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INTEGRAL REPRESENTATIONS INVOLVING NEW HYPERGEOMETRIC FUNCTIONS OF FOUR VARIABLES

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Abstract. In this paper, we define some new quadruple hypergeometric functions, which we denoted by $X_i^{(4)}(i = 31, 32, ..., 50)$. Then, we obtain its integral representations of Euler-type and Laplace-type.

1. Introduction

Gauss hypergeometric function has many applications in number theory, combinatorics, partitions, mathematical physics, etc. This gives motivation to the research authors to study hypergeometric functions of two or more variables (for example [1]-[5], [7], [8], [10], [11], [15], [16]). The studies in representation theory, geometry, algebraic geometry, combinatorics, number theory, mirror symmetry, etc. have led to increasing interest in the study of hypergeometric functions of several variables. Also, hypergeometric functions have several applications in physical and chemical problems ([12], [13], [17]) for example, they are seen in the solutions of degenerate second-order partial diferential equations in many problems in gas dynamics, in the problem of adiabatic flat-parallel gas flow without whirlwind and in many other problems [9].

Very recently, Bin-Saad et al. ([2], [3]) investigated that there exist ten additional complete quadruple hypergeometric functions, which are $X_1^{(4)}, X_2^{(4)}, ..., X_{10}^{(4)}$ and they had not been included in Exton's [7] and Sharma's and Parihar's [16] set, and then obtained their Euler-type and Laplace-type integral representations. In [1] Bin-Saad and Younis established new integral representations of Euler-type for some hypergeometric functions of four variables, whose kernels include the quadruple functions $X_6^{(4)}, X_7^{(4)}, ..., X_{10}^{(4)}$.

Motivated by the works [7], [16] and ([2], [3]), in the present paper, we introduce twenty new hypergeometric functions of four variables as below:

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$$X_{31}^{(4)}(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_3,c_4;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_{n+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1}$$

$$X_{32}^{(4)}(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_{n+q}}{(c_1)_{m+p}(c_2)_n(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{2}$$

$$X_{33}^{(4)}(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_3,c_1;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_{n+q}}{(c_1)_{m+q}(c_2)_n(c_3)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{3}$$

$$X_{34}^{(4)}(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_1,c_2;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_{n+q}}{(c_1)_{m+p}(c_2)_{n+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{4}$$

$$X_{35}^{(4)}(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_2,c_1;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_{n+q}}{(c_1)_{m+q}(c_2)_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{5}$$

$$X_{36}^{(4)}(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_1,c_1,c_2;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_{n+q}}{(c_1)_{m+n+p}(c_2)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{6}$$

$$X_{37}^{(4)}(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_1,c_2,c_1;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_{n+q}}{(c_1)_{m+n+q}(c_2)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{7}$$

$$X_{38}^{(4)}(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c,c,c,c;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_{n+q}}{(c)_{m+n+p+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{8}$$

$$X_{39}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_3,c_4;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2n+p}(a_2)_{p+q}(a_3)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{9}$$

$$X_{40}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2n+p}(a_2)_{p+q}(a_3)_q}{(c_1)_{m+p}(c_2)_n(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{10}$$

$$X_{41}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_3,c_1;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2n+p}(a_2)_{p+q}(a_3)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{11}$$

$$X_{42}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_1,c_2;x,y,z,u\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2n+p}(a_2)_{p+q}(a_3)_q}{(c_1)_{m+p}(c_2)_{n+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{12}$$

$$X_{43}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_1,c_1;x,y,z,u\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2n+p}(a_2)_{p+q}(a_3)_q}{(c_1)_{m+p+q}(c_2)_{n}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{13}$$

$$X_{44}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_3,c_1;x,y,z,u\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2n+p}(a_2)_{p}(a_3)_q(a_4)_q}{(c_1)_{m+q}(c_2)_{n}(c_3)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{14}$$

$$X_{45}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_3,c_3;x,y,z,u\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2n+p}(a_2)_{p}(a_3)_q(a_4)_q}{(c_1)_{m}(c_2)_{n}(c_3)_{p+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{15}$$

$$X_{46}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_2,c_1;x,y,z,u\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2n+p}(a_2)_{p}(a_3)_q(a_4)_q}{(c_1)_{m+q}(c_2)_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{16}$$

$$X_{47}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_1,c_1;x,y,z,u\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2n+p}(a_2)_{p}(a_3)_q(a_4)_q}{(c_1)_{m+p+q}(c_2)_{n}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{17}$$

$$X_{48}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_2,a_2;c_1,c_2,c_3,c_4;x,y,z,u\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p+q}(a_2)_{n+p+q}}{(c_1)_{m}(c_2)_{n}(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{18}$$

$$X_{49}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_2,a_2;c_1,c_1,c_2,c_3;x,y,z,u\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p+q}(a_2)_{n+p+q}}{(c_1)_{m+n}(c_2)_{p}(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{19}$$

$$X_{50}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_1,a_1,a_2;c_1,c_2,c_3,c_1;x,y,z,u\right) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2n+2p+q}(a_2)_{q}}{(c_1)_{m+q}(c_2)_{n}(c_3)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}. \tag{20}$$

