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EXISTENCE OF MILD SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS OF COUPLED HYBRID FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS

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Abstract. In this paper we investigate the study of a system of two non-homogeneous boundary value problems of coupled hybrid integro-differential equations of fractional order. Our main existence result is based on a hybrid fixed point theorem due to Dhage in Banach algebras describing the sufficient conditions for the existence of a mild coupled solution of the systems of two fractional integrodifferential equations.

1. Introduction

Fractional calculus is the study of mathematical modeling of systems and processes occurring in many engineering and scientific phenomena in the form of fractional differential and integral equations. The popularity of this subject is the non-local nature of fractional order operators. Due to this reason, fractional order operators are used for describing the hereditary properties of many materials and processes. For applications in applied and biomedical sciences and engineering, we refer the reader to the research monographs [16, 17, 18]. Before stating the main problem of this paper, we recall the following basic definitions of fractional calculus [18, 20] which are useful in what follows.

Definition 1.1. If $J_\infty = [t_0, \infty)$ be an interval of the real line $\mathbb{R}$ for some $t_0 \in \mathbb{R}$ with $t_0 \geq 0$, then for any $x \in C(J_\infty, \mathbb{R})$, the Riemann-Liouville fractional integral of order $q > 0$ is defined as

$$I^q x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{x(s)}{(t-s)^{1-q}}\, ds,\ t \in J_\infty,$$

provided the right hand side is pointwise defined on (t_0, ∞).

Definition 1.2. If $x \in AC^n(J_\infty, \mathbb{R})$, then the Caputo derivative ${}^cD^q x$ of x of fractional order q is defined as

$${}^cD^q x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} x^{(n)}(s)\, ds,\ n-1 < q < n, n = [q]+1,$$

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where $[q]$ denotes the integer part of the real number q, and Γ is the Euler's gamma function. Here $AC^n(J_\infty, \mathbb{R})$ denote the space of real valued functions $x(t)$ which have continuous derivatives up to order $n-1$ on J_∞ such that $x^n(t) \in AC(J_\infty, \mathbb{R})$.

Now we state a couple of useful lemmas which are helpful in transforming the fractional differential equation into an equivalent Riemann-Louville integral equation.

Lemma 1.1 (Kilbas et.al. [16])**.** *Suppose that $x \in AC^n[0,1]$ and $\alpha \in (n-1, n)$, $n \in \mathbb{N}$. Then, the general solution of the fractional differential equation ${}^cD^\alpha x(t) = 0$ is*

$$x(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1},$$

for all $t \in [0,1]$, where C_i, $i = 0, 1, \ldots, n-1$ are constants and $AC^n[0,1]$ is the space of $(n-1)$ times continuously differentiable real-valued functions $x : [0,1] \to \mathbb{R}$ such that $x^{(n)} \in AC(J, \mathbb{R})$.

Lemma 1.2 (Podlumny [20])**.** *Let $x \in AC^n[0,1]$ and $\alpha > 0$, then*

$$I_t^{\alpha}\, {}^cD_t^\alpha x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}, i = 0, 1, \ldots, n-1,\ n = [\alpha] + 1$.

In recent days, existence of solutions of boundary value problems for coupled system of fractional order differential equations have attracted more attentions. See for example, [1, 4, 5, 6, 20] and references therein. It is because, the class of hybrid fractional differential equations includes the perturbations of original differential equations in different ways. See Dhage [11]and references therein.

To the best of our knowledge, the area concerned with the study of coupled system of hybrid fractional order differential equations is not analyzed in a well manner and very few articles are available see [5, 6, 7, 8, 9, 20, 21]. In [2], the authors studied existence and uniqueness results for the following coupled system of boundary value problems for hybrid fractional differential equations.

