\alpha > 0$. Then

$$|\mathfrak{u}(\tau,\omega)| \leq a(\tau,\omega)+\int_0^\tau\left[\sum_{n=1}^{\infty}\frac{(g(\tau,\omega)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\psi'(s)\left(\psi(\tau)-\psi(s)\right)^{n\alpha-1}\right]\mathfrak{u}(s,\omega)\Delta s,\ \ 0 \leq \tau < b.$$

Remark 2.13. *Under the hypothesis of Lemma 2.12 let $a(t,\omega)$ be a nondecreasing function on $[0,b)$. Then $\mathfrak{u}(t,\omega) \leq a(t,\omega)E_\alpha(g(t,\omega)\Gamma(\alpha)\left(\psi(\tau)-\psi(0)\right)^{\alpha})$, where E_α is the Mittag-Leffler function defined by*

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)},\ \ z \in C,\ \ Re(\alpha) > 0.$$

Lemma 2.14. *Let $\mathfrak{u} \in PC_{1-\gamma,\psi}(J)$ satisfies the following inequality*

$$|\mathfrak{u}(\tau,\omega)| \leq c_1+c_2\int_0^\tau \psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}|\mathfrak{u}(s,\omega)|\,ds+\sum_{0<t_k<t}|\mathfrak{u}_{\tau_k}(\omega)|,$$

where c_1 and c_2 are positive constants. Then

$$|\mathfrak{u}(\tau,\omega)| \leq c_1\left(1+\psi E_\alpha(c_2\Gamma(\alpha)\left(\psi(\tau)-\psi(0)\right)^{\alpha})^k E_\alpha(c_2\Gamma(\alpha)\left(\psi(\tau)-\psi(0)\right)^{\alpha}\right)\ \ for\ \tau \in (t_k.t_{k+1}],$$

where $\psi = \sup\{\psi_k : k = 1,2,3,...,m\}$.

Theorem 2.15. [6]*(Schauder's Fixed Point Theorem) Let E be a Banach space and Q be a nonempty bounded convex and closed subset of E and $N : Q \to Q$ is compact, and continuous map. Then N has at least one fixed point in Q.*

Theorem 2.16. [6]*(Schaefer's Fixed Point Theorem) Let K be a Banach space and let $\mathfrak{P} : K \to K$ be completely continuous operator. If the set $\{\mathfrak{h} \in K : \mathfrak{h} = \delta\mathfrak{P}\mathfrak{h}$ for some $\delta \in (0,1)\}$ is bounded, then $\mathfrak{P}$ has a fixed point.*

3. Existence results

Here we list the following assumptions which are going to be useful in proving the results:

(H1) The function $\mathfrak{g} : J \times R \to R$ is a rd-continuous.
(H2) Let $\ell(\tau, \omega)$ be a positive constant satisfies

$$|\mathfrak{g}(\tau, \mathfrak{u}, \omega) - \mathfrak{g}(\tau, \mathfrak{v}, \omega)| \leq \ell(\tau, \omega) |\mathfrak{u} - \mathfrak{v}| .$$

(H3) Let m, n be a positive constants and $M(\omega) = \sup m(\tau, \omega),\ N(\omega) = \sup n(\tau, \omega)$, such that

$$|\mathfrak{g}(\tau, \mathfrak{u}, \omega) - \mathfrak{g}(\tau, \mathfrak{v}, \omega)| \leq m(\tau, \omega) + n(\tau, \omega) |\mathfrak{u}(\tau, \omega)| .$$

(H4) For the increasing function $\varphi \in C_{1-\gamma,\psi}(J)$, there exists $\lambda_\varphi > 0$ such that

$$^{\mathbb{T}}\mathfrak{I}^{\alpha}\varphi(t) \leq \lambda_\varphi \varphi(\tau, \omega).$$

Theorem 3.1. *Assume that (H1) and (H3) are fulfilled. Then, equation* (1) *has at least one solution.*