Here, $(a)_m$ denotes the Pochhammer symbol given by

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)...(a+m-1)\ (m \in \mathbb{N} := \{1,2,...\})\ \textit{and}\ (a)_0 = 1.$$

We have organized the rest of this paper in the following way: Section 2 introduces some integral representations of Euler-type which include the Gaussian hypergeometric function ${}_2F_1$, Appell's functions of two variables F_3 and F_4, the Horn's function of two variables H_3, the Exton's triple functions $X_1, X_2, X_9, X_{17}, X_{18}, X_{19}$ and X_{20}, the Lauricell's triple functions $F_C^{(3)}$ and F_R and the quadruple functions

$F_C^{(4)}, F_{14}^{(4)}, X_8^{(4)}$ and $X_9^{(4)}$ for the new hypergeometric functions of four variables $X_i^{(4)} (i = 31, 32, ..., 50)$. Section 3 deals with the derivation of Laplace integral representations of these quadruple functions.

2. Integral representations of Euler-Type

We first recall the Gaussian hypergeometric function defined by (see [18], [19])

$$ {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \ (|x| < 1). $$

Appell's functions F_3 and F_4 of two variables and the Horn's function H_3 of two variables are given by

$$ F_3(a, b, c, d; e; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_n (c)_m (d)_n}{(e)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \ (\max\{|x|, |y|\} < 1), \quad (21) $$

$$ F_4(a, b; c, d; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(c)_m (d)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \ (\sqrt{|x|} + \sqrt{|y|} < 1) $$

and

$$ H_3(a, b; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n} (b)_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, $$

$$ \left(|x| < r, |y| < s, r + \left(s - \frac{1}{2} \right)^2 = \frac{1}{4} \right), $$

respectively (see [18]). The Exton's triple functions $X_1, X_2, X_9, X_{17}, X_{18}, X_{19}$ and X_{20} (see [8]) are defined by as follows:

$$ X_1(a_1, a_2; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+2n+p} (a_2)_p}{(c_1)_m (c_2)_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, $$

$$ \left(\sqrt{r} + \sqrt{s} < \frac{1}{2} \wedge t < \frac{1}{2} \left(1 - 2\sqrt{r}\ \right) + \frac{1}{2} \sqrt{(1 - 2\sqrt{r}\ )^2 - 4s}, |x| \le r, |y| \le s, |z| \le t \right), $$

$$ X_2(a_1, a_2; c_1, c_2, c_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+2n+p} (a_2)_p}{(c_1)_m (c_2)_n (c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, $$

$$ \left(2\sqrt{r} + 2\sqrt{s} + t < 1, |x| \le r, |y| \le s, |z| \le t \right), $$

$$ X_9(a_1, a_2; c; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n} \ (a_2)_{n+2p}}{(c)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, $$

$$ \left(r < \frac{1}{4} \wedge t < \frac{1}{4} \wedge s < \frac{1}{2} + \frac{1}{2} \sqrt{(1 - 4r)(1 - 4t)}, |x| \le r, |y| \le s, |z| \le t \right), $$

$$ X_{17}(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n} (a_2)_{n+p} (a_3)_p}{(c_1)_m (c_2)_n (c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, $$

$$ \left(r < \frac{1}{4} \wedge t < 1 \wedge s < (1 - 2\sqrt{r})(1 - t), |x| \le r, |y| \le s, |z| \le t \right), $$

$$X_{18}\left(a_1,a_2,a_3,a_4;c;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(a_4)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$\left(r<\frac{1}{4}\wedge t<1\wedge s<\frac{1}{2}+\frac{1}{2}\sqrt{(1-4r)},|x|\leq r,|y|\leq s,|z|\leq t\right),$$

$$X_{19}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(a_4)_p}{(c_1)_m(c_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$\left(s+2\sqrt{r}<1\wedge t<,|x|\leq r,|y|\leq s,|z|\leq t\right)$$

and

$$X_{20}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(a_4)_p}{(c_1)_{m+p}(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$\left(s+2\sqrt{r}<1\wedge t<1,|x|\leq r,|y|\leq s,|z|\leq t\right).$$