$$\left.\begin{aligned} {}^cD^\alpha \left(\frac{x(t)}{f(t, x(t), y(t))} \right) &= h_1(t, x(t), y(t)), \quad 0 < t < 1, \\ {}^cD^\beta \left(\frac{y(t)}{g(t, x(t), y(t))} \right) &= h_2(t, x(t), y(t)), \quad 0 < t < 1, \\ x(0) = x(1) = 0, &\quad y(0) = y(1) = 0, \end{aligned}\right\} \tag{1.1}$$

where $\alpha, \beta \in (1, 2]$, $J = [0, 1]$, ${}^cD$ is the Caputo's fractional derivative, $f, g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $h_1, h_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. The system was extended by Baleanu et.al. [6] to multi-point hybrid system and studied sufficient conditions for existence and uniqueness of solutions. Amjad Ali, et.al. [3] extended the result to the following coupled system

$$\left.\begin{aligned} {}^cD^\alpha \left(\frac{x(t) - f_1(t, x(t), y(t))}{f_2(t, x(t), y(t))} \right) &= \phi(t, x(t), y(t)) \ \text{ a.e. } t \in J = [0, 1], \\ {}^cD^\beta \left(\frac{y(t) - g_1(t, x(t), y(t))}{g_2(t, x(t), y(t))} \right) &= \psi(t, x(t), y(t)) \ \text{ a.e. } t \in J = [0, 1], \\ x(0) = a, x(1) = b, y(0) = c, y(1) = d, & \end{aligned}\right\} \tag{1.2}$$

where $\alpha, \beta \in (1, 2]$, ${}^cD$ is the Caputo fractional derivative, a, b, c, d are real constants and the nonlinear functions $f_2, g_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $f_1, g_1, \psi : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Motivated by the work mentioned above, in this paper, we study the existence of coupled solutions to the following nonhomogeneous boundary value problem of coupled hybrid integro differential equations of fractional order,

$$\left.\begin{aligned} &{}^cD^{\omega}\left(\frac{x(t) - \sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t), y(t))}{f(t, x(t), y(t))}\right) = \phi(t, x(t), y(t)) \text{ a.e. } t \in J = [0, 1], \\ &{}^cD^{\delta}\left(\frac{y(t) - \sum_{j=1}^{n} I^{\gamma_j} k_j(t, x(t), y(t))}{g(t, x(t), y(t))}\right) = \psi(t, x(t), y(t)) \text{ a.e. } t \in J = [0, 1], \\ &\qquad x(0) = a, x(1) = b, y(0) = c, y(1) = d, \end{aligned}\right\} \tag{1.3}$$

where $\omega, \delta \in (1, 2]$, ${}^cD$ is the Caputo fractional derivative a, b, c, d are real constants and the non-linear functions $f, g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $\phi, \psi : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $h_i, k_j : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ and $\beta_i > 0$ and $\gamma_j > 0$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$.

By a **coupled solution** of the coupled boundary value problem of fractional differential equations (1.3) we mean a pair of functions $(x, y) \in C^2(J, \mathbb{R}) \times C^2(J, \mathbb{R})$ that satisfies the equations in (1.3), where $C^2(J, \mathbb{R})$ is the space of twice continuously differentiable real-valued functions defined on J.

We use standard hybrid fixed point theory developed in [7, 8, 9, 10] involving the three operators in a Banach algebra to establish the sufficient conditions for existence of the coupled solutions to coupled system of quadratic fractional differential equations (1.3). We also give a numerical example to illustrate our main result of this paper.

2. Auxiliary Results

We place the nonhomogeneous boundary value problems of coupled fractional differential equations in the function space $X = C([0, 1], \mathbb{R})$ of continuous real-valued functions $f : [0, 1] \to \mathbb{R}$. Clearly, $X = C([0, 1], \mathbb{R})$ is a Banach space under the supremum norm

$$\|x\| = \sup\{|x(t)| : t \in [0, 1]\} \tag{2.1}$$

which is again a Banach algebra w.r.t. the multiplication " $\cdot$ " defined by

$$(x \cdot y)(t) = x(t) \cdot y(t). \tag{2.2}$$

Given the Banach algebra X, consider the product space $E = X \times X$ which is a vector space w.r.t. the co-ordinatewise addition and scalar multiplication. Define a norm $\|\cdot\|$ in the product linear space E by

$$\|(x, y)\| = \|x\| + \|y\|. \tag{2.3}$$

Then, the normed linear space $(E, \|(\cdot,\cdot)\|)$ is a Banach space which further becomes a Banach algebra w.r.t. the multiplication " $\cdot$ " defined by

$$\big((x,y)\cdot(u,v)\big)(t) = (x,y)(t)\cdot(u,v)(t) = \big(x(t)u(t), y(t)v(t)\big) \tag{2.4}$$

for all $t \in J$, where $(x,y),(u,v) \in X \times X = E$. The following result concerning this fact of algebraic structure of the product space $E = X \times X$ is proved in Dhage [13].