Proof. Consider the operator $\mathscr{P} : C_{1-\gamma,\psi}(J) \to C_{1-\gamma,\psi}(J)$. The equivalent integral of (2) is of the operator form

$$(\mathscr{P}\mathfrak{u})(\tau, \omega) = \frac{\mathfrak{u}_0(\omega)}{\Gamma(\gamma)} (\psi(\tau) - \psi(0))^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi^{'}(s) (\psi(\tau) - \psi(s))^{\alpha-1} \mathfrak{g}(s, \mathfrak{u}(s, \omega), \omega)\Delta s \quad (5)$$

Define $B_r = \left\{ \mathfrak{u} \in C_{1-\gamma,\psi}(J) : \|\mathfrak{u}\|_{C_{1-\gamma,\psi}} \leq r \right\}$. Set

$$\sigma := \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{M(\omega)}{\Gamma(\alpha+1)} (\psi(b) - \psi(0))^{\alpha-\gamma+1} ,$$

$$\rho := \frac{B(\gamma, \alpha)}{\Gamma(\alpha)} N(\omega) (\psi(b) - \psi(0))^{\alpha} .$$

In order to prove the fixed point here we utilize Theorem 2.15. We prove the result in the following steps

Step 1: We check that $\mathscr{P}(B_r) \subset B_r$.

$$\begin{aligned}
&\left|(\psi(\tau)-\psi(0))^{1-\gamma}(\mathscr{P}\mathfrak{u})(\tau,\omega)\right| \\
&\leq \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi^{'}(s)\,(\psi(\tau)-\psi(s))^{\alpha-1}\,|\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)|\,\Delta s \\
&\leq \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi^{'}(s)\,(\psi(\tau)-\psi(s))^{\alpha-1}\,|m(s,\omega)+n(s,\omega)\mathfrak{u}(s,\omega)|\,\Delta s \\
&\leq \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} M(\omega) \int_0^\tau \psi^{'}(s)\,(\psi(\tau)-\psi(s))^{\alpha-1}\,\Delta s \\
&\quad + \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} N(\omega) \int_0^\tau \psi^{'}(s)\,(\psi(\tau)-\psi(s))^{\alpha-1}\,(\psi(s)-\psi(a))^{\gamma-1}\,\Delta s\,\|\mathfrak{u}\|_{C_{1-\gamma,\psi}} \\
&\leq \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha+1)} M(\omega)\,(\psi(\tau)-\psi(0))^{\alpha} \\
&\quad + \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} B(\gamma,\alpha) N(\omega)\,(\psi(\tau)-\psi(0))^{\alpha+\gamma-1}\,\|\mathfrak{u}\|_{C_{1-\gamma,\psi}} \\
&\leq \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{M(\omega)}{\Gamma(\alpha+1)}\,(\psi(b)-\psi(0))^{\alpha-\gamma+1} + \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} N(\omega)\,(\psi(b)-\psi(0))^{\alpha}\,\|\mathfrak{u}\|_{C_{1-\gamma,\psi}}
\end{aligned}$$

Hence

$$\|(\mathscr{P}\mathfrak{u})\|_{C_{1-\gamma,\psi}} \leq \sigma + \rho r \leq r.$$

Which yields that $\mathscr{P}(B_r) \subset B_r$.
Next we prove that the operator $\mathscr{P}$ is completely continuous.
Step 2: The operator $\mathscr{P}$ is continuous.
Let $\mathfrak{u}_n$ be a sequence such that $\mathfrak{u}_n \to \mathfrak{u}$ in $C_{1-\gamma,\psi}(J)$. Then for each $\tau \in J$,

$$\begin{aligned}
&\left|(\psi(\tau)-\psi(0))^{1-\gamma}((\mathscr{P}\mathfrak{u}_n)(\tau,\omega)-(\mathscr{P}\mathfrak{u})(\tau,\omega))\right| \\
&\leq \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi^{'}(s)\,(\psi(\tau)-\psi(s))^{\alpha-1}\,|\mathfrak{g}(s,\mathfrak{u}_n(s,\omega),\omega)-\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)|\,\Delta s \\
&\leq \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi^{'}(s)\,(\psi(\tau)-\psi(s))^{\alpha-1}\,|\mathfrak{g}(s,\mathfrak{u}_n(s,\omega),\omega)-\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)|\,ds, \quad \text{(by Proposition 2.4)} \\
&\leq \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\,(\psi(b)-\psi(0))^{\alpha}\,\|\mathfrak{g}(\cdot,\mathfrak{u}_n(\cdot,\omega),\omega)-\mathfrak{g}(\cdot,\mathfrak{u}(\cdot,\omega),\omega)\|_{C_{1-\gamma,\psi}},
\end{aligned}$$