Lauricella hypergeometric functions of three variables $F_C^{(3)}$ and F_R are as below (see [11], [15])

$$F_C^{(3)}\left(a,b;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}(b)_{m+n+p}}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p},$$

$$(\sqrt{|x|}+\sqrt{|y|}+\sqrt{|z|}<1)$$

and

$$F_R\left(a_1,a_2,a_1,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+p}(a_2)_n(b_1)_{m+p}(b_2)_n}{(c_1)_m(c_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$\left(\sqrt{r}+\sqrt{s}<1\wedge t<1,|x|\leq r,|y|\leq s,|z|\leq t\right).$$

The Lauricella's quadruple function $F_C^{(4)}$ (see [11]) is in the following form:

$$F_C^{(4)}\left(a,b;c_1,c_2,c_3,c_4;x,y,z,u\right)=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+p+q}(b)_{m+n+p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

$$(\sqrt{|x|}+\sqrt{|y|}+\sqrt{|z|}+\sqrt{|u|}<1).$$

Sharma and Parihar hypergeometric function of four variables $F_{14}^{(4)}$ is as follows (see [16]):

$$F_{14}^{(4)}\left(a_1,a_1,a_1,a_2,b_1,b_1,b_1,b_2;c_1,c_2,c_3,c_1;x,y,z\right)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n+p}(b_2)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

$$(\sqrt{|x|}+\sqrt{|y|}+\sqrt{|z|}<1\wedge\sqrt{|u|}<1).$$

The quadruple hypergeometric functions $X_8^{(4)}$ and $X_9^{(4)}$ are given by

$$X_8^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2,c_3;x,y,z,u\right)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+2q+n+p}(a_2)_n(a_3)_p}{(c_1)_{m+n}(c_2)_p(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}$$

and

$$X_9^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_2,c_1,c_1,c_3;x,y,z,u\right)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+2q+n+p}(a_2)_n(a_3)_p}{(c_1)_{n+p}(c_2)_m(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

respectively (see [2]).

Now, by means of the Gaussian hypergeometric function ${}_2F_1$, Appell's functions F_3 and F_4, Horn's function H_3, the Exton's triple fuctions $X_1, X_2, X_9, X_{17}, X_{18}, X_{19}$ and X_{20}, the Lauricell's triple functions $F_C^{(3)}$ and F_R and the quadruple functions $F_C^{(4)}, F_{14}^{(4)}, X_8^{(4)}$ and $X_9^{(4)}$, we investigate some integral representations of Euler-type for $X_{31}^{(4)}, X_{32}^{(4)}, ..., X_{50}^{(4)}$ as follows:

$$X_{31}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_3,c_4;x,y,z,u\right)=\frac{\Gamma(c_3)}{\Gamma(a_2)\Gamma(c_3-a_2)}\int_0^1\alpha^{a_2-1}$$
$$\times[(1-\alpha)+\alpha^2 z]^{c_3-a_2-1}X_{17}\left(a_1,a_3,1+a_2-c_3;c_1,c_2,c_4;x,y,\frac{-\alpha u}{[(1-\alpha)+\alpha^2 z]}\right)d\alpha$$
$$\left(\Re(a_2)>0,\Re(c_3-a_2)>0\right), \tag{22}$$

$$X_{32}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_1,c_3;x,y,z,u\right)=\frac{\Gamma(c_2)}{\Gamma(a_3)\Gamma(c_2-a_3)}\int_0^1\alpha^{a_3-1}$$
$$\times(1-\alpha)^{c_2-a_3-1}(1-\alpha y)^{-a_1}X_{20}\left(a_2,1+a_3-c_2,\frac{a_1}{2},\frac{a_1+1}{2};c_1,c_2;z,\frac{-\alpha u}{(1-\alpha)},\right.$$
$$\left.\frac{4x}{(1-\alpha y)}\right)d\alpha$$
$$\left(\Re(a_3)>0,\Re(c_2-a_3)>0\right), \tag{23}$$

$$X_{33}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_3,c_1;x,y,z,u\right)=\frac{\Gamma(c_2)}{\Gamma(a_1)\Gamma(c_2-a_1)}\int_0^1\alpha^{a_1-1}$$
$$\times(1-\alpha)^{c_2-a_1-1}(1-\alpha y)^{-a_3}X_{19}\left(a_2,a_3,\frac{1+a_1-c_2}{2},\frac{a_1-c_2}{2}+1;c_3,c_1;z,\frac{u}{(1-\alpha y)},\right.$$
$$\left.\frac{4\alpha^2 x}{(1-\alpha)^2}\right)d\alpha$$
$$\left(\Re(a_1)>0,\Re(c_2-a_1)>0\right), \tag{24}$$

