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SOME COMPARATIVE GROWTH RATES OF WRONSKIANS GENERATED BY ENTIRE AND MEROMORPHIC FUNCTIONS ON THE BASIS OF THEIR RELATIVE (p,q)-TH TYPE AND RELATIVE (p,q)-TH WEAK TYPE

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Abstract. The main aim of this paper is to prove some results related to the growth rates of composite entire and meromorphic functions on the basis of their relative (p,q)-th order, relative (p,q)-th lower order, relative (p,q)-th type and relative (p,q)-th weak type where p and q are any two positive integers and that of wronskian generated by one of the factors.

1. Introduction, Definitions and Notations

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to f is defined on $|z| = r$ as $M_f(r) = \max|z| = r\,|f(z)|$. If an entire function f is non-constant then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f{}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exist and is such that $\lim\limits_{s\to\infty} M_f^{-1}(s) = \infty$. In this connection we just recall the following definition which is relevant:

Definition 1. [1] A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $(M_f(r))^2 \le M_f(r^\sigma)$ holds. For examples of functions with or without the Property (A), one may see [1].

When f is meromorphic, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of f, playing the same role as $M_f(r)$. The integrated counting function $N_f(r,a)\left(\overline{N}_f(r,a)\right)$ of a-points (distinct a-points) of f is defined as

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t}\,dt + n_f(0,a)\log r$$

Key words and phrases. Entire function, meromorphic function, relative (p,q)-th order, relative (p,q)-th lower order, relative (p,q)-th type, relative (p,q)-th weak type, growth, property (A), wronskian.

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$$\left(\overline{N}_f(r,a)=\int_0^r \frac{\overline{n}_f(t,a)-\overline{n}_f(r,a)}{t}dt+\overline{n}_f(0,a)\log r\right),$$

where we denote by $n_f(t,a)\,(\overline{n}_f(t,a))$ the number of a-points (distinct a-points) of f in $|z|\leq t$ and an ∞ -point is a pole of f . In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\overline{N}_f(r)$ respectively. The function $N_f(r,a)$ is called the enumerative function. On the other hand, the function $m_f(r)\equiv m_f(r,\infty)$ known as the proximity function is defined as

$$m_f(r)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta,$$

$$\text{where } \log^+ x=\max(\log x,0) \text{ for all } x\geqslant 0$$

and an ∞ -point is a pole of f .

Analogously, $m_{\frac{1}{f-a}}(r)\equiv m_f(r,a)$ is defined when a is not an ∞-point of f.

Thus the Nevanlinna's characteristic function $T_f(r)$ corresponding to f is defined as

$$T_f(r)=N_f(r)+m_f(r)\ .$$

When f is entire, $T_f(r)$ coincides with $m_f(r)$ as $N_f(r)=0$.

However, for a meromorphic function f, the Wronskian determinant $W(f)=W(a_1,a_2,....a_k,f)$ is defined as

$$W(f)=\begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1^{'} & a_2^{'} & . & . & . & a_k^{'} & f^{'} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix}$$

where $a_1,a_2,....a_k$ are linearly independent meromorphic functions and small with respect to f (i.e., $T_{a_i}(r)=S(r,f)$ or in other words $\frac{T_{a_i}(r)}{S(r,f)}\to 0$ as $r\to\infty$ for $i=1,2,3...k$). From the Nevanlinna's second fundamental theorem, it follows that the set of values of $a\in\mathbb{C}\cup\{\infty\}$ for which $\delta(a;f)>0$ is countable and $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)\leq 2$ (cf [7],.$p.43$) where $\delta(a;f)=1-\limsup\limits_{r\to\infty}\frac{N(r,a;f)}{T_f(r)}=\liminf\limits_{r\to\infty}\frac{m(r,a;f)}{T_f(r)}$. If in particular $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, we say that f has the maximum deficiency sum.

If f is non-constant entire then $T_f(r)$ is strictly increasing and continuous functions of r. Also its inverse $T_f^{-1}:(T_f(0),\infty)\to(0,\infty)$ exist and is such that $\lim\limits_{s\to\infty}T_f^{-1}(s)=\infty$. Further the ratio $\frac{T_f(r)}{T_g(r)}$ as $r\to\infty$ is called the growth of f with respect to g in terms of the Nevanlinna's Characteristic functions of the meromorphic functions f and g.

However let us consider that $x\in[0,\infty)$ and $k\in\mathbb{N}$ where $\mathbb{N}$ be the set of all positive integers. We define $\exp^{[k]}x=\exp\left(\exp^{[k-1]}x\right)$ and $\log^{[k]}x=\log\left(\log^{[k-1]}x\right)$. We also denote $\log^{[0]}x=x$, $\log^{[-1]}x=\exp x$, $\exp^{[0]}x=x$ and $\exp^{[-1]}x=\log x$. Further we assume that throughout the present paper a,p,q,m,n and l always denote positive integers. Now considering this, we introduce the definition of the

(p,q)-th order and (p,q)-th lower order of an entire or meromorphic function which are as follows:

Definition 2. The (p,q)-th order and (p,q)-th lower order of an entire function f are defined as:

$$\rho^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},$$

If f is a meromorphic function, then

$$\rho^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r},$$

Definition 2 avoids the restriction $p \geq q$ of the original definition of (p,q)-th order (respectively (p,q)-th lower order) of entire functions introduced by Juneja et al. [8]. Moreover for entire and meromorphic functions when $p < q$, then Definition 2 is a special case of Proposition 1.2 and Definition 1.6 of [12] respectively for $\varphi(r) = \log^{[l]} r$ where $l > p - q$. If $p = l$ and $q = 1$ then we write $\rho^{(l,1)}(f) = \rho_f^{[l]}$ and $\lambda^{(l,1)}(f) = \lambda_f^{[l]}$ where $\rho_f^{[l]}$ and $\lambda_f^{[l]}$ are respectively known as generalized order and generalized lower order of f. Also for $p = 2$ and $q = 1$ we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by ρ_f and λ_f where ρ_f and λ_f are the classical growth indicator known as order and lower order of f.

In this connection we just recall the following definition of index-pair where we will give a minor modification to the original definition (see e.g. [8]) :

Definition 3. An entire function f is said to have index-pair (p,q) if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho^{(p,q)}(f) < \infty$, then

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Analogously one can easily verify that Definition 3 of index-pair can also be applicable to a meromorphic function f.

However, the function f is said to be of regular (p,q) growth when (p,q)-th order and (p,q)-th lower order of f are the same. Functions which are not of regular (p,q) growth are said to be of irregular (p,q) growth.

In order to compare the growth of entire functions having the same (p,q)-th order, Juneja, Kapoor and Bajpai [9] also introduced the concepts of (p,q)-th type and (p,q)-th lower type of entire function. Next we recall the definitions of (p,q)-th type and (p,q)-th lower type of entire and meromorphic function where we will give a minor modification to the original definition (see e.g. [9]):

Definition 4. The (p,q)-th type and the (p,q)-th lower type of entire function f having non-zero finite positive (p,q)-th order $\rho_f(p,q)$ are defined as :

$$\sigma^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}(f)}} \text{ and } \overline{\sigma}^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}(f)}},$$

$$0 \leq \overline{\sigma}^{(p,q)}(f) \leq \sigma^{(p,q)}(f) \leq \infty .$$

If f is meromorphic function with $0 < \rho^{(p,q)}(f) < \infty$, then

$$\sigma^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-2]} T_f(r)}{\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}(f)}} \text{ and } \overline{\sigma}^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-2]} T_f(r)}{\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}(f)}},$$

$$0 \leq \overline{\sigma}^{(p,q)}(f) \leq \sigma^{(p,q)}(f) \leq \infty .$$

Likewise, to compare the growth of entire functions having the same (p,q)-th lower order, one can also introduced the concept of (p,q)-th weak type in the following manner :

Definition 5. The (p,q)-th weak type of entire function f having non-zero finite positive (p,q)-th tower order $\lambda_f(p,q)$ is defined as :

$$\tau^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}}.$$

Similarly one may define the growth indicator $\overline{\tau}^{(p,q)}(f)$ of an entire function f in the following way :

$$\overline{\tau}^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}} \ , \ 0 < \lambda^{(p,q)}(f) < \infty.$$

If f is meromorphic function with $0 < \lambda^{(p,q)}(f) < \infty$, then

$$\tau^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-2]} T_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}} \text{ and } \overline{\tau}^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-2]} T_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}},$$

where $0 < \lambda^{(p,q)}(f) < \infty$. It is also obvious that $0 \leq \tau^{(p,q)}(f) \leq \overline{\tau}^{(p,q)}(f) \leq \infty$.