Since $\mathfrak{g}$ is continuous, Lebesgue dominated convergence theorem implies

$$\|\mathscr{P}\mathfrak{u}_n - \mathscr{P}\mathfrak{u}\|_{C_{1-\gamma,\psi}} \to 0 \quad \text{as} \quad n \to \infty.$$

Step 3: $\mathscr{P}(B_r)$ is relatively compact.

Thus $\mathscr{P}(B_r)$ is uniformly bounded. Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, then

$$\begin{aligned}
&\left|(\mathscr{P}\mathfrak{u})(\tau_2,\omega)\left(\psi(\tau_2)-\psi(0)\right)^{1-\gamma}-(\mathscr{P}\mathfrak{u})(\tau_1,\omega)\left(\psi(\tau_1)-\psi(0)\right)^{1-\gamma}\right| \\
&\leq \left|\frac{\left(\psi(\tau_2)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^{\tau_2}\psi^{'}(s)\left(\psi(\tau_2)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s\right. \\
&\quad\left.-\frac{\left(\psi(\tau_1)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^{\tau_1}\psi^{'}(s)\left(\psi(\tau_1)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\psi^{'}(s)\left|\left(\psi(\tau_2)-\psi(0)\right)^{1-\gamma}\left(\psi(\tau_2)-\psi(s)\right)^{\alpha-1}\right. \\
&\quad\left.-\left(\psi(\tau_1)-\psi(0)\right)^{1-\gamma}\left(\psi(\tau_1)-\psi(s)\right)^{\alpha-1}\right|\left|\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\right|\Delta s \\
&\quad+\frac{\left(\psi(\tau_2)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\psi^{'}(s)\left(\psi(\tau_2)-\psi(s)\right)^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\right|\Delta s \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\psi^{'}(s)\left|\left(\psi(\tau_2)-\psi(0)\right)^{1-\gamma}\left(\psi(\tau_2)-\psi(s)\right)^{\alpha-1}\right. \\
&\quad\left.-\left(\psi(\tau_2)-\psi(0)\right)^{1-\gamma}\left(\psi(\tau_1)-\psi(s)\right)^{\alpha-1}\right|\left|\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\right|ds \\
&\quad\quad+\frac{\left(\psi(\tau_2)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\left(\psi(\tau_2)-\psi(\tau_1)\right)^{\alpha+\gamma-1}B(\gamma,\alpha)\left\|\mathfrak{g}\right\|_{C_{1-\gamma,\psi}}.
\end{aligned}$$

Thus, the right-hand side of the above inequality tends to zero. Hence, along with the Arzëla-Ascoli theorem and from Step 1-3, it is concluded that $\mathscr{P}$ is completely continuous. Thus the proposed problem has at least one solution. □

Lemma 3.2. *Assume that (H1) and (H3) are fulfilled. If*

$$\left(\frac{\ell(\tau,\omega)B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(b)-\psi(0)\right)^{\alpha}\right)<1 \tag{6}$$

then the problem (1) *has a unique solution.*

Proof. Consider the operator $\mathscr{P}: C_{1-\gamma,\psi}(J)\to C_{1-\gamma,\psi}(J)$.

$$(\mathscr{P}\mathfrak{u})(\tau,\omega)=\frac{\mathfrak{u}_0(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_0^{\tau}\psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s \tag{7}$$