$$X_{34}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_1,c_2;x,y,z,u\right)=\frac{\Gamma(c_2)}{\Gamma(a_3)\Gamma(c_2-a_3)}\int_0^1\alpha^{a_3-1}$$
$$\times(1-\alpha)^{c_2-a_3-1}(1-\alpha y)^{-a_1}(1-\alpha u)^{-a_2}F_3\left(\frac{a_1}{2},\frac{a_2}{2},\frac{a_1+1}{2},\frac{a_2+1}{2};c_1;\frac{4x}{(1-\alpha y)^2},\right.$$
$$\left.\frac{4z}{(1-\alpha u)^2}\right)d\alpha$$
$$\left(\Re(a_3)>0,\Re(c_2-a_3)>0\right), \tag{25}$$

$$X_{35}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_2,c_1;x,y,z,u\right)=\frac{\Gamma(a_1+a_2+a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\int_0^1\int_0^1$$
$$\times\alpha^{a_1-1}(1-\alpha)^{a_2-1}\beta^{a_1+a_2-1}(1-\beta)^{a_3-1}F_4\left(\frac{a_1+a_2+a_3}{2},\frac{a_1+a_2+a_3+1}{2};c_1,\right.$$
$$\left.c_2;4\alpha^2\beta^2x+4(1-\alpha)\beta(1-\beta)u,4\alpha\beta(1-\beta)y+4\beta^2(1-\alpha)^2z\right)d\alpha d\beta$$
$$\left(\Re(a_i)>0,(i=1,2,3)\right), \tag{26}$$

$$X_{36}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_1,c_1,c_2;x,y,z,u\right)=\frac{\Gamma(c_2)}{\Gamma(a_3)\Gamma(c_2-a_3)}\int_0^1\alpha^{a_3-1}$$
$$\times(1-\alpha)^{c_2-a_3-1}(1-\alpha u)^{-a_2}X_{18}\left(a_1,1+a_3-c_2,\frac{a_2}{2},\frac{a_2+1}{2};c_1;x,\frac{-\alpha y}{(1-\alpha)},\right.$$
$$\left.\frac{4z}{(1-\alpha u)^2}\right)d\alpha$$
$$\left(\Re(a_3)>0,\Re(c_2-a_3)>0\right), \tag{27}$$

$$X_{37}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_1,c_2,c_1;x,y,z,u\right)=\frac{\Gamma(c_1)}{\Gamma(a_3)\Gamma(c_1-a_3)}\int_0^1\alpha^{a_3-1}$$
$$\times(1-\alpha)^{c_1-a_3-1}(1-\alpha y)^{-a_1}(1-\alpha u)^{-a_2}{}_2F_1\left(\frac{a_1}{2},\frac{a_1+1}{2};c_1-a_3;\frac{4(1-\alpha)x}{(1-\alpha y)^2}\right)$$
$$\times{}_2F_1\left(\frac{a_2}{2},\frac{a_2+1}{2};c_2;\frac{4z}{(1-\alpha u)^2}\right)d\alpha$$
$$\left(\Re(a_3)>0,\Re(c_1-a_3)>0\right), \tag{28}$$

$$X_{38}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c,c,c,c;x,y,z,u\right)=\frac{\Gamma(a_2+a_3)}{\Gamma(a_2)\Gamma(a_3)}\int_0^1\alpha^{a_2-1}$$
$$\times(1-\alpha)^{a_3-1}X_9\left(a_1,a_2+a_3;c;x,(1-\alpha)y,\alpha^2z+\alpha(1-\alpha)u\right)d\alpha$$
$$\left(\Re(a_2)>0,\Re(a_3)>0\right), \tag{29}$$

$$X_{39}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_3,c_4;x,y,z,u\right)=\frac{\Gamma(c_4)}{\Gamma(a_3)\Gamma(c_4-a_3)}\int_0^1\alpha^{a_3-1}$$
$$\times(1-\alpha)^{c_4-a_3-1}(1-\alpha u)^{-a_2}X_2\left(a_1,a_2;c_1,c_2,c_3;x,y,\frac{z}{(1-\alpha u)}\right)d\alpha$$
$$\left(\Re(a_3)>0,\Re(c_4-a_3)>0\right), \tag{30}$$

$$X_{40}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_1,c_3;x,y,z,u\right)=\frac{\Gamma(c_3)}{\Gamma(a_3)\Gamma(c_3-a_3)}\int_0^1$$
$$\times\alpha^{c_3-a_3-1}(1-\alpha)^{a_3-1}[1-(1-\alpha)u]^{-a_2}X_1\left(a_1,a_2;c_2,c_1;y,x,\frac{z}{[1-(1-\alpha)u]}\right)d\alpha$$
$$\left(\Re(a_3)>0,\Re(c_3-a_3)>0\right), \tag{31}$$