Lemma 2.1 (Dhage [13])**.** *The product space E is a Banach algebra w.r.t. the norm $\|(\cdot,\cdot)\|$ and the multiplication " $\cdot$ " defined by* (2.3) *and* (2.4) *respectively.*

Proof. Let (x,y) and (u,v) be any two elements of E. Then, by definitions of the the norm $\|(\cdot,\cdot)\|$ and the multiplication " $\cdot$ " in E, we obtain

$$\|\big((x,y)\cdot(u,v)\big)\| = \|\big(xu, yv\big)\| = \|x\,u\| + \|y\,v\| \leq \|x\|\,\|u\| + \|y\|\,\|v\|, \tag{2.5}$$

and

$$\|(x,y)\|\,\|(u,v)\| = \big[\|x\| + \|u\|\big]\big[\|y\| + \|v\|\big]. \tag{2.6}$$

From (2.5) and (2.6) it follows that

$$\|\big((x,y)\cdot(u,v)\big)\| \leq \|(x,y)\|\,\|(u,v)\|.$$

This shows that $(E, \|\cdot\|, \cdot)$ is a Banach algebra and the proof of lemma is complete. □

We employ the following hybrid fixed point theorem of Dhage [7, 8] as a basic tool for proving the main existence result of coupled solutions of this paper. We need the following definition in what follows.

Definition 2.1 (Dhage [6, 7])**.** Let E be a Banach space An operator $T : E \to E$ is called Lipschitz if there exists a constant $L_T > 0$ such that

$$\|T(x) - T(y)\| \leq L_T\,\|x - y\|$$

for all elements $x, y \in X$.

Theorem 2.1 (Dhage [7, 8])**.** *Let S be a closed convex and bounded subset of the Banach algebra X and let $\mathcal{A}, \mathcal{C} : X \to X$ and $\mathcal{B} : S \to X$ be three operators such that*

(a) *$\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian with Lipschitz constants $L_{\mathcal{A}}$ and $L_{\mathcal{C}}$ respectively,*
(b) *$\mathcal{B}$ is compact and continuous,*
(c) *$x = \mathcal{A}x\mathcal{B}y + \mathcal{C}x\ \forall y \in S \implies x \in S$, and*
(d) *$L_{\mathcal{A}}M_{\mathcal{B}} + L_{\mathcal{C}} < 1$, where $M_{\mathcal{B}} = \|\mathcal{B}(S)\| = \sup\{\|\mathcal{B}x\| : x \in S\}$.*

Then the operator equation $x = \mathcal{A}x\mathcal{B}x + \mathcal{C}x$ has a solution in S.

We need the following assumptions in the sequel.

(H1) There exists a constant $L_f > 0$ such that

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \leq L_f\left(|x - \bar{x}| + |y - \bar{y}|\right),$$

for all $t \in J$ and $x, \bar{x}, y, \bar{y} \in \mathbb{R}$.

(H2) There exists a constant $L_g > 0$ such that

$$|g(t,x,y) - g(t,\bar{x},\bar{y})| \leq L_g\left(|x - \bar{x}| + |y - \bar{y}|\right)$$

for all $t \in J$ and $x, \bar{x}, y, \bar{y} \in \mathbb{R}$.

(H3) There exists bounded functions $L_{h_i}, L_{k_j} : J \to \mathbb{R}_+$, with bounds $\|L_{h_i}\|$ and $\|L_{k_j}\|$ such that

$$|h_i(t,x,y) - h_i(t,\bar{x},\bar{y})| \le L_{h_i}(t)\left(|x-\bar{x}| + |y-\bar{y}|\right), \quad i = 1,\dots,m,$$

and

$$|k_j(t,x,y) - k_j(t,\bar{x},\bar{y})| \le L_{k_j}(t)\left(|x-\bar{x}| + |y-\bar{y}|\right), \quad j = 1,\dots,n.$$

for $t \in J$ and $x, \bar{x}, y, \bar{y} \in \mathbb{R}$.