Let $\mathfrak{u}_1,\mathfrak{u}_2\in C_{1-\gamma,\psi}(J)$ and $t\in J$, then we have

$$\begin{aligned}
&\left|\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}\left((\mathscr{P}\mathfrak{u}_1)(\tau,\omega)-(\mathscr{P}\mathfrak{u}_2)(\tau,\omega)\right)\right| \\
&\leq\frac{\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^{\tau}\psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{u}_1(s,\omega),\omega)-\mathfrak{g}(s,\mathfrak{u}_2(s,\omega),\omega)\right|\Delta s \\
&\leq\frac{\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^{\tau}\psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{u}_1(s,\omega),\omega)-\mathfrak{g}(s,\mathfrak{u}_2(s,\omega),\omega)\right|ds \\
&\leq\frac{\ell(\tau,\omega)\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^{\tau}\psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\left|\mathfrak{u}_1(s,\omega)-\mathfrak{u}_2(s,\omega)\right|ds \\
&\leq\frac{\ell(\tau,\omega)B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(b)-\psi(0)\right)^{\alpha}\left\|\mathfrak{u}_1-\mathfrak{u}_2\right\|_{C_{1-\gamma,\psi}}.
\end{aligned}$$

Then,

$$\|\mathscr{P}\mathfrak{u}_1 - \mathscr{P}\mathfrak{u}_2\|_{C_{1-\gamma,\psi}} \leq \frac{\ell(\tau,\omega)B(\gamma,\alpha)}{\Gamma(\alpha)} \left(\psi(b)-\psi(0)\right)^{\alpha} \|\mathfrak{u}_1 - \mathfrak{u}_2\|_{C_{1-\gamma,\psi}}.$$

From (6), it follows that $\mathscr{P}$ has a unique solution. □

4. Stability analysis

Next, we shall give the definitions and the criteria generalized U-H-R stability for ψ-HFD of dynamic equations on time scales.

Definition 4.1. *Equation* (1) *is generalized U-H-R stable with respect to* $\varphi \in C_{1-\gamma}(J)$ *if there exists a real number* $c_{\mathfrak{g},\varphi} > 0$ *such that for each solution* $\mathfrak{v} \in C_{1-\gamma}(J)$ *of the inequality*

$$\left| {}^{\mathbb{T}}\Delta^{\alpha,\beta}\mathfrak{v}(\tau,\omega) - \mathfrak{g}(\tau,\mathfrak{v}(\tau,\omega),\omega)\right| \leq \varphi(\tau), \tag{8}$$

there exists a solution $\mathfrak{u} \in C_{1-\gamma}(J)$ *of equation* (1) *with*

$$|\mathfrak{v}(\tau,\omega) - \mathfrak{u}(\tau,\omega)| \leq c_{\mathfrak{g},\varphi}\varphi(\tau,\omega), \quad t \in J.$$

Theorem 4.2. *Assume that (H1), (H3), (H4) and* (6) *are satisfied. Then, the problem* (1) *is generalized U-H-R stable.*

Proof. Let $\mathfrak{v} \in C_{1-\gamma}(J)$ be solution of the following inequality (8) and let $\mathfrak{u} \in C_{1-\gamma}(J)$ be the unique solution of the Hilfer type dynamics equation (1).

$$\mathfrak{u}(\tau,\omega) = \frac{\mathfrak{u}_0(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^{\tau}\psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s.$$

By integration of (8) we obtain

$$\left|\mathfrak{v}(\tau,\omega) - \frac{\mathfrak{u}_0(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_0^{\tau}\psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{v}(s,\omega),\omega)\Delta s\right| \leq \lambda_{\varphi}\varphi(\tau,\omega). \tag{9}$$

On the other hand, we have

$$\begin{aligned}
|\mathfrak{v}(\tau,\omega) - \mathfrak{u}(\tau,\omega)| &\leq \left|\mathfrak{v}(\tau,\omega) - \frac{\mathfrak{u}_0(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_0^{\tau}\psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{v}(s,\omega),\omega)\Delta s\right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{\tau}\psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{v}(s,\omega),\omega) - \mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\right|\Delta s \\
&\leq \left|\mathfrak{v}(\tau) - \frac{\mathfrak{u}_0(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_0^{\tau}\psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{v}(s,\omega),\omega)\Delta s\right| \\
&\quad + \frac{\ell(\tau,\omega)}{\Gamma(\alpha)}\int_0^{\tau}\psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}|\mathfrak{v}(s,\omega) - \mathfrak{u}(s,\omega)|\,ds \\
&\leq \lambda_{\varphi}\varphi(\tau,\omega) + \frac{\ell(\tau,\omega)}{\Gamma(\alpha)}\int_0^{\tau}\psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}|\mathfrak{v}(s,\omega) - \mathfrak{u}(s,\omega)|\,ds.
\end{aligned}$$