Bernal [1, 2] introduced the definition of relative order of an entire function f with respect to another entire function g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0.\} \\ &= \overline{\lim_{r\to\infty}} \frac{\log M_g^{-1}(M_f(r))}{\log r} . \end{aligned}$$

The definition coincides with the classical one [13] if $g(z) = \exp z$. Similarly one can define the relative lower order of an entire function f with respect to another entire function g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \underline{\lim_{r\to\infty}} \frac{\log M_g^{-1}(M_f(r))}{\log r} .$$

Sánchez Ruiz et al. [11] gave the definitions of relative (p,q)-th order and relative (p,q)-th lower order of an entire function with respect to another entire function and Debnath et al. [5] introduced the definitions of relative (p, q)-th order and relative (p, q)-th lower order of a meromorphic function with respect to another entire function in the light of index-pair. In order to keep accordance with Definition 2 and Definition 3, we will give a minor modification to the original definition of relative (p, q)-th order and relative (p, q)-th lower order of entire and meromorphic function (see e.g. [5, 11]).

Definition 6. Let f and g be any two entire functions with index-pairs (m, q) and (m, p) respectively. Then the relative (p, q)-th order and relative (p, q)-th lower order of f with respect to g are defined as

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_g^{-1}(M_f(r))}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_g^{-1}(M_f(r))}{\log^{[q]} r}.$$

If f is a meromorphic and g is entire, then

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r}.$$

Further an entire or meromorphic function f, for which relative (p, q)-th order and relative (p, q)-th lower order with respect to an entire function g are the same is called a function of regular relative (p, q) growth with respect to g. Otherwise, f is said to be irregular relative (p, q) growth with respect to g.

Now in order to refine the above growth scale, one may introduce the definitions of an another growth indicators, such as relative (p, q)-th type and relative (p, q)-th lower type of entire or meromorphic functions with respect to another entire function in the light of their index-pair which are as follows:

Definition 7. Let f and g be any two entire functions with index-pairs (m, q) and (m, p) respectively. The relative (p, q)-th type and the relative (p, q)-th lower type of f with respect to g when $0 < \rho_g^{(p,q)}(f) < \infty$ are defined as:

$$\sigma_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_g^{-1}(M_f(r))}{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}} \text{ and } \overline{\sigma}_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_g^{-1}(M_f(r))}{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}}.$$

If f is a meromorphic and g is entire, then

$$\sigma_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}} \text{ and } \overline{\sigma}_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}},$$

where $0 < \rho_g^{(p,q)}(f) < \infty$.

Analogously, to determine the relative growth of f having same non zero finite relative (p, q)-th lower order with respect to g, one can introduced the definition of relative (p, q)-th weak type $\tau_g^{(p,q)}(f)$ and the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of f with respect to g of finite positive relative (p, q)-th lower order $\lambda_g^{(p,q)}(f)$ in the following way:

Definition 8. Let f and g be any two entire functions with index-pairs (m, q) and (m, p) respectively. The relative (p, q)-th weak type $\tau_g^{(p,q)}(f)$ and the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of f with respect to g when $0<\lambda_g^{(p,q)}(f)<\infty$ are defined as:

$$\tau_g^{(p,q)}(f)=\underline{\lim}_{r\to\infty}\frac{\log^{[p-1]}M_g^{-1}(M_f(r))}{\left(\log^{[q-1]}r\right)^{\lambda_g^{(p,q)}(f)}}\text{ and }\overline{\tau}_g^{(p,q)}(f)=\overline{\lim}_{r\to\infty}\frac{\log^{[p-1]}M_g^{-1}(M_f(r))}{\left(\log^{[q-1]}r\right)^{\lambda_g^{(p,q)}(f)}}.$$

If f is a meromorphic and g is entire, then

$$\tau_g^{(p,q)}(f)=\underline{\lim}_{r\to\infty}\frac{\log^{[p-1]}T_g^{-1}(T_f(r))}{\left(\log^{[q-1]}r\right)^{\lambda_g^{(p,q)}(f)}}\text{ and }\overline{\tau}_g^{(p,q)}(f)=\overline{\lim}_{r\to\infty}\frac{\log^{[p-1]}T_g^{-1}(T_f(r))}{\left(\log^{[q-1]}r\right)^{\lambda_g^{(p,q)}(f)}},$$

where $0<\lambda_g^{(p,q)}(f)<\infty$.

Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the Wronskians. Actually in the paper we establish some new results depending on the comparative growth properties of composite transcendental entire and meromorphic functions using relative (p,q)-th order, relative (p,q)-th type and relative (p,q)-th weak type and that of wronskian generated by one of the factors. We use the standard notations and definitions of the theory of entire and meromorphic functions which are available in [7] and [14].

2. **Lemmas**

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [3] Let f be meromorphic and g be entire then for all sufficiently large values of r,

$$T_{f\circ g}(r)\leqslant\{1+o(1)\}\frac{T_g(r)}{\log M_g(r)}T_f(M_g(r)) .$$

Lemma 2. [6] Let f be an entire function which satisfy the Property (A), $\beta>0$, $\delta>1$ and $\alpha>2$. Then

$$\beta T_f(r)<T_f\left(\alpha r^{\delta}\right) .$$

Lemma 3. [10] Let f be a transcendental meromorphic function having the maximum deficiency sum. Then

$$\lim_{r\to\infty}\frac{T_{W(f)}(r)}{T_f(r)}=1+k_1-k_1\delta(\infty;f) .$$

Lemma 4. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and g be a transcendental entire function having the maximum deficiency sum with $0<\lambda^{(m,p)}(g)\leq\rho^{(m,p)}(g)<\infty$ where $m>1$. Then

$$\frac{\lambda^{(m,p)}(g)}{\rho^{(m,p)}(g)}\leq\underline{\lim}_{r\to\infty}\frac{\log^{[p]}T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p]}T_g^{-1}(T_f(r))}\leq\overline{\lim}_{r\to\infty}\frac{\log^{[p]}T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p]}T_g^{-1}(T_f(r))}\leq\frac{\rho^{(m,p)}(g)}{\lambda^{(m,p)}(g)} .$$

Proof. For any $\varepsilon(>0)$, we get from Lemma 3 for all sufficiently large values of r that

$$T_{W(f)}(r)\leq((1+k_1-k_1\delta(\infty;f))+\varepsilon)T_f(r) \tag{1}$$

and

$$T_{W(f)}(r)\geq((1+k_1-k_1\delta(\infty;f))-\varepsilon)T_f(r) . \tag{2}$$

Also from Lemma 3, we get for all sufficiently large values of r that

$$T_{W(g)}(r) \geq ((1+k_2-k_2\delta(\infty;g))-\varepsilon)\, T_g(r)$$
$$i.e.,\ r \geq T_{W(g)}^{-1}(((1+k_2-k_2\delta(\infty;g))-\varepsilon)\, T_g(r))$$
$$i.e.,\ T_g^{-1}\left(\frac{r}{(1+k_2-k_2\delta(\infty;g))-\varepsilon}\right) \geq T_{W(g)}^{-1}(r) \tag{3}$$

and

$$T_{W(g)}(r) \leq ((1+k_2-k_2\delta(\infty;g))+\varepsilon)\, T_g(r)$$
$$i.e.,\ r \leq T_{W(g)}^{-1}(((1+k_2-k_2\delta(\infty;g))+\varepsilon)\, T_g(r))$$
$$i.e.,\ T_g^{-1}\left(\frac{r}{(1+k_2-k_2\delta(\infty;g))+\varepsilon}\right) \leq T_{W(g)}^{-1}(r)\ . \tag{4}$$