(H4) There exist constants $M_1 > 0$ and $M_2 > 0$ such that

$$\left|\frac{a}{f(0,a,c)}\right| \le M_1 \quad \text{and} \quad \left|\frac{c}{g(0,a,c)}\right| \le M_2$$

and

$$\begin{aligned}
F_0 &= \sup_{t\in J}|f(t,0,0)|, \quad G_0 = \sup_{t\in J}|g(t,0,0)| \\
G_1(t,s) &= \sup_{t\in[0,1]}\left[\int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)}ds - t\int_0^1 \frac{(1-s)^{\omega-1}}{\Gamma(\omega)}ds\right], \\
G_2(t,s) &= \sup_{t\in[0,1]}\left[\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}ds - t\int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)}ds\right] \\
M_{G_i} &= \max\left\{|G_i(t,s)| : (t,s) \in [0,1]\times[0,1]\right\}, \ i = 1,2; \\
H_0 &= \sup_{t\in J}|h_i(t,0,0)| \ \text{ for all } , i = 1,2,\cdots,m \quad \text{and} \\
K_0 &= \sup_{t\in J}|k_j(t,0,0)| \ \text{ for all } j = 1,2,\cdots,n.
\end{aligned}$$

(H5) There exist constants $M_{h_i} > 0, M_{k_j} > 0$ such that

$$|h_i(t,x,y)| \le M_{h_i} \quad \text{and} \quad |k_j(t,x,y)| \le M_{k_j}$$

for all $(t,x,y) \in J\times\mathbb{R}\times\mathbb{R}$ and $i = 1,\dots,m,\ j = 1,\dots,n$.

(H6) The real functions ϕ and ψ are bounded on $J\times\mathbb{R}\times\mathbb{R}$ with bounds M_ϕ and M_ψ respectively.

(H7) The constants in the hypotheses (H1) through (H6) satisfy the following conditions,

$$\Omega = L_f\left(M_{G_1}M_\phi + M_1 + \frac{|b| + \sum_{i=1}^{m}\frac{M_{h_i}}{\Gamma(\beta_{i+i})}}{|f(1,a,c)|}\right) + L_g\left(M_{G_2}M_\psi + M_2 + \frac{|d| + \sum_{j=1}^{n}\frac{M_{k_j}}{\Gamma(\gamma_{j+1})}}{|g(1,b,d)|}\right) < 1.$$

3. Existence Result

In this section, we prove our main existence result for mild coupled solutions of the coupled differential equations of fractional order (1.3). The following useful lemma is immediate and follows from the theory of fractional calculus.

Lemma 3.1. *If a function $x \in C^m([0,1],\mathbb{R},\mathbb{R})$ is a solution of the hybrid fractional integrodifferential equation*

$$\left.\begin{array}{r} {}^cD^{\omega}\left(\dfrac{x(t)-\sum\limits_{i=1}^{m} I^{\beta_i} h_i(t,x(t),y(t))}{f(t,x(t),y(t))}\right) = \phi(t,x(t),y(t)) \quad a.e.\ t \in J=[0,1], \\ x(0)=a, x(1)=b, \end{array}\right\} \tag{3.1}$$

then it satisfies the following hybrid fractional integral equation

$$\begin{aligned} x(t) &= \sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t),y(t)) + \Big[f(t,x(t),y(t))\Big]\times \\ &\quad \times \Bigg[\int_0^1 G_1(t,s)\phi(s,x(s),y(s))ds + (1-t)\frac{a}{f(0,a,y(0))} \\ &\quad + t\left(\frac{b-\sum\limits_{i=1}^{m}\int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,x(s),y(s))ds}{f(1,b,y(1))}\right)\Bigg], \quad t \in J. \end{aligned} \tag{3.2}$$

Proof. Applying Riemann-Liouville fractional integral of order ω on both sides of (3.1) and using Lemmas 1.1 and1.2, we obtain

$$\frac{x(t)-\sum\limits_{i=1}^{m} I^{\beta_i} h_i(t,x(t),y(t))}{f(t,x(t),y(t))} = \int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)}\phi(s,x(s),y(s))ds + C_1 + C_2 t \tag{3.3}$$

which implies

$$\begin{aligned} x(t) &= \sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t),y(t)) + \Big[f(t,x(t),y(t))\Big]\times \\ &\quad \times \left(\int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)}\phi(s,x(s),y(s))ds + C_1 + C_2 t\right). \end{aligned} \tag{3.4}$$