$$X_{41}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_3,c_1;x,y,z,u\right)=\frac{\Gamma(c_3)}{\Gamma(a_2)\Gamma(c_3-a_2)}\int_0^1\alpha^{a_2-1}$$
$$\times(1-\alpha)^{c_3-a_2-1}(1-\alpha z)^{-a_1}F_R\left(\frac{a_1}{2},1+a_2-c_3,\frac{a_1}{2},\frac{a_1+1}{2},a_3,\frac{a_1+1}{2};c_2,c_1,c_1;\right.$$
$$\left.\frac{4y}{(1-\alpha z)^2},\frac{-\alpha u}{(1-\alpha)},\frac{4x}{(1-\alpha z)^2}\right)d\alpha$$
$$\left(\Re(a_2)>0,\Re(c_3-a_2)>0\right), \tag{32}$$

$$X_{42}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_1,c_2;x,y,z,u\right)=\frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)}\int_0^1\alpha^{a_1-1}$$
$$\times[(1-\alpha)+\alpha^2x]^{c_1-a_1-1}(1-\alpha z)^{-a_2}F_3\left(\frac{1+a_1-c_1}{2},a_2,\frac{a_1-c_1}{2}+1,a_3;c_2;\right.$$
$$\left.\frac{4\alpha^2y}{[(1-\alpha)+\alpha^2x]^2},\frac{u}{(1-\alpha z)}\right)d\alpha$$
$$\left(\Re(a_1)>0,\Re(c_1-a_1)>0\right), \tag{33}$$

$$X_{43}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_1,c_1;x,y,z,u\right)=\frac{\Gamma(c_1)}{\Gamma(a_2)\Gamma(c_1-a_2)}\int_0^1\alpha^{a_2-1}$$
$$\times(1-\alpha)^{c_1-a_2-1}(1-\alpha z)^{-a_1}(1-\alpha u)^{-a_3}F_4\left(\frac{a_1}{2},\frac{a_1+1}{2};c_1-a_2,c_2;\frac{4(1-\alpha)x}{(1-\alpha z)^2},\right.$$
$$\left.\frac{4y}{(1-\alpha z)^2}\right)d\alpha$$
$$\left(\Re(a_2)>0,\Re(c_1-a_2)>0\right), \tag{34}$$

$$X_{44}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_3,c_1;x,y,z,u\right)=\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\int_0^1\alpha^{a_1-1}$$
$$\times(1-\alpha)^{a_2-1}F_{14}^{(4)}\left(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},a_3,\frac{a_1+a_2+1}{2},\frac{a_1+a_2+1}{2},\right.$$
$$\left.\frac{a_1+a_2+1}{2},a_4;c_1,c_2,c_3,c_1;4\alpha^2x,4\alpha^2y,4\alpha(1-\alpha)z,u\right)d\alpha$$
$$\left(\Re(a_1)>0,\Re(a_2)>0\right), \tag{35}$$

$$X_{45}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_3,c_3;x,y,z,u\right)=\frac{\Gamma(a_1+a_3)}{\Gamma(a_1)\Gamma(a_3)}\int_0^1\alpha^{a_1-1}$$
$$\times(1-\alpha)^{a_3-1}X_9^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_4,a_1;c_1,c_3,c_3,c_2;\alpha^2x,\alpha z,(1-\alpha)u,\right.$$
$$\left.\alpha^2y\right)d\alpha$$
$$\left(\Re(a_1)>0,\Re(a_3)>0\right), \tag{36}$$

$$X_{46}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_2,c_1;x,y,z,u\right)=\frac{\Gamma(a_1+a_4)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_4)\Gamma(a)\Gamma(c_2-a)}$$

$$\times\int_0^1\int_0^1\alpha^{a_1-1}(1-\alpha)^{a_4-1}\beta^{a-1}(1-\beta)^{c_2-a-1}X_8^{(4)}\left(a_1+a_4,a_1+a_4,a_1+a_4,a_1+a_4,\right.$$

$$\left.a_1+a_4,a_3,a_2,a_1+a_4;c_1,c_1,c_2-a,a;\alpha^2x,(1-\alpha)u,\alpha(1-\beta)z,\alpha^2\beta y\right)d\alpha d\beta$$

$$\left(\Re(a_1)>0,\Re(a_4)>0,\Re(a)>0,\Re(c_2-a)>0\right), \tag{37}$$

$$X_{47}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_1,c_1;x,y,z,u\right)=\frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(a_3)\Gamma(c_1-a_3)}$$

$$\times\frac{\Gamma(c_2)}{\Gamma(c_2-a_1)}\int_0^1\int_0^1\alpha^{a_1-1}\beta^{a_3-1}(1-\beta)^{c_1-a_3-1}[(1-\alpha)+\alpha^2y]^{c_2-a_1-1}(1-\beta u)^{-a_4}$$