Now from (1) and (3) it follows for all sufficiently large values of r that

$$T_{W(g)}^{-1}\left(T_{W(f)}(r)\right) \leq T_{W(g)}^{-1}(((1+k_1-k_1\delta(\infty;f))+\varepsilon)\, T_f(r))$$
$$i.e.,\ T_{W(g)}^{-1}\left(T_{W(f)}(r)\right) \leq T_g^{-1}\left(\left(\frac{(1+k_1-k_1\delta(\infty;f))+\varepsilon}{(1+k_2-k_2\delta(\infty;g))-\varepsilon}\right) T_f(r)\right)\ . \tag{5}$$

Again from (2) and (4), it follows for all sufficiently large values of r that

$$T_{W(g)}^{-1}\left(T_{W(f)}(r)\right) \geq T_{W(g)}^{-1}(((1+k_1-k_1\delta(\infty;f))-\varepsilon)\, T_f(r))$$
$$i.e.,\ T_{W(g)}^{-1}\left(T_{W(f)}(r)\right) \geq T_g^{-1}\left(\left(\frac{(1+k_1-k_1\delta(\infty;f))-\varepsilon}{(1+k_2-k_2\delta(\infty;g))+\varepsilon}\right) T_f(r)\right)\ . \tag{6}$$

Now from (5) and (6), we get for all sufficiently large values of r that

$$\log^{[p]} T_{W(g)}^{-1}\left(T_{W(f)}(r)\right) \leq \log^{[p]} T_g^{-1}\left(\left(\frac{(1+k_1-k_1\delta(\infty;f))+\varepsilon}{(1+k_2-k_2\delta(\infty;g))-\varepsilon}\right) T_f(r)\right) \tag{7}$$

and

$$\log^{[p]} T_{W(g)}^{-1}\left(T_{W(f)}(r)\right) \geq \log^{[p]} T_g^{-1}\left(\left(\frac{(1+k_1-k_1\delta(\infty;f))-\varepsilon}{(1+k_2-k_2\delta(\infty;g))+\varepsilon}\right) T_f(r)\right)\ . \tag{8}$$

Now for the definition of (m,p)-th order and (m,p)-th lower order of g, we get for all sufficiently large values of r that

$$T_g\left(\exp^{[p-1]}\left[\log^{[m-2]} T_f(r)\right]^{\frac{1}{\rho^{(m,p)}(g)+\varepsilon}}\right) \leq T_f(r)$$
$$i.e.,\ \log^{[p]} T_g^{-1}(T_f(r)) \geq \frac{1}{\left(\rho^{(m,p)}(g)+\varepsilon\right)} \log^{[m-1]} T_f(r) \tag{9}$$

and

$$T_g\left[\exp^{[p-1]}\left[\log^{[m-2]}\left[\left(\frac{(1+k_1-k_1\delta(\infty;f))+\varepsilon}{((1+k_2-k_2\delta(\infty;g))-\varepsilon)}\right) T_f(r)\right]\right]^{\frac{1}{\lambda^{(m,p)}(g)-\varepsilon}}\right]$$
$$\geq \left[\left(\frac{(1+k_1-k_1\delta(\infty;f))+\varepsilon}{(1+k_2-k_2\delta(\infty;g))-\varepsilon}\right) T_f(r)\right]$$
$$i.e.,\ \exp^{[p-1]}\left[\log^{[m-2]}\left[\left(\frac{(1+k_1-k_1\delta(\infty;f))+\varepsilon}{((1+k_2-k_2\delta(\infty;g))-\varepsilon)}\right) T_f(r)\right]\right]^{\frac{1}{\lambda^{(m,p)}(g)-\varepsilon}}$$
$$\geq T_g^{-1}\left[\left(\frac{(1+k_1-k_1\delta(\infty;f))+\varepsilon}{((1+k_2-k_2\delta(\infty;g))-\varepsilon)}\right) T_f(r)\right]$$

$$i.e., \ \frac{1}{\left(\lambda^{(m,p)}(g)-\varepsilon\right)}\log^{[m-1]}T_f(r)+O(1)\geq$$

$$\log^{[p]}T_g^{-1}\left[\left(\frac{(1+k_1-k_1\delta(\infty;f))+\varepsilon}{((1+k_2-k_2\delta(\infty;g))-\varepsilon)}\right)T_f(r)\right]. \quad (10)$$

Therefore from (7) and (10), it follows for all sufficiently large values of r that

$$\log^{[p]}T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)\leq\frac{1}{\left(\lambda^{(m,p)}(g)-\varepsilon\right)}\log^{[m-1]}T_f(r)+O(1). \quad (11)$$

Therefore from (9) and (11), it follows for all sufficiently large values of r that

$$\frac{\log^{[p]}T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p]}T_g^{-1}(T_f(r))}\leq\left(\frac{\rho^{(m,p)}(g)+\varepsilon}{\lambda^{(m,p)}(g)-\varepsilon}\right)\cdot\frac{\log^{[m-1]}T_f(r)+O(1)}{\log^{[m-1]}T_f(r)}$$

$$i.e., \ \overline{\lim_{r\to\infty}}\frac{\log^{[p]}T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p]}T_g^{-1}(T_f(r))}\leq\frac{\rho^{(m,p)}(g)}{\lambda^{(m,p)}(g)}. \quad (12)$$

Similarly, from (8) it can be shown for all sufficiently large values of r that

$$\underline{\lim_{r\to\infty}}\frac{\log^{[p]}T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p]}T_g^{-1}(T_f(r))}\geq\frac{\lambda^{(m,p)}(g)}{\rho^{(m,p)}(g)}. \quad (13)$$

Therefore from (12) and (13), we obtain that

$$\frac{\lambda^{(m,p)}(g)}{\rho^{(m,p)}(g)}\leq\underline{\lim_{r\to\infty}}\frac{\log^{[p]}T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p]}T_g^{-1}(T_f(r))}\leq\overline{\lim_{r\to\infty}}\frac{\log^{[p]}T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p]}T_g^{-1}(T_f(r))}\leq\frac{\rho^{(m,p)}(g)}{\lambda^{(m,p)}(g)}.$$

Thus the lemma follows from above.

Lemma 5. Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and g be a transcendental entire function having the maximum deficiency sum with regular (m,p)-growth where $m>1$. Then the relative (p,q)-th order and relative (p,q)-th lower order of $W(f)$ with respect to $W(g)$ are same as those of f with respect to g.

Proof. If g is of regular (m,p)-growth, then from Lemma 4 get that

$$\lim_{r\to\infty}\frac{\log^{[p]}T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p]}T_g^{-1}(T_f(r))}=1. \quad (14)$$

Now in view of (14), we obtain that

$$\begin{aligned}\rho_{W(g)}^{(p,q)}(W(f))&=\overline{\lim_{r\to\infty}}\frac{\log^{[p]}T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[q]}r}\\&=\overline{\lim_{r\to\infty}}\frac{\log^{[p]}T_g^{-1}(T_f(r))}{\log^{[q]}r}\cdot\lim_{r\to\infty}\frac{\log^{[p]}T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p]}T_g^{-1}(T_f(r))}\\&=\rho_g^{(p,q)}(f)\cdot 1=\rho_g^{(p,q)}(f).\end{aligned}$$

In a similar manner, $\lambda_{W(g)}^{(p,q)}(W(f))=\lambda_g^{(p,q)}(f)$.