By applying Lemma 2.12, we obtain

$$|\mathfrak{v}(\tau,\omega) - \mathfrak{u}(\tau,\omega)| \leq \left[(1+\nu_1\ell(\tau,\omega)\lambda_{\varphi})\lambda_{\varphi}\right]\varphi(\tau,\omega),$$

where $\nu_1 = \nu_1(\alpha)$ is a constant,then

$$|\mathfrak{v}(\tau,\omega) - \mathfrak{u}(\tau,\omega)| \leq c_{\mathfrak{g}}\epsilon\varphi(\tau,\omega),$$

thus the proof is complete. □

5. Random Differential Equation with Impulsive Effect

Here we declare some assumption that will be useful in this section

(H5) Let m^* be a positive constant $M^*(\omega) = \sup m^*(\tau, \omega)$

$$|\mathfrak{g}(\tau, \mathfrak{u}, \omega) - \mathfrak{g}(\tau, \mathfrak{v}, \omega)| \leq m^*(\tau, \omega) |\mathfrak{u}(\tau, \omega)| .$$

(H6) Let n^* be a positive constant and $N^*(\omega) = \sup n^*(\tau, \omega)$

$$\mathfrak{u}_{\tau_k}(\omega) \leq n*(\tau, \omega)$$

(H7) Let ℓ^* be a positive constant, such that

$$|\mathfrak{u}_{\tau_k}(\omega) - \mathfrak{v}_{\tau_k}(\omega)| \leq \ell*(\tau, \omega)$$

(H8) For the increasing function $\varphi \in PC_{1-\gamma,\psi}(J)$, there exists $\lambda_\varphi > 0$ such that

$$^{\mathbb{T}}\mathfrak{J}^{\alpha;\psi}\varphi(\tau, \omega) \leq \lambda_\varphi \varphi(\tau, \omega).$$

Theorem 5.1. *Assume that [H5] and [H6] are satisfied. Then, Eq.*(3) *has at least one solution.*

Proof. Consider the operator $\mathfrak{P} : \Omega \times PC_{1-\gamma,\psi} \to PC_{1-\gamma,\psi}$. The operator form of integral equation (6) is written as follows

$$\mathfrak{u}(\tau, \omega) = \mathfrak{P}\mathfrak{u}(\tau, \omega),$$

where

$$(\mathfrak{P}\mathfrak{u})(\tau, \omega) = \frac{\mathfrak{u}_0(\omega)}{\Gamma(\gamma)} (\psi(\tau) - \psi(0))^{\gamma-1} + \frac{\sum_{0<t_k<\tau} \mathfrak{u}_{\tau_k}(\omega)}{\Gamma(\gamma)} (\psi(\tau) - \psi(0))^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi^{'}(s) (\psi(\tau) - \psi(s))^{\alpha-1} \mathfrak{g}(s, \mathfrak{u}(s, \omega), \omega) ds. \tag{10}$$

First, we prove that the operator $\mathfrak{P}$ defined by (7) verifies the conditions of Theorem 2.16.
Step 1: The operator $\mathfrak{P}$ is continuous.

Let $\mathfrak{u}_n$ be a sequence such that $\mathfrak{u}_n \to \mathfrak{u}$ in $PC_{1-\gamma,\psi}$. Then for each $t \in J$,

$$\begin{aligned}
&\left|\mathfrak{P}\mathfrak{u}_n(\tau, \omega) - \mathfrak{P}\mathfrak{u}(\tau, \omega)) (\psi(\tau) - \psi(0))^{1-\gamma}\right| \\
&\leq \frac{1}{\Gamma(\gamma)} \sum_{0<\tau_k<\tau} |\psi_k(\mathfrak{u}_{k_n}(\tau_k)) - \psi_k(\mathfrak{u}(\tau_k))| \\
&\quad + \frac{(\psi(\tau) - \psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi^{'}(s) (\psi(\tau) - \psi(s))^{\alpha-1} |\mathfrak{g}(s, \mathfrak{u}_n(s, \omega), \omega) - \mathfrak{g}(s, \mathfrak{u}(s, \omega), \omega)| ds.
\end{aligned}$$

Since $\mathfrak{g}$ is continuous, then we have

$$\|\mathfrak{P}\mathfrak{u}_n - \mathfrak{P}\mathfrak{u}\|_{PC_{1-\gamma,\psi}} \to 0 \text{ as } n \to \infty.$$

This proves the continuity of $\mathfrak{P}$.
Step 2: The operator $\mathfrak{P}$ maps bounded sets into bounded sets in $PC_{1-\gamma,\psi}$.