Using the boundary conditions $x(0)=a$, $x(1)=b$, we have

$$C_1 = \frac{x(0)}{f(0,x(0),y(0)} = \frac{a}{f(0,a,y(0))}$$

$$\begin{aligned} C_2 = &\frac{b-\sum\limits_{i=1}^{m}\int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,x(s),y(s))ds}{f(1,b,y(1))} \\ &- \int_0^1 \frac{(1-s)^{\omega-1}}{\Gamma(\omega)}\phi(s,x(s),y(s))ds - \frac{a}{f(0,a,y(0))}. \end{aligned}$$

Substituting C_1, C_2 in (3.4),

$$x(t) = \sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t),y(t)) + \Big[f(t,x(t),y(t))\Big]\times$$

$$
\begin{aligned}
&\times \Bigg[\int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)}\phi(s,x(s),y(s))ds + \frac{a}{f(0,a,y(0))} \\
&+ t\left(\frac{b - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,x(s),y(s))ds}{f(1,b,y(1))} \right. \\
&\left. - \int_0^1 \frac{(1-s)^{\omega-1}}{\Gamma(\omega)}\phi(s,x(s),y(s))ds - \frac{a}{f(0,a,y(0))} \right) \Bigg] \\
= &\sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t),y(t)) + \Big[f(t,x(t),y(t)) \Big] \times \\
&\times \Bigg[\int_0^1 G(t,s)\phi(s,x(s),y(s))ds + (1-t)\frac{a}{f(0,a,y(0))} \\
&+ t\left(\frac{b - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,x(s),y(s))ds}{f(1,b,y(1))} \right) \Bigg] \qquad (3.5)
\end{aligned}
$$

where,

$$
G(t,s) = \left[\int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)} ds - t\int_0^1 \frac{(1-s)^{\omega-1}}{\Gamma(\omega)} ds \right].
$$

The proof is complete. □

Theorem 3.1. *A solution $u \in C(J,\mathbb{R})$ of the hybrid fractional integral equation 3.2 is called a mild solution of the fractional differential equation 3.1 defined on J. Similarly, a mild coupled solution $(u,v) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ of a system of the coupled fractional differential equations 1.3 is defined on J.*

We use the notations

$$
A = \left| \frac{a}{f(0,a,c)} \right| + \left| \frac{b + \sum_{i=1}^{m} \frac{M_{h_i}}{\Gamma(\beta_i+1)}}{f(1,b,d)} \right|,
$$

$$
B = \left| \frac{c}{g(0,a,c)} \right| + \left| \frac{d + \sum_{j=1}^{n} \frac{M_{k_j}}{\Gamma(\gamma_j+1)}}{g(1,b,d)} \right|.
$$

Theorem 3.2. *Assume that the hypotheses $(H1)-(H7)$ hold. Furthermore, if*

$$
\begin{aligned}
&\big(L_f + L_g\big)\Big(A + M_{G_1} M_\phi + B + M_{G_2} M_\psi\Big) \\
&\qquad + \sum_{i=1}^{m} \frac{\|L_{h_i}\|}{\Gamma(\beta_i+1)} + \sum_{j=1}^{n} \frac{\big\|L_{k_j}\big\|}{\Gamma(\gamma_j+1)} < 1, \qquad (3.6)
\end{aligned}
$$

then the coupled system (1.3) has a mild coupled solution defined on J.

Proof. By Lemma 3.1, the mild coupld solutions of the coupled fractional integrodifferential equations in (1.3) are the solutions to the coupled fractional integral equations,

$$\begin{aligned} x(t) = \sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t),y(t)) + \Big[f(t,x(t),y(t))\Big] \times \\ \times \left[\int_0^1 G_1(t,s)\phi(s,x(s),y(s))ds + (1-t)\frac{a}{f(0,a,c)} \right. \\ \left. + t\left(\frac{b - \sum_{i=1}^{m}\int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,x(s),y(s))ds}{f(1,b,d)}\right)\right] \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} y(t) = \sum_{j=1}^{n} I^{\gamma_j} k_j(t,x(t),y(t)) + \Big[f(t,x(t),y(t))\Big] \times \\ \times \left[\int_0^1 G_2(t,s)\psi(s,x(s),y(s))ds + (1-t)\frac{a}{f(0,a,c)} \right. \\ \left. + t\left(\frac{b - \sum_{j=1}^{n}\int_0^1 \frac{(1-s)^{\gamma_j-1}}{\Gamma(\gamma_j)} k_j(s,x(s),y(s))ds}{f(1,b,d)}\right)\right]. \end{aligned} \tag{3.8}$$