Thus the lemma follows.

Lemma 6. If $f(z)$ be a meromorphic function of regular (p,q)-th growth i.e., if $\rho^{(p,q)}(f)=\lambda^{(p,q)}(f)$, then

$$\sigma^{(p,q)}(f)=\overline{\sigma}^{(p,q)}(f)=\tau^{(p,q)}(f)=\overline{\tau}^{(p,q)}(f).$$

We omit the proof of Lemma 6 because it can be carried out in the line of Theorem 6 of [4].

Lemma 7. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and g be a transcendental entire function having the maximum deficiency sum with $0<\tau^{(m,p)}(g)\leq\overline{\tau}^{(m,p)}(g)<\infty$ and $0<\overline{\sigma}^{(m,p)}(g)\leq\sigma^{(m,p)}(g)<\infty$ where $m>2$. Then

$$\max\left\{\left(\frac{\tau^{(m,p)}(g)}{\overline{\tau}^{(m,p)}(g)}\right)^{\frac{1}{\lambda^{(m,p)}(g)}},\left(\frac{\overline{\sigma}^{(m,p)}(g)}{\sigma^{(m,p)}(g)}\right)^{\frac{1}{\rho^{(m,p)}(g)}}\right\}\leq\underline{\lim}_{r\to\infty}\frac{\log^{[p-1]}T^{-1}_{W(g)}\left(T_{W(f)}(r)\right)}{\log^{[p-1]}T_g^{-1}\left(T_f(r)\right)}$$

$$\leq\overline{\lim}_{r\to\infty}\frac{\log^{[p-1]}T^{-1}_{W(g)}\left(T_{W(f)}(r)\right)}{\log^{[p-1]}T_g^{-1}\left(T_f(r)\right)}\leq\min\left\{\left(\frac{\overline{\tau}^{(m,p)}(g)}{\tau^{(m,p)}(g)}\right)^{\frac{1}{\lambda^{(m,p)}(g)}},\left(\frac{\sigma^{(m,p)}(g)}{\overline{\sigma}^{(m,p)}(g)}\right)^{\frac{1}{\rho^{(m,p)}(g)}}\right\}.$$

Proof. From the definition of (m,p)-th type and (m,p)-th lower type, we get for all sufficiently large values of r that

$$T_g\left(\exp^{[p-1]}\left\{\frac{\log^{[m-2]}T_f(r)}{\left(\sigma^{(m,p)}(g)+\varepsilon\right)}\right\}^{\frac{1}{\rho^{(m,p)}(g)}}\right)\leq T_f(r)$$

$$i.e.,\ \log^{[p-1]}T_g^{-1}\left(T_f(r)\right)\geq\left\{\frac{\log^{[m-2]}T_f(r)}{\left(\sigma^{(m,p)}(g)+\varepsilon\right)}\right\}^{\frac{1}{\rho^{(m,p)}(g)}} \tag{15}$$

and

$$T_g\left(\exp^{[p-1]}\left\{\frac{\log^{[m-2]}\left(\frac{(1+k_1-k_1\delta(\infty;f))-\varepsilon}{(1+k_2-k_2\delta(\infty;g))+\varepsilon}\right)T_f(r)}{\left(\overline{\sigma}^{(m,p)}(g)-\varepsilon\right)}\right\}^{\frac{1}{\rho^{(m,p)}(g)}}\right)\geq$$

$$\left[\left(\frac{(1+k_1-k_1\delta(\infty;f))-\varepsilon}{(1+k_2-k_2\delta(\infty;g))+\varepsilon}\right)T_f(r)\right]$$

$$i.e.,\ \exp^{[p-1]}\left\{\frac{\log^{[m-2]}\left(\frac{(1+k_1-k_1\delta(\infty;f))-\varepsilon}{(1+k_2-k_2\delta(\infty;g))+\varepsilon}\right)T_f(r)}{\left(\overline{\sigma}^{(m,p)}(g)-\varepsilon\right)}\right\}^{\frac{1}{\rho^{(m,p)}(g)}}\geq$$

$$T_g^{-1}\left[\left(\frac{(1+k_1-k_1\delta(\infty;f))-\varepsilon}{(1+k_2-k_2\delta(\infty;g))+\varepsilon}\right)T_f(r)\right]. \tag{16}$$

Therefore from (5) and (16), it follows for all sufficiently large values of r that

$$T^{-1}_{W(g)}\left(T_{W(f)}(r)\right)\leq\exp^{[p-1]}\left\{\frac{\log^{[m-2]}\left(\frac{(1+k_1-k_1\delta(\infty;f))-\varepsilon}{(1+k_2-k_2\delta(\infty;g))+\varepsilon}\right)T_f(r)}{\left(\overline{\sigma}^{(m,p)}(g)-\varepsilon\right)}\right\}^{\frac{1}{\rho^{(m,p)}(g)}}$$

$$i.e.,\ \log^{[p-1]}T^{-1}_{W(g)}\left(T_{W(f)}(r)\right)\leq\left\{\frac{\log^{[m-2]}\left(\frac{(1+k_1-k_1\delta(\infty;f))-\varepsilon}{(1+k_2-k_2\delta(\infty;g))+\varepsilon}\right)T_f(r)}{\left(\overline{\sigma}^{(m,p)}(g)-\varepsilon\right)}\right\}^{\frac{1}{\rho^{(m,p)}(g)}}. \tag{17}$$

Therefore from (15) and (17), it follows for all sufficiently large values of r that

$$\frac{\log^{[p-1]} T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p-1]} T_g^{-1}\left(T_f\left(r\right)\right)} \leq \frac{\left\{\frac{\log^{[m-2]}\left(\frac{(1+k_1-k_1\delta(\infty;f))-\varepsilon}{(1+k_2-k_2\delta(\infty;g))+\varepsilon}\right)T_f(r)}{\left(\overline{\sigma}^{(m,p)}(g)-\varepsilon\right)}\right\}^{\frac{1}{\rho^{(m,p)}(g)}}}{\left\{\frac{\log^{[m-2]} T_f(r)}{\left(\sigma^{(m,p)}(g)+\varepsilon\right)}\right\}^{\frac{1}{\rho^{(m,p)}(g)}}}$$

$$i.e.,\ \frac{\log^{[p-1]} T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p-1]} T_g^{-1}\left(T_f\left(r\right)\right)} \leq \left(\frac{\sigma^{(m,p)}\left(g\right)+\varepsilon}{\overline{\sigma}^{(m,p)}\left(g\right)-\varepsilon}\right)^{\frac{1}{\rho^{(m,p)}(g)}} \cdot \left(\frac{\log^{[m-2]} T_f\left(r\right)+O(1)}{\log^{[m-2]} T_f\left(r\right)}\right)^{\frac{1}{\rho^{(m,p)}(g)}}$$

$$i.e.,\ \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p-1]} T_g^{-1}\left(T_f\left(r\right)\right)} \leq \left(\frac{\sigma^{(m,p)}\left(g\right)}{\overline{\sigma}^{(m,p)}\left(g\right)}\right)^{\frac{1}{\rho^{(m,p)}(g)}} . \tag{18}$$

Similarly from (6), it can be shown for all sufficiently large values of r that

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p-1]} T_g^{-1}\left(T_f\left(r\right)\right)} \geq \left(\frac{\overline{\sigma}^{(m,p)}\left(g\right)}{\sigma^{(m,p)}\left(g\right)}\right)^{\frac{1}{\rho^{(m,p)}(g)}} . \tag{19}$$