$$H_3\left(1+a_1-c_2,a_2;c_1-a_3;\frac{\alpha^2(1-\beta)x}{[(1-\alpha)+\alpha^2y]^2},\frac{-\alpha(1-\beta)z}{[(1-\alpha)+\alpha^2y]}\right)d\alpha d\beta$$

$$\left(\Re(a_1)>0,\Re(a_3)>0,\Re(c_1-a_3)>0,\Re(c_2-a_1)>0\right), \tag{38}$$

$$X_{48}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_2,a_2;c_1,c_2,c_3,c_4;x,y,z,u\right)=\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\int_0^1\alpha^{a_1-1}$$

$$\times(1-\alpha)^{a_2-1}F_C^{(4)}\left(\frac{a_1+a_2}{2},\frac{a_1+a_2+1}{2};c_1,c_2,c_3,c_4;4\alpha^2x,4\alpha(1-\alpha)y,\right.$$

$$\left.4\alpha(1-\alpha)z,4\alpha(1-\alpha)u\right)d\alpha$$

$$\left(\Re(a_1)>0,\Re(a_2)>0\right), \tag{39}$$

$$X_{49}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_2,a_2;c_1,c_1,c_2,c_3;x,y,z,u\right)=\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\int_0^1\alpha^{a_1-1}$$

$$\times(1-\alpha)^{a_2-1}F_C^{(3)}\left(\frac{a_1+a_2}{2},\frac{a_1+a_2+1}{2};c_1,c_2,c_3;4\alpha^2x+4\alpha(1-\alpha)y,\right.$$

$$\left.4\alpha(1-\alpha)z,4\alpha(1-\alpha)u\right)d\alpha$$

$$\left(\Re(a_1)>0,\Re(a_2)>0\right), \tag{40}$$

$$X_{50}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_1,a_1,a_2;c_1,c_2,c_3,c_1;x,y,z,u\right)=\frac{\Gamma(c_1)}{\Gamma(a_1))\Gamma(c_1-a_1)}\int_0^1$$

$$\times\alpha^{a_1-1}[(1-\alpha)+\alpha^2x]^{c_1-a_1-1}(1-\alpha u)^{-a_2}F_4\left(\frac{1+a_1-c_1}{2},\frac{a_1-c_1}{2}+1;\right.$$

$$\left.c_2,c_3;\frac{4\alpha^2y}{[(1-\alpha)+\alpha^2x]^2},\frac{4\alpha^2z}{[(1-\alpha)+\alpha^2x]^2}\right)d\alpha$$

$$\left(\Re(a_1)>0,\Re(c_1-a_1)>0\right). \tag{41}$$

Proof. First of all, we recall the Beta function (see [6], [14])

$$B(a,b)=\begin{cases}\displaystyle\int_0^1 t^{a-1}(1-t)^{b-1}\,dt & (\Re(a)>0,\Re(b)>0)\,,\\ \dfrac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} & (a,b\in\mathbb{C}\setminus\mathbb{Z}_0^-).\end{cases}\tag{42}$$

For convenience, denote by Θ the right side of the equation (22). Then, by substituting the expression of the X_{17} from the definition (21) into the right hand side of (22) and using the equality (42), it follows that

$$\Theta=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_3)_{n+q}(1+a_1-c_3)_q(1+a_1-c_3+q)_p(-1)^{p+q}}{(c_1)_m(c_2)_n(c_4)_q}$$

$$\times\frac{\Gamma(c_3)}{\Gamma(a_2)\Gamma(c_3-a_2)}\int_0^1\alpha^{a_2+2p+q-1}(1-\alpha)^{c_3-a_2-q-p-1}d\alpha\times\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}$$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(1+a_1-c_3)_q(1+a_1-c_3+q)_p\Gamma(c_3-a_2-p-q)(-1)^{p+q}}{\Gamma(c_3-a_2)}$$

$$\times\frac{(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_{n+q}}{(c_1)_m(c_2)_n(c_3)_q(c_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}$$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_{n+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}$$

$$=X_{31}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_3,c_4;x,y,z,u\right)$$

which proves the equation(22). From the relation (42), one can easily obtain the other integral representations.