Choose

$$\rho \geq \frac{F_0\,[M_{G_1}\,M_\phi + A] + G_0\,[M_{G_2}\,M_\psi + B] + \sum_{i=1}^{n}\frac{\|L_{\phi_i}\|}{\Gamma(\rho_{i+1})} + \sum_{j=1}^{m}\frac{\|L_{\psi_j}\|}{\Gamma(\gamma_{j+1})}}{1-\Omega}$$

and define a subset S of the Banach space $X \times X$ by

$$S = \{(x,y) \in X \times X : \|(x,y)\| \leq \rho\}. \tag{3.9}$$

Clearly, S is a closed, convex and bounded subset of the Banach space $E = X \times X$. Define the operators $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2) : E \to E$, $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2) : E \to E$ and $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2) : S \to E$ by

$$\left.\begin{aligned} \mathcal{A}_1(x,y) = f(t,x(t),y(t)),\,,\ t \in J, \\ \mathcal{A}_2(x,y) = g(t,x(t),y(t)),\ t \in J, \end{aligned}\right\} \tag{3.10}$$

$$\left.\begin{aligned}\mathcal{B}_1(x,y) &= \int_0^1 G_1(t,s)\phi(s,x(s),y(s))ds + (1-t)\frac{a}{f(0,a,c)}\\ &\quad + t\left(\frac{b-\sum_{i=1}^m \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} h_i(s,x(s),y(s))ds}{f(1,b,d)}\right),\ t\in J,\\ \mathcal{B}_2(x,y) &= \int_0^1 G_2(t,s)\psi(s,x(s),y(s))ds + (1-t)\frac{c}{g(0,a,c))}\\ &\quad + t\left(\frac{d-\sum_{j=1}^n \int_0^1 \frac{(1-s)^{\gamma_j-1}}{\Gamma(\gamma_j)} k_j(s,x(s),y(s))ds}{g(1,b,d)}\right),\ t\in J,\end{aligned}\right\} \tag{3.11}$$

and

$$\left.\begin{aligned}\mathcal{C}_1(x,y) &= \sum_{i=1}^m I^{\beta_i} h_i(t,x(t),y(t)),\ t\in J,\\ \mathcal{C}_2(x,y) &= \sum_{j=1}^n I^{\gamma_j} k_j(t,x(t),y(t)),\ t\in J.\end{aligned}\right\} \tag{3.12}$$

Then the coupled system of hybrid integral equations (3.7) and (3.8) can be written as the system of operator equations as

$$\mathcal{A}(x,y)(t)\,\mathcal{B}(x,y)(t) + \mathcal{C}(x,y)(t) = (x,y)(t),\ t\in J, \tag{3.13}$$

which further in view of the multiplication (2.4) of two elements in E yields

$$\begin{aligned}&\Big(\mathcal{A}_1(x,y)(t)\mathcal{B}_1(x,y)(t) + \mathcal{C}_1(x,y)(t)\,,\ \mathcal{A}_2(x,y)(t)\mathcal{B}_2(x,y)(t) + \mathcal{C}_2(x,y)(t)\Big)\\ &\qquad = (x,y)(t), \quad t\in[0,1],\end{aligned} \tag{3.14}$$

This further implies that

$$\left.\begin{aligned}\mathcal{A}_1(x,y)(t)\mathcal{B}_1(x,y)(t) + \mathcal{C}_1(x,y)(t) = x(t), \quad t\in[0,1],\\ \mathcal{A}_2(x,y)(t)\mathcal{B}_2(x,y)(t) + \mathcal{C}_2(x,y)(t) = y(t), \quad t\in[0,1].\end{aligned}\right\} \tag{3.15}$$

Now we prove that the operators, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ satisfy the conditions of Theorem 2.1 in a series of following steps.