Therefore from (18) and (19), we obtain that

$$\left(\frac{\overline{\sigma}^{(m,p)}\left(g\right)}{\sigma^{(m,p)}\left(g\right)}\right)^{\frac{1}{\rho^{(m,p)}(g)}} \leq \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p-1]} T_g^{-1}\left(T_f\left(r\right)\right)} \leq \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p-1]} T_g^{-1}\left(T_f\left(r\right)\right)}$$

$$\leq \left(\frac{\sigma^{(m,p)}\left(g\right)}{\overline{\sigma}^{(m,p)}\left(g\right)}\right)^{\frac{1}{\rho^{(m,p)}(g)}} . \tag{20}$$

Similarly, using the weak type one can easily verify that

$$\left(\frac{\tau^{(m,p)}\left(g\right)}{\overline{\tau}^{(m,p)}\left(g\right)}\right)^{\frac{1}{\lambda^{(m,p)}(g)}} \leq \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p-1]} T_g^{-1}\left(T_f\left(r\right)\right)} \leq \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p-1]} T_g^{-1}\left(T_f\left(r\right)\right)}$$

$$\leq \left(\frac{\overline{\tau}^{(m,p)}\left(g\right)}{\tau^{(m,p)}\left(g\right)}\right)^{\frac{1}{\lambda^{(m,p)}(g)}} . \tag{21}$$

Thus the lemma follows from (20) and (21).

Lemma 8. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty} \delta\left(a;f\right) + \delta\left(\infty;f\right) = 2$ and g be a transcendental entire function having the maximum deficiency sum with regular (m,p)-growth and non zero finite (m,p)-th type where $m > 2$. Then the relative (p,q)-th type and relative (p,q)-th lower type of $W\left(f\right)$ with respect to $W\left(g\right)$ are same as those of f with respect to g if $\rho_g^{(p,q)}\left(f\right)$ is positive finite.

Proof. If g is of regular (m,p)-th growth with non zero finite (m,p)-th type, then from Lemma 6 and Lemma 7 we get that

$$\lim_{r\to\infty} \frac{\log^{[p-1]} T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p-1]} T_g^{-1}\left(T_f\left(r\right)\right)} = 1 . \tag{22}$$

Now from Lemma 5 and (22), we obtain that

$$\sigma_{W(g)}^{(p,q)}(W(f)) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\left[\log^{[q-1]} r\right]^{\rho_{W(g)}^{(p,q)}(W(f))}}$$

$$= \lim_{r\to\infty} \frac{\log^{[p-1]} T_{W(g)}^{-1}\left(T_{W(f)}(r)\right)}{\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)} \cdot \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)}{\left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)}}$$

$$= 1 \cdot \sigma_g^{(p,q)}(f) = \sigma_g^{(p,q)}(f) \ .$$

Similarly, $\overline{\sigma}_{W(g)}^{(p,q)}(W(f)) = \overline{\sigma}_g^{(p,q)}(f)$.

This proves the theorem.

Lemma 9. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$ and g be a transcendental entire function having the maximum deficiency sum with regular (m,p)-growth and non zero finite (m,p)-th type where $m > 2$. Then $\tau_{W(g)}^{(p,q)}(W(f))$ and $\overline{\tau}_{W(g)}^{(p,q)}(W(f))$ are same as those of f with respect to g i.e.,

$$\tau_{W(g)}^{(p,q)}(W(f)) = \tau_g^{(p,q)}(f) \text{ and } \overline{\tau}_{W(g)}^{(p,q)}(W(f)) = \overline{\tau}_g^{(p,q)}(f) \ .$$

when $\lambda_g^{(p,q)}(f)$ is positive finite.

We omit the proof of Lemma 9 because it can be carried out in the line of Lemma 8.

3. Main Results

In this section we present the main results of the paper.

Theorem 1. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\sigma^{(m,n)}(g) < \infty$ where $a > 1$ and $q = m-1$. If h satisfies the Property (A), then

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p]} T_{W(h)}^{-1}\left(T_{W(f)}\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \leq \frac{\sigma^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)} \ .$$

Proof. Let us suppose that $\alpha > 2$ and $\delta \to 1^+$ in Lemma 2. Since $T_h^{-1}(r)$ is an increasing function r, it follows from Lemma 1, Lemma 2 and the inequality $T_g(r) \leq \log M_g(r)$ $\{cf.\ [7]\}$ for all sufficiently large values of r that

$$T_h^{-1}\left(T_{f\circ g}(r)\right) \leqslant T_h^{-1}\left[\{1+o(1)\} T_f\left(M_g(r)\right)\right]$$

$$i.e.,\ T_h^{-1}\left(T_{f\circ g}(r)\right) \leqslant \alpha\left[T_h^{-1}\left(T_f\left(M_g(r)\right)\right)\right]^{\delta}$$

$$i.e.,\ \log^{[p]} T_h^{-1}\left(T_{f\circ g}(r)\right) \leqslant \log^{[p]} T_h^{-1}\left(T_f\left(M_g(r)\right)\right) + O(1)$$

$$i.e.,\ \log^{[p]} T_h^{-1}\left(T_{f\circ g}(r)\right) \leqslant \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]} M_g(r) + O(1)$$

$$i.e.,\ \log^{[p]} T_h^{-1}\left(T_{f\circ g}(r)\right) \leqslant \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[m-1]} M_g(r) + O(1)$$

$$i.e.,\ \log^{[p]} T_h^{-1}\left(T_{f\circ g}(r)\right) \leqslant \left(\rho_h^{(p,q)}(f)+\varepsilon\right)\left(\sigma^{(m,n)}(g)+\varepsilon\right)\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)} + O(1)\ . \quad (23)$$

Now from the definition of $\lambda_{W(h)}^{(p,q)}(W(f))$ and in view of Lemma 5, we obtain for all sufficiently large values of r that

$$\log^{[p]} T_{W(h)}^{-1}\left(T_{W(f)}\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right) \geq \left(\lambda_{W(h)}^{(p,q)}(W(f))-\varepsilon\right)\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}$$

$$i.e.,\ \log^{[p]} T_{W(h)}^{-1}\left(T_{W(f)}\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right) \geq \left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\ . \quad (24)$$

Therefore from (23) and (24), it follows for all sufficiently large values of r that

$$\frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p]} T_{W(h)}^{-1}\left(T_{W(f)}\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \leq \frac{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\left(\sigma^{(m,n)}(g)+\varepsilon\right)\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)} + O(1)}{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}}$$

$$i.e.,\ \overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p]} T_{W(h)}^{-1}\left(T_{W(f)}\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \leq \frac{\sigma^{(m,n)}(g)\cdot\rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}\ .$$

Thus the theorem is established.

In the line of Theorem 1 the following theorem can be proved :

Theorem 2. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$ and $\overline{\tau}^{(m,n)}(g)<\infty$ where $a>1$ and $q=m-1$. If h satisfies the Property (A), then

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p]} T_{W(h)}^{-1}\left(T_{W(f)}\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \leq \frac{\overline{\tau}^{(m,n)}(g)\cdot\rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}\ .$$

Now we state the following two theorems without their proofs as those can easily be carried out in the line of Theorem 1 and Theorem 2 respectively.

Theorem 3. Let f be meromorphic, g be a transcendental entire function with $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and h be a transcendental entire function having the maximum deficiency sum with regular (m,p) growth such that $\lambda_h^{(p,n)}(g)>0$,

$\rho_h^{(p,q)}(f) < \infty$ and $\sigma^{(m,n)}(g) < \infty$ where $m > 1$ and $q = m - 1$. If h satisfies the Property (A), then

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]} T_{W(h)}^{-1}\left(T_{W(g)}\left(\exp^{[n]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \leq \frac{\sigma^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)} .$$

Theorem 4. Let f be meromorphic, g be a transcendental entire function with $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$ and h be a transcendental entire function having the maximum deficiency sum with regular (m,p) growth such that $\lambda_h^{(p,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ and $\overline{\tau}^{(m,n)}(g) < \infty$ where $m > 1$ and $q = m - 1$. If h satisfies the Property (A), then

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]} T_{W(h)}^{-1}\left(T_{W(g)}\left(\exp^{[n]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \leq \frac{\overline{\tau}^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)} .$$

Using the notion of (p,q)-th lower type we may state the following two theorems without proof because it can be carried out in the line of Theorem 1 and Theorem 3 respectively.