3. Integral representations of Laplace-Type

In this section, we establish the following integral representations of Laplace-type for the quadruple hypergeometric functions $X_i^{(4)}(i=31,32,...,50)$:

$$X_{31}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_3,c_4;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^{\infty}\int_0^{\infty}$$
$$\times e^{-(s+t)}s^{a_1-1}\;t^{a_2-1}{}_0F_1\left(-;c_1;s^2x\right){}_0F_1\left(-;c_3;t^2z\right)\Psi_2\left(a_3;c_2,c_4;sy,tu\right)dsdt$$
$$\left(\Re(a_1)>0,\Re(a_2)>0\right),\tag{43}$$

$$X_{32}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_1,c_3;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^{\infty}\int_0^{\infty}$$
$$\times e^{-(s+t)}s^{a_1-1}\;t^{a_2-1}{}_0F_1\left(-;c_1;s^2x+t^2z\right)\Psi_2\left(a_3;c_2,c_3;sy,tu\right)dsdt$$
$$\left(\Re(a_1)>0,\Re(a_2)>0\right),\tag{44}$$

$$X_{33}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_3,c_1;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}t^{a_2-1}v^{a_3-1}{}_0F_1\left(-;c_1;s^2x+tvu\right){}_0F_1\left(-;c_2;svy\right)$$
$$\times{}_0F_1\left(-;c_3;t^2z\right)dsdtdv$$
$$\left(\Re(a_i)>0,(i=1,2,3)\right), \tag{45}$$

$$X_{34}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_1,c_2;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty$$
$$\times e^{-(s+t)}s^{a_1-1}\ t^{a_2-1}{}_0F_1\left(-;c_1;s^2x+t^2z\right){}_1F_1\left(a_3;c_2;sy+tu\right)dsdt$$
$$\left(\Re(a_1)>0,\Re(a_2)>0\right), \tag{46}$$

$$X_{35}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_2,c_2,c_1;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}t^{a_2-1}v^{a_3-1}{}_0F_1\left(-;c_1;s^2x+tvu\right)$$
$$\times{}_0F_1\left(-;c_2;svy+t^2z\right)dsdtdv$$
$$\left(\Re(a_i)>0,(i=1,2,3)\right), \tag{47}$$

$$X_{36}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_1,c_1,c_2;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}t^{a_2-1}v^{a_3-1}{}_0F_1\left(-;c_1;s^2x+svy+t^2z\right)$$
$$\times{}_0F_1\left(-;c_2;tvu\right)dsdtdv$$
$$\left(\Re(a_i)>0,(i=1,2,3)\right), \tag{48}$$

$$X_{37}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c_1,c_1,c_2,c_1;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}t^{a_2-1}v^{a_3-1}{}_0F_1\left(-;c_1;s^2x+svy+tvu\right)$$
$$\times{}_0F_1\left(-;c_2;t^2z\right)dsdtdv$$
$$\left(\Re(a_i)>0,(i=1,2,3)\right), \tag{49}$$

$$X_{38}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c,c,c,c;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}t^{a_2-1}v^{a_3-1}{}_0F_1\left(-;c;s^2x+svy+t^2z+tvu\right)$$
$$\times dsdtdv$$
$$\left(\Re(a_i)>0,(i=1,2,3)\right), \tag{50}$$

$$X_{39}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_3,c_4;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_3)}\int_0^\infty\int_0^\infty$$
$$\times e^{-(s+t)}s^{a_1-1}\;t^{a_3-1}{}_0F_1\left(-;c_1;s^2x\right){}_0F_1\left(-;c_2;s^2y\right)\Psi_2\left(a_2;c_3,c_4;sz,tu\right)dsdt$$
$$\left(\Re(a_1)>0,\Re(a_3)>0\right),\tag{51}$$

$$X_{40}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_1,c_3;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty$$
$$\times e^{-(s+t)}s^{a_1-1}\;t^{a_2-1}{}_0F_1\left(-;c_1;s^2x+stz\right){}_0F_1\left(-;c_2;s^2y\right){}_1F_1\left(a_3;c_3;tu\right)dsdt$$
$$\left(\Re(a_1)>0,\Re(a_2)>0\right),\tag{52}$$

$$X_{41}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_3,c_1;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}t^{a_2-1}v^{a_3-1}{}_0F_1\left(-;c_1;s^2x+tvu\right){}_0F_1\left(-;c_2;t^2y\right)$$
$$\times{}_0F_1\left(-;c_3;stz\right)dsdtdv$$
$$\left(\Re(a_i)>0,(i=1,2,3)\right),\tag{53}$$

$$X_{42}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_1,c_2;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty$$
$$\times e^{-(s+t)}s^{a_1-1}\;t^{a_2-1}{}_0F_1\left(-;c_1;s^2x+stz\right)\Phi_3\left(a_3;c_2;tu,s^2y\right)dsdt$$
$$\left(\Re(a_1)>0,\Re(a_2)>0\right),\tag{54}$$

$$X_{43}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_1,c_1;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}t^{a_2-1}v^{a_3-1}{}_0F_1\left(-;c_1;s^2x+stz+tvu\right)$$
$$\times{}_0F_1\left(-;c_2;s^2y\right)dsdtdv$$
$$\left(\Re(a_i)>0,(i=1,2,3)\right),\tag{55}$$