Indeed, it is enough to show that for $r > 0$, there exists a positive constant l such that

$B_r = \left\{ \mathfrak{u} \in PC_{1-\gamma} : \|\mathfrak{u}\|_{PC_{1-\gamma,\psi}} \leq r \right\}$, we have $\|\mathfrak{P}(\mathfrak{u})\|_{PC_{1-\gamma,\psi}} \leq l$.

$$\begin{aligned}
&\left|(\mathfrak{P}\mathfrak{u})(\tau,\omega)\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}\right| \\
\leq &\frac{\mathfrak{u}_0(\omega)}{\Gamma(\gamma)} + \frac{\sum_{0<\tau_k<\tau}|\mathfrak{u}_{\tau_k}(\omega)|}{\Gamma(\gamma)} + \frac{\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)|\,ds \\
\leq &\frac{|\mathfrak{u}_0(\omega)|}{\Gamma(\gamma)} + \frac{m\,|n^*(\tau,\omega)|}{\Gamma(\gamma)} + \frac{\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}|m^*(s,\omega)\mathfrak{u}(s,\omega)|\,ds \\
\leq &\frac{|\mathfrak{u}_0(\omega)|}{\Gamma(\gamma)} + \frac{mN^*(\omega)}{\Gamma(\gamma)} + \frac{\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}M^*(\omega)\int_0^\tau \psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}|\mathfrak{u}(s,\omega)|\,ds \\
\leq &\frac{|\mathfrak{u}_0(\omega)|}{\Gamma(\gamma)} + \frac{mN^*(\omega)}{\Gamma(\gamma)} + \frac{\left(\psi(\tau)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha+1)}M^*(\omega)\left(\psi(\tau)-\psi(0)\right)^{\alpha}\|\mathfrak{u}\|_{PC_{1-\gamma,\psi}} \\
\leq &\frac{|\mathfrak{u}_0(\omega)|}{\Gamma(\gamma)} + \frac{mN^*(\omega)}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha+1)}M^*(\omega)\left(\psi(b)-\psi(0)\right)^{\alpha-1+\gamma}\|\mathfrak{u}\|_{PC_{1-\gamma,\psi}} \\
:= &l
\end{aligned}$$

That is $\mathfrak{P}$ is bounded.
Step 3: The operator $\mathfrak{P}$ maps bounded sets into equicontinuous set of $PC_{1-\gamma,\psi}$.
Let $\tau_l, \tau_m \in J,\ \tau_l > \tau_m, B_r$ be a bounded set of $PC_{1-\gamma,\psi}$ as in Step 2, and $\mathfrak{u} \in B_r$.
Then,

$$\begin{aligned}
&\left|\left(\psi(\tau_l)-\psi(0)\right)^{1-\gamma}(\mathfrak{P}\mathfrak{u})(\tau_l,\omega) - \left(\psi(\tau_m)-\psi(0)\right)^{1-\gamma}(\mathfrak{P}\mathfrak{u})(\tau_m,\omega))\right| \\
= &\left| \frac{\sum\limits_{0<\tau_k<\tau_l}\mathfrak{u}_{\tau_k}(\omega)}{\Gamma(\gamma)} + \frac{\left(\psi(\tau_l)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^{\tau_l}\psi^{'}(s)\left(\psi(\tau_l)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)ds \right. \\
&\left. - \frac{\sum\limits_{0<\tau_k<\tau_m}\mathfrak{u}_{\tau_k}(\omega)}{\Gamma(\gamma)} - \frac{\left(\psi(\tau_m)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_0^{\tau_m}\psi^{'}(s)\left(\psi(\tau_m)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)ds \right|.
\end{aligned}$$