Theorem 5. Let f be a transcendental meromorphic function with $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\overline{\sigma}^{(m,n)}(g) < \infty$ where $a > 1$ and $q = m - 1$. If h satisfies the Property (A), then

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]} T_{W(h)}^{-1}\left(T_{W(f)}\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \leq \frac{\overline{\sigma}^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)} .$$

Theorem 6. Let f be meromorphic, g be a transcendental entire function with $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$ and h be a transcendental entire function having the maximum deficiency sum with regular (m,p) growth such that $\lambda_h^{(p,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ and $\overline{\sigma}^{(m,n)}(g) < \infty$ where $m > 1$ and $q = m - 1$. If h satisfies the Property (A), then

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]} T_{W(h)}^{-1}\left(T_{W(g)}\left(\exp^{[n]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \leq \frac{\overline{\sigma}^{(m,n)}(g) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)} .$$

Further using the notion of (p,q)-th weak type we may also state the following two theorems without proof because it can be carried out in the line of Theorem 2 and Theorem 4 respectively.

Theorem 7. Let f be a transcendental meromorphic function with $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\tau^{(m,n)}(g) < \infty$ where $a > 1$ and $q = m - 1$. If h

satisfies the Property (A), then

$$\varliminf_{r\to\infty}\frac{\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p]}T_{W(h)}^{-1}\left(T_{W(f)}\left(\exp^{[q]}\left(\log^{[n-1]}r\right)^{\rho^{(m,n)}(g)}\right)\right)}\leq\frac{\tau^{(m,n)}\left(g\right)\cdot\rho_h^{(p,q)}\left(f\right)}{\lambda_h^{(p,q)}\left(f\right)}\ .$$

Theorem 8. Let f be meromorphic, g be a transcendental entire function with $\sum\limits_{a\neq\infty}\delta\left(a;g\right)+\delta\left(\infty;g\right)=2$ and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $\lambda_h^{(p,n)}\left(g\right)>0$, $\rho_h^{(p,q)}\left(f\right)<\infty$ and $\tau^{(m,n)}\left(g\right)<\infty$ where $m>1$ and $q=m-1$. If h satisfies the Property (A), then

$$\varliminf_{r\to\infty}\frac{\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p]}T_{W(h)}^{-1}\left(T_{W(g)}\left(\exp^{[n]}\left(\log^{[n-1]}r\right)^{\rho^{(m,n)}(g)}\right)\right)}\leq\frac{\tau^{(m,n)}\left(g\right)\cdot\rho_h^{(p,q)}\left(f\right)}{\lambda_h^{(p,n)}\left(g\right)}\ .$$

Now we state the following six theorems without their proofs as those can easily be carried out in the line in the line of Theorem 1.

Theorem 9. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta\left(a;f\right)+\delta\left(\infty;f\right)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $0<\lambda_h^{(p,q)}\left(f\right)<\infty$ or $0<\rho_h^{(p,q)}\left(f\right)<\infty$ and $\sigma^{(m,n)}\left(g\right)<\infty$ where $a>1$ and $q=m-1$. If h satisfies the Property (A), then

$$\varliminf_{r\to\infty}\frac{\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p]}T_{W(h)}^{-1}\left(T_{W(f)}\left(\exp^{[q]}\left(\log^{[n-1]}r\right)^{\rho^{(m,n)}(g)}\right)\right)}\leq\sigma^{(m,n)}\left(g\right)\ .$$

Theorem 10. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta\left(a;f\right)+\delta\left(\infty;f\right)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth such that $0<\lambda_h^{(p,q)}\left(f\right)<\infty$ or $0<\rho_h^{(p,q)}\left(f\right)<\infty$ and $\overline{\tau}^{(m,n)}\left(g\right)<\infty$ where $a>1$ and $q=m-1$. If h satisfies the Property (A), then

$$\varliminf_{r\to\infty}\frac{\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p]}T_{W(h)}^{-1}\left(T_{W(f)}\left(\exp^{[q]}\left(\log^{[n-1]}r\right)^{\rho^{(m,n)}(g)}\right)\right)}\leq\overline{\tau}^{(m,n)}\left(g\right)\ .$$

Theorem 11. Let f be meromorphic, g be a transcendental entire function with $\sum\limits_{a\neq\infty}\delta\left(a;g\right)+\delta\left(\infty;g\right)=2$ and h be a transcendental entire function having the maximum deficiency sum with regular (m,p) growth such that $\lambda_h^{(p,n)}\left(g\right)>0$, $\lambda_h^{(p,q)}\left(f\right)<\infty$ and $\sigma^{(m,n)}\left(g\right)<\infty$ where $m>1$ and $q=m-1$. If h satisfies the Property (A), then

$$\varliminf_{r\to\infty}\frac{\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p]}T_{W(h)}^{-1}\left(T_{W(g)}\left(\exp^{[n]}\left(\log^{[n-1]}r\right)^{\rho^{(m,n)}(g)}\right)\right)}\leq\frac{\sigma^{(m,n)}\left(g\right)\cdot\lambda_h^{(p,q)}\left(f\right)}{\lambda_h^{(p,n)}\left(g\right)}\ .$$

Theorem 12. Let f be meromorphic, g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and h be a transcendental entire function having the maximum deficiency sum with regular (m,p) growth such that $\rho_h^{(p,n)}(g)>0$, $\rho_h^{(p,q)}(f)<\infty$ and $\sigma^{(m,n)}(g)<\infty$ where $m>1$ and $q=m-1$. If h satisfies the Property (A), then

$$\varliminf_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]}T_{W(h)}^{-1}\left(T_{W(g)}\left(\exp^{[n]}\left(\log^{[n-1]}r\right)^{\rho^{(m,n)}(g)}\right)\right)}\leq\frac{\sigma^{(m,n)}(g)\cdot\rho_h^{(p,q)}(f)}{\rho_h^{(p,n)}(g)}\ .$$

Theorem 13. Let f be meromorphic, g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and h be a transcendental entire function having the maximum deficiency sum with regular (m,p) growth such that $\lambda_h^{(p,n)}(g)>0$, $\lambda_h^{(p,q)}(f)<\infty$ and ${}^{(m,n)}(g)<\infty$ where $m>1$ and $q=m-1$. If h satisfies the Property (A), then

$$\varliminf_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]}T_{W(h)}^{-1}\left(T_{W(g)}\left(\exp^{[n]}\left(\log^{[n-1]}r\right)^{\rho^{(m,n)}(g)}\right)\right)}\leq\frac{\overline{\tau}^{(m,n)}(g)\cdot\lambda_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)}\ .$$

Theorem 14. Let f be meromorphic, g be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and h be a transcendental entire function having the maximum deficiency sum with regular (m,p) growth such that $\rho_h^{(p,n)}(g)>0$, $\rho_h^{(p,q)}(f)<\infty$ and $\overline{\tau}^{(m,n)}(g)<\infty$ where $m>1$ and $q=m-1$. If h satisfies the Property (A), then

$$\varliminf_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p]}T_{W(h)}^{-1}\left(T_{W(g)}\left(\exp^{[n]}\left(\log^{[n-1]}r\right)^{\rho^{(m,n)}(g)}\right)\right)}\leq\frac{\overline{\tau}^{(m,n)}(g)\cdot\rho_h^{(p,q)}(f)}{\rho_h^{(p,n)}(g)}\ .$$