Step I: First we show that $\mathcal{A} = (\mathcal{A}_1,\mathcal{A}_2)$ and $\mathcal{C} = (\mathcal{C}_1,\mathcal{C}_2)$ are Lipschitzian on E with Lipschitz constants $(L_f + L_g)$ and $\left(\sum_{i=1}^m \frac{\|L_{h_i}\|}{\Gamma(\beta_i+1)} + \sum_{j=1}^n \frac{\|L_{k_j}\|}{\Gamma(\gamma_j+1)}\right)$ respectively. Let $(x,y),(\bar{x},\bar{y})\in E$ be arbitrary. Then, using (H4), we have

$$\begin{aligned}|\mathcal{A}_1(x,y)(t) - \mathcal{A}_1(\bar{x},\bar{y})(t) &= |f(t,x(t),y(t)) - f(t,\bar{x}(t),\bar{y}(t))|\\ &\le L_f\left(|x(t)-\bar{x}(t)| + |y(t)-\bar{y}(t)|\right)\\ &\le L_f\big(\|x-\bar{x}\| + \|y-\bar{y}\|\big)\end{aligned}$$

for all $t\in J$. Taking the supremum over t, we obtain

$$\|\mathcal{A}_1(x,y) - \mathcal{A}_1(\bar{x},\bar{y})\| \le L_f\big(\|x-\bar{x}\| + \|y-\bar{y}\|\big)$$

for all $(x, y), (\bar{x}, \bar{y}) \in E$. Similarly, we obtain

$$\|\mathcal{A}_2(x, y) - \mathcal{A}_2(\bar{x}, \bar{y})\| \leq L_g\big(\|x - \bar{x}\| + \|y - \bar{y}\|\big)$$

for all $(x, y), (\bar{x}, \bar{y}) \in E$. Therefore, by definition of the operator $\mathcal{A}$, we obtain

$$\begin{aligned}
&\|\mathcal{A}(x, y) - \mathcal{A}(\bar{x}, \bar{y})\| \\
&\qquad = \Big\|\big(\mathcal{A}_1(x, y), \mathcal{A}_2(x, y)\big) - \big(\mathcal{A}_1(\bar{x}, \bar{y}), \mathcal{A}_2(\bar{x}, \bar{y})\big)\Big\| \\
&\qquad = \Big\|\big(\mathcal{A}_1(x, y) - \mathcal{A}_1(\bar{x}, \bar{y})\,,\, \mathcal{A}_2(x, y) - \mathcal{A}_2(\bar{x}, \bar{y})\big)\Big\| \\
&\qquad \leq \|\mathcal{A}_1(x, y) - \mathcal{A}_1(\bar{x}, \bar{y})\| + \|\mathcal{A}_2(x, y) - \mathcal{A}_2(\bar{x}, \bar{y})\| \\
&\qquad \leq L_f\big(\|x - \bar{x}\| + \|y - \bar{y}\|\big) + L_g\big(\|x - \bar{x}\| + \|y - \bar{y}\|\big) \\
&\qquad = (L_f + L_g)\big(\|x - \bar{x}\| + \|y - \bar{y}\|\big)
\end{aligned}$$

for all $(x, y), (\bar{x}, \bar{y}) \in E$, where $L_{\mathcal{A}} = (L_f + L_g)$.

Similarly, by definition of the operator $\mathcal{C}$, we obtain

$$\begin{aligned}
&\big|\mathcal{C}_1(x, y)(t) - \mathcal{C}_1(\bar{x}, \bar{y})(t)\big| \\
&\qquad = \left|\sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t), y(t)) - \sum_{i=1}^{m} I^{\beta_i} h_i(t, \bar{x}(t), \bar{y}(t))\right| \\
&\qquad \leq \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\beta_i - 1}}{\Gamma(\beta_i)} L_{h_i}(s)\left(|x(s) - \bar{x}(s)| + |y(s) - \bar{y}(s)|\right) ds \\
&\qquad \leq \sum_{i=1}^{m} \frac{\|L_{h_i}\|}{\Gamma(\beta_i + 1)}\left(\|(x - \bar{x})\| + \|(y - \bar{y}\|\right),
\end{aligned}$$

for all $t \in [0, 1]$. Taking the supremum over t, we get

$$\|\mathcal{C}_1(x, y) - \mathcal{C}_1(\bar{x}, \bar{y})\| \leq \sum_{i=1}^{m} \frac{\|L_{h_i}\|}{\Gamma(\beta_i + 1)}\left(\|(x - \bar{x})\| + \|(y - \bar{y}\|\right)$$

for all $(x, y), (\bar{x}, \bar{y}) \in E$.

Similarly, we can prove that $\mathcal{C}_2$ is also a Lipschitzian with Lipschitz constant $\sum\limits_{j=1}^{n} \dfrac{\big\|\|L_{k_j}\|\big\|}{\Gamma