As $t_l \to t_m$, the right hand side tends to zero. As a outcome of Step 1 - 3 together with Arzelä-Ascoli theorem, we can conclude that $\mathfrak{P} : PC_{1-\gamma,\psi} \to PC_{1-\gamma,\psi}$ is continuous and completely continuous. Now, it remains to show that the set

$$\Omega = \{\mathfrak{u} \in PC_{1-\gamma,\psi} : \mathfrak{u}(\tau,\omega) = \eta\mathfrak{P}\mathfrak{u}(\tau,\omega), 0 < \eta < 1\}$$

is bounded set. Let $\mathfrak{u} \in \omega,\ \mathfrak{u} = \eta\mathfrak{P}(t,\omega)$ for some $0 < \eta < 1$. Thus for each $\tau \in J$.
We have

$$\begin{aligned}
\mathfrak{u}(\tau,\omega) = &\eta\left[\frac{\mathfrak{u}_0(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} + \frac{\sum_{0<\tau_k<\tau}\mathfrak{u}_{\tau_k}(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1}\right. \\
&\left. + \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)ds\right].
\end{aligned}$$

This shows that the set Ω is bounded. As a consequence of Theorem 2.16, Eq. (3) has at least one solution. □

Theorem 5.2. *Assume that [H2] and [H7] are satisfied. If*

$$\left(\frac{m\ell^*(\tau,\omega)}{\Gamma(\gamma)}\left(\psi(b)-\psi(0)\right)^{1-\gamma}+\frac{\ell(\tau,\omega)}{\Gamma(\alpha)}\left(\psi(b)-\psi(0)\right)^{\alpha}B(\gamma,\alpha)\right)<1, \tag{11}$$

then, the Eq. (3) *has a unique solution.*

Definition 5.3. *Eq.* (3) *is generalized U-H-R stable with respect to* φ *if there exists a real number* $C_{f,\varphi}>0$ *such that for each solution* $\mathfrak{u}:\Omega\to PC_{1-\gamma,\psi}$ *of the inequality*

$$\begin{cases}\left|{}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{u}(\tau,\omega)-\mathfrak{g}(\tau,\mathfrak{y}(\tau,\omega),\omega)\right|\le\varphi(\tau,\omega),\\ \left|\Delta^{\mathbb{T}}\mathfrak{J}^{1-\gamma;\psi}\mathfrak{y}(\tau,\omega)|_{\tau=\tau_k}-\mathfrak{y}_{\tau_k}(\omega)\right|\le\varphi(\tau,\omega).\end{cases} \tag{12}$$

there exists a solution $\mathfrak{y}:\Omega\to PC_{1-\gamma,\psi}$ *of Eq.* (3) *with*

$$|\mathfrak{y}(\tau,\omega)-\mathfrak{u}(\tau,\omega)|\le C_{f,\varphi}\varphi(\tau,\omega),\quad \tau\in J,\omega\in\Omega.$$

Remark 5.4. *A function* $\mathfrak{y}\in PC_{1-\gamma,\psi}$ *is a solution of the inequality*

$$\left|{}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{y}(\tau,\omega)-\mathfrak{g}(\tau,\mathfrak{y}(\tau,\omega),\omega)\right|\le\epsilon,$$

if and only if there exist a function $g\in PC_{1-\gamma,\psi}$ *and a sequence* g_k*,* $k=1,2,...,m$ *such that*

(i) $|g(\tau)|\le\epsilon,\ |g_k|<\epsilon.$
(ii) ${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{y}(\tau,\omega)=\mathfrak{g}(\tau,\mathfrak{y}(\tau,\omega),\omega)+g(\tau).$
(iii) $\Delta^{\mathbb{T}}\mathfrak{J}^{1-\gamma;\psi}\mathfrak{y}(\tau,\omega)|_{\tau=\tau_k}=\mathfrak{y}_{\tau_k}(\omega)+g_k.$

Theorem 5.5. *Let the assumptions [H2], [H7], [H8] and* (11) *hold. Then, Eq.*(3) *is generalized U-H-R stable.*