Theorem 15. Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite (a,p)-th type such that (i) $0<\rho_h^{(p,q)}(f)<\infty$, (ii) $\rho_h^{(p,q)}(f)=\rho^{(m,n)}(g)$, (iii) $\sigma^{(m,n)}(g)<\infty$ and (iv) $0<\sigma_h^{(p,q)}(f)<\infty$ where $q=n=m-1$ and $a>2$. Also let h satisfies the Property (A). Then

$$\varliminf_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}\left(T_{W(f)}(r)\right)}\leq\frac{\rho_h^{(p,q)}(f)\cdot\sigma^{(m,n)}(g)}{\sigma_h^{(p,q)}(f)}\ .$$

Proof. In view of condition (ii) ,we obtain from (23) for all sufficiently large values of r that

$$\log^{[p]}T_h^{-1}(T_{f\circ g}(r))\leqslant$$
$$\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\left(\sigma^{(m,n)}(g)+\varepsilon\right)\left[\log^{[n-1]}r\right]^{\rho_h^{(p,q)}(f)}+O(1)\ . \qquad (25)$$

Again in view of Definition 7, Lemma 5 and Lemma 8, we get for a sequence of values of r tending to infinity that

$$\log^{[p-1]} T_{W(h)}^{-1}\left(T_{W(f)}\left(r\right)\right) \geq \left(\sigma_{W(h)}^{(p,q)}\left(W\left(f\right)\right)-\varepsilon\right)\left[\log^{[n-1]} r\right]^{\rho_{W(h)}^{(p,q)}(W(f))}$$

$$i.e.,\ \log^{[p-1]} T_{W(h)}^{-1}\left(T_{W(f)}\left(r\right)\right) \geq \left(\sigma_{h}^{(p,q)}\left(f\right)-\varepsilon\right)\left[\log^{[n-1]} r\right]^{\rho_{h}^{(p,q)}(f)} . \quad (26)$$

Now from (25) and (26), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p-1]} T_{W(h)}^{-1}\left(T_{W(f)}\left(r\right)\right)} \leq \frac{\left(\rho_h^{(p,q)}\left(f\right)+\varepsilon\right)\left(\sigma^{(m,n)}\left(g\right)+\varepsilon\right)\left[\log^{[n-1]} r\right]^{\rho_h^{(p,q)}(f)}+O(1)}{\left(\sigma_h^{(p,q)}\left(f\right)-\varepsilon\right)\left[\log^{[n-1]} r\right]^{\rho_h^{(p,q)}(f)}} .$$

Since $\varepsilon\left(>0\right)$ is arbitrary, it follows from above that

$$\varliminf_{r\to\infty} \frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p-1]} T_{W(h)}^{-1}\left(T_{W(f)}\left(r\right)\right)} \leq \frac{\rho_h^{(p,q)}\left(f\right)\cdot\sigma^{(m,n)}\left(g\right)}{\sigma_h^{(p,q)}\left(f\right)} .$$

Using the notion of (p,q)-th lower type and relative (p,q)-th lower type , we may state the following theorem without its proof as it can be carried out in the line of Theorem 15 and in view of Lemma 8.

Theorem 16. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta\left(a;f\right)+\delta\left(\infty;f\right)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite (a,p)-th type such that (i) $0<\rho_h^{(p,q)}\left(f\right)<\infty$, (ii) $\rho_h^{(p,q)}\left(f\right)=\rho^{(m,n)}\left(g\right)$, (iii) $\overline{\sigma}^{(m,n)}\left(g\right)<\infty$ and (iv) $0<\overline{\sigma}_h^{(p,q)}\left(f\right)<\infty$ where $q=n=m-1$ and $a>2$. Also let h satisfies the Property (A). Then

$$\varliminf_{r\to\infty} \frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p-1]} T_{W(h)}^{-1}\left(T_{W(f)}\left(r\right)\right)} \leq \frac{\rho_h^{(p,q)}\left(f\right)\cdot\overline{\sigma}^{(m,n)}\left(g\right)}{\overline{\sigma}_h^{(p,q)}\left(f\right)} .$$

Similarly using the notion of (p,q)-th type and relative (p,q)-th lower type one may state the following two theorems without their proofs because those can also be carried out in the line of Theorem 15.

Theorem 17. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta\left(a;f\right)+\delta\left(\infty;f\right)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite (a,p)-th type such that (i) $0<\lambda_h^{(p,q)}\left(f\right)\leq\rho_h^{(p,q)}\left(f\right)<\infty$, (ii) $\rho_h^{(p,q)}\left(f\right)=\rho^{(m,n)}\left(g\right)$, (iii) $\sigma^{(m,n)}\left(g\right)<\infty$ and (iv) $0<\overline{\sigma}_h^{(p,q)}\left(f\right)<\infty$ where $q=n=m-1$ and $a>2$. Also let h satisfies the Property (A). Then

$$\varliminf_{r\to\infty} \frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p-1]} T_{W(h)}^{-1}\left(T_{W(f)}\left(r\right)\right)} \leq \frac{\lambda_h^{(p,q)}\left(f\right)\cdot\sigma^{(m,n)}\left(g\right)}{\overline{\sigma}_h^{(p,q)}\left(f\right)} .$$

Theorem 18. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta\left(a;f\right)+\delta\left(\infty;f\right)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite

(a,p)-th type such that (i) $0<\rho_h^{(p,q)}(f)<\infty$, (ii) $\rho_h^{(p,q)}(f)=\rho^{(m,n)}(g)$, (iii) $\sigma^{(m,n)}(g)<\infty$ and (iv) $0<\overline{\sigma}_h^{(p,q)}(f)<\infty$ where $q=n=m-1$ and $a>2$. Also let h satisfies the Property (A). Then

$$\overline{\lim_{r\to\infty}}\frac{\log^{[p]}T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p-1]}T_{W(h)}^{-1}\left(T_{W(f)}(r)\right)}\leq\frac{\rho_h^{(p,q)}(f)\cdot\sigma^{(m,n)}(g)}{\overline{\sigma}_h^{(p,q)}(f)}.$$

Now using the concept of relative (p,q)-th weak type, we may state the subsequent four theorems without their proofs since those can be carried out in the line of Theorem 15, Theorem 16, Theorem 17 and Theorem 18 respectively and with help of Lemma 9.

Theorem 19. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite (a,p)-th type such that (i) $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, (ii) $\lambda_h^{(p,q)}(f)=\lambda^{(m,n)}(g)$, (iii) $\overline{\tau}^{(m,n)}(g)<\infty$ and (iv) $0<\overline{\tau}_h^{(p,q)}(f)<\infty$ where $q=n=m-1$ and $a>2$. Also let h satisfies the Property (A). Then

$$\underline{\lim_{r\to\infty}}\frac{\log^{[p]}T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p-1]}T_{W(h)}^{-1}\left(T_{W(f)}(r)\right)}\leq\frac{\rho_h^{(p,q)}(f)\cdot\overline{\tau}^{(m,n)}(g)}{\overline{\tau}_h^{(p,q)}(f)}.$$

Theorem 20. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite (a,p)-th type such that (i) $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, (ii) $\lambda_h^{(p,q)}(f)=\lambda^{(m,n)}(g)$, (iii) $\tau^{(m,n)}(g)<\infty$ and (iv) $0<\tau_h^{(p,q)}(f)<\infty$ where $q=n=m-1$ and $a>2$. Also let h satisfies the Property (A). Then

$$\underline{\lim_{r\to\infty}}\frac{\log^{[p]}T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p-1]}T_{W(h)}^{-1}\left(T_{W(f)}(r)\right)}\leq\frac{\rho_h^{(p,q)}(f)\cdot\tau^{(m,n)}(g)}{\tau_h^{(p,q)}(f)}.$$