$$X_{44}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_3,c_1;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_3)\Gamma(a_4)}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}t^{a_3-1}v^{a_4-1}{}_0F_1\left(-;c_1;s^2x+tvu\right){}_0F_1\left(-;c_2;s^2y\right)$$
$$\times{}_1F_1\left(a_2;c_3;sz\right)dsdtdv$$
$$\left(\Re(a_i)>0,(i=1,3,4)\right),\tag{56}$$

$$X_{45}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_3,c_3;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_4)}\int_0^\infty\int_0^\infty$$
$$\times e^{-(s+t)}s^{a_1-1}\;t^{a_4-1}{}_0F_1\left(-;c_1;s^2x\right){}_0F_1\left(-;c_2;s^2y\right)\Phi_2\left(a_2,a_3;c_3;sz,tu\right)$$
$$\times dsdt$$
$$\left(\Re(a_1)>0,\Re(a_4)>0\right),\tag{57}$$

$$X_{46}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_2,c_1;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_3)\Gamma(a_4)}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}t^{a_3-1}v^{a_4-1}{}_0F_1\left(-;c_1;s^2x+tvu\right)\Phi_3\left(a_2;c_2;sz,s^2y\right)$$
$$\times dsdtdv$$
$$\left(\Re(a_i)>0,(i=1,3,4)\right),\tag{58}$$

$$X_{47}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_1,c_1;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v+w)}s^{a_1-1}t^{a_2-1}v^{a_3-1}w^{a_4-1}{}_0F_1\left(-;c_1;s^2x+stz+vwu\right)$$
$$\times{}_0F_1\left(-;c_2;s^2y\right)dsdtdvdw$$
$$\left(\Re(a_i)>0,(i=1,2,3,4)\right),\tag{59}$$

$$X_{48}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_2,a_2;c_1,c_2,c_3,c_4;x,y,z,u\right)=\frac{1}{\Gamma(a_1)}\int_0^\infty e^{-s}s^{a_1-1}$$
$$\times{}_0F_1\left(-;c_1;s^2x\right)\Psi_2^{(3)}\left(a_2;c_2,c_3,c_4;sy,sz,su\right)ds$$
$$\left(\Re(a_1)>0\right),\tag{60}$$

$$X_{49}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_2,a_2;c_1,c_1,c_2,c_3;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty$$
$$\times e^{-(s+t)}s^{a_1-1}\ t^{a_2-1}{}_0F_1\left(-;c_1;s^2x+sty\right){}_0F_1\left(-;c_2;stz\right){}_0F_1\left(-;c_3;stu\right)dsdt$$
$$\left(\Re(a_1)>0,\Re(a_2)>0\right),\tag{61}$$

$$X_{50}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_1,a_1,a_2;c_1,c_2,c_3,c_1;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty$$
$$\times e^{-(s+t)}s^{a_1-1}\ t^{a_2-1}{}_0F_1\left(-;c_1;s^2x+stu\right){}_0F_1\left(-;c_2;s^2y\right){}_0F_1\left(-;c_3;s^2z\right)dsdt$$
$$\left(\Re(a_1)>0,\Re(a_2)>0\right),\tag{62}$$

where ${}_0F_1,{}_1F_1,\Phi_2,\Phi_3,\Psi_2$ and $\Psi_2^{(3)}$ are the confluent hypergeometric functions defined, respectively, by

$${}_0F_1\left(-;c;x\right)=\sum_{m=0}^{\infty}\frac{1}{(c)_m}\frac{x^m}{m!},\ |x|<\infty,\tag{63}$$

$${}_1F_1\left(a;c;x\right)=\sum_{m=0}^{\infty}\frac{(a)_m}{(c)_m}\frac{x^m}{m!},\ |x|<\infty,\tag{64}$$

$$\Phi_2\left(a,b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<\infty,\ |y|<\infty,\tag{65}$$

$$\Phi_3\left(a;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<\infty,\ |y|<\infty,\tag{66}$$

$$\Psi_2\left(a;b,c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}}{(b)_m(c)_n}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<\infty,\ |y|<\infty \tag{67}$$

and

$$\Psi_2^{(3)}\left(a;b,c,d;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}}{(b)_m(c)_n(d)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$
$$|x|<\infty,\ |y|<\infty,\ |z|<\infty. \tag{68}$$

Proof. To prove each of the integral representations from (43) to (62), it is enough to consider the expressions of confluent hypergeometric functions given above and then to use Gamma integral formula

$$\Gamma(a)=\int_0^{\infty}e^{-t}\,t^{a-1}\,dt,\ (\Re(a)>0).$$

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