Proof. Let $\mathfrak{y}$ be solution of inequality (12) and by Theorem 5.2 there $\mathfrak{u}$ is unique solution of the problem

$$\begin{aligned}{}^{\mathbb{T}}\Delta^{\mathfrak{u}}(\tau,\omega)&=\mathfrak{g}(\tau,\mathfrak{u}(\tau,\omega),\omega),\\ \Delta^{\mathbb{T}}\mathfrak{J}^{1-\gamma;\psi}\mathfrak{u}(\tau,\omega)|_{\tau=\tau_k}&=\mathfrak{u}_{\tau_k}(\omega),\\ {}^{\mathbb{T}}\mathfrak{J}^{1-\gamma;\psi}\mathfrak{u}(\tau,\omega)|_{\tau=0}&=\mathfrak{u}_0(\omega),\end{aligned}$$

Then, we have

$$\begin{aligned}\mathfrak{u}(\tau,\omega)=&\frac{\mathfrak{u}_0(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1}+\frac{\sum_{0<\tau_k<\tau}\mathfrak{u}_{\tau_k}(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1}\\ &+\frac{1}{\Gamma(\alpha)}\int_0^{\tau}\psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)ds.\end{aligned}$$

By differentiating inequality (12), for each $\tau \in (\tau_k, \tau_{k+1}]$, we have

$$\begin{aligned}
&\Bigg|\mathfrak{y}(\tau,\omega) - \frac{\mathfrak{u}_0(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} + \frac{\sum_{0<\tau_k<\tau}\mathfrak{y}_{\tau_k}(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} \\
&+\frac{1}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{y}(s,\omega),\omega)ds\Bigg| \\
&\leq \left|\frac{\sum\limits_{0<\tau_k<\tau} g_k}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\varphi(s,\omega)ds\right| \\
&\leq \frac{m}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1}\varphi(\tau,\omega) + \lambda_\varphi\varphi(\tau,\omega) \\
&\leq \left(\frac{m}{\Gamma(\gamma)}\left(\psi(b)-\psi(0)\right)^{\gamma-1} + \lambda_\varphi\right)\varphi(\tau,\omega).
\end{aligned}$$

Hence for each $\tau \in (\tau_k, \tau_{k+1}]$, it follows

$$\begin{aligned}
&|\mathfrak{y}(\tau,\omega) - \mathfrak{u}(\tau,\omega)| \\
&\leq \Bigg|\mathfrak{y}(\tau,\omega) - \frac{\mathfrak{u}_0(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} - \frac{\sum_{0<\tau_k<\tau}\mathfrak{u}_{t_k}(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} \\
&\quad -\frac{1}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)ds\Bigg| \\
&\leq \Bigg|\mathfrak{y}(\tau,\omega) - \frac{\mathfrak{u}_0(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} - \frac{\sum\limits_{0<\tau_k<\tau}\mathfrak{y}_k(\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} \\
&\quad -\frac{1}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{y}(s,\omega),\omega)ds\Bigg| + \frac{\sum\limits_{0<\tau_k<\tau}|\mathfrak{y}_k(\omega)-\mathfrak{u}_k(\omega)|}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{y}(s,\omega),\omega) - \mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\right|ds \\
&\leq \left(\frac{m}{\Gamma(\gamma)}\left(\psi(b)-\psi(0)\right)^{\gamma-1} + \lambda_\varphi\right)\varphi(\tau,\omega) + \frac{m\ell^*(\tau,\omega)}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1}|\mathfrak{y}(\tau,\omega)-\mathfrak{u}(\tau,\omega)| \\
&\quad + \frac{\ell(\tau,\omega)}{\Gamma(\alpha)}\int_0^\tau \psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}|\mathfrak{y}(s,\omega)-\mathfrak{u}(s,\omega)|ds
\end{aligned}$$

By Lemma 2.12, there exists a constant $K > 0$ independent of $\lambda_\varphi\varphi(\tau,\omega)$ such that

$$|\mathfrak{y}(\tau,\omega) - \mathfrak{u}(\tau,\omega)| \leq K\left(\frac{m}{\Gamma(\gamma)}\left(\psi(b)-\psi(0)\right)^{\gamma-1} + \lambda_\varphi\right)\varphi(\tau,\omega) := C_{f,\varphi}\varphi(\tau,\omega).$$

Thus, Eq.(3) is generalized U-H-R stable. □

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