Theorem 21. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite (a,p)-th type such that (i) $0<\lambda_h^{(p,q)}(f)<\infty$, (ii) $\lambda_h^{(p,q)}(f)=\lambda^{(m,n)}(g)$, (iii) $\overline{\tau}^{(m,n)}(g)<\infty$ and (iv) $0<\tau_h^{(p,q)}(f)<\infty$ where $q=n=m-1$ and $a>2$. Also let h satisfies the Property (A). Then

$$\underline{\lim_{r\to\infty}}\frac{\log^{[p]}T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p-1]}T_{W(h)}^{-1}\left(T_{W(f)}(r)\right)}\leq\frac{\lambda_h^{(p,q)}(f)\cdot\overline{\tau}^{(m,n)}(g)}{\tau_h^{(p,q)}(f)}.$$

Theorem 22. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite (a,p)-th type such that (i) $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, (ii) $\lambda_h^{(p,q)}(f)=\lambda^{(m,n)}(g)$, (iii) $\overline{\tau}^{(m,n)}(g)<\infty$ and (iv) $0<\tau_h^{(p,q)}(f)<\infty$ where $q=n=m-1$

and $a > 2$. Also let h satisfies the Property (A). Then

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p-1]} T_{W(h)}^{-1}\left(T_{W(f)}\left(r\right)\right)} \leq \frac{\rho_h^{(p,q)}\left(f\right)\cdot \overline{\tau}^{(m,n)}\left(g\right)}{\tau_h^{(p,q)}\left(f\right)} .$$

We may now state the following theorems without their proofs based on relative (p,q)-th type and relative (p,q)-th weak type:

Theorem 23. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta\left(a;f\right)+\delta\left(\infty;f\right)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite (a,p)-th type such that (i) $0<\lambda_h^{(p,q)}\left(f\right)\leq\rho_h^{(p,q)}\left(f\right)<\infty$, (ii) $\lambda_h^{(p,q)}\left(f\right)=\rho^{(m,n)}\left(g\right)$, (iii) $\sigma^{(m,n)}\left(g\right)<\infty$ and (iv) $0<\overline{\tau}_h^{(p,q)}\left(f\right)<\infty$ where $q=n=m-1$ and $a>2$. Also let h satisfies the Property (A). Then

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p-1]} T_{W(h)}^{-1}\left(T_{W(f)}\left(r\right)\right)} \leq \frac{\rho_h^{(p,q)}\left(f\right)\cdot \sigma^{(m,n)}\left(g\right)}{\overline{\tau}_h^{(p,q)}\left(f\right)} .$$

Theorem 24. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta\left(a;f\right)+\delta\left(\infty;f\right)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite (a,p)-th type such that (i) $0<\rho_h^{(p,q)}\left(f\right)<\infty$, (ii) $\rho_h^{(p,q)}\left(f\right)=\lambda^{(m,n)}\left(g\right)$, (iii) $\overline{\tau}^{(m,n)}\left(g\right)<\infty$ and (iv) $0<\sigma_h^{(p,q)}\left(f\right)<\infty$ where $q=n=m-1$ and $a>2$. Also let h satisfies the Property (A). Then

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p-1]} T_{W(h)}^{-1}\left(T_{W(f)}\left(r\right)\right)} \leq \frac{\rho_h^{(p,q)}\left(f\right)\cdot \overline{\tau}^{(m,n)}\left(g\right)}{\sigma_h^{(p,q)}\left(f\right)} .$$

Theorem 25. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta\left(a;f\right)+\delta\left(\infty;f\right)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite (a,p)-th type such that (i) $0<\lambda_h^{(p,q)}\left(f\right)\leq\rho_h^{(p,q)}\left(f\right)<\infty$, (ii) $\lambda_h^{(p,q)}\left(f\right)=\rho^{(m,n)}\left(g\right)$, (iii) $\overline{\sigma}^{(m,n)}\left(g\right)<\infty$ and (iv) $0<\tau_h^{(p,q)}\left(f\right)<\infty$ where $q=n=m-1$ and $a>2$. Also let h satisfies the Property (A). Then

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p-1]} T_{W(h)}^{-1}\left(T_{W(f)}\left(r\right)\right)} \leq \frac{\rho_h^{(p,q)}\left(f\right)\cdot \overline{\sigma}^{(m,n)}\left(g\right)}{\tau_h^{(p,q)}\left(f\right)} .$$

Theorem 26. Let f be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta\left(a;f\right)+\delta\left(\infty;f\right)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite (a,p)-th type such that (i) $0<\rho_h^{(p,q)}\left(f\right)<\infty$, (ii) $\rho_h^{(p,q)}\left(f\right)=\lambda^{(m,n)}\left(g\right)$, (iii) $\tau^{(m,n)}\left(g\right)<\infty$ and (iv) $0<\overline{\sigma}_h^{(p,q)}\left(f\right)<\infty$ where $q=n=m-1$ and $a>2$. Also let h satisfies the Property (A). Then

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p-1]} T_{W(h)}^{-1}\left(T_{W(f)}\left(r\right)\right)} \leq \frac{\rho_h^{(p,q)}\left(f\right)\cdot \tau^{(m,n)}\left(g\right)}{\overline{\sigma}_h^{(p,q)}\left(f\right)} .$$

Theorem 27. Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite (a,p)-th type such that (i) $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, (ii) $\lambda_h^{(p,q)}(f)=\rho^{(m,n)}(g)$, (iii) $\sigma^{(m,n)}(g)<\infty$ and (iv) $0<\tau_h^{(p,q)}(f)<\infty$ where $q=n=m-1$ and $a>2$. Also let h satisfies the Property (A). Then

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}\left(T_{W(f)}(r)\right)}\leq\frac{\lambda_h^{(p,q)}(f)\cdot\sigma^{(m,n)}(g)}{\tau_h^{(p,q)}(f)}.$$

Theorem 28. Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite (a,p)-th type such that (i) $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, (ii) $\rho_h^{(p,q)}(f)=\lambda^{(m,n)}(g)$, (iii) $\overline{\tau}^{(m,n)}(g)<\infty$ and (iv) $0<\overline{\sigma}_h^{(p,q)}(f)<\infty$ where $q=n=m-1$ and $a>2$. Also let h satisfies the Property (A). Then

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}\left(T_{W(f)}(r)\right)}\leq\frac{\lambda_h^{(p,q)}(f)\cdot\overline{\tau}^{(m,n)}(g)}{\overline{\sigma}_h^{(p,q)}(f)}.$$

Theorem 29. Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite (a,p)-th type such that (i) $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, (ii) $\lambda_h^{(p,q)}(f)=\rho^{(m,n)}(g)$, (iii) $\sigma^{(m,n)}(g)<\infty$ and (iv) $0<\tau_h^{(p,q)}(f)<\infty$ where $q=n=m-1$ and $a>2$. Also let h satisfies the Property (A). Then

$$\overline{\lim}_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}\left(T_{W(f)}(r)\right)}\leq\frac{\rho_h^{(p,q)}(f)\cdot\sigma^{(m,n)}(g)}{\tau_h^{(p,q)}(f)}.$$

Theorem 30. Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, g be an entire function and h be a transcendental entire function having the maximum deficiency sum with regular (a,p) growth and non zero finite (a,p)-th type such that (i) $0<\rho_h^{(p,q)}(f)<\infty$, (ii) $\rho_h^{(p,q)}(f)=\lambda^{(m,n)}(g)$, (iii) $\overline{\tau}^{(m,n)}(g)<\infty$ and (iv) $0<\overline{\sigma}_h^{(p,q)}(f)<\infty$ where $q=n=m-1$ and $a>2$. Also let h satisfies the Property (A). Then

$$\overline{\lim}_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}\left(T_{W(f)}(r)\right)}\leq\frac{\rho_h^{(p,q)}(f)\cdot\overline{\tau}^{(m,n)}(g)}{\overline{\sigma}_h^{(p,q)}(f)}.$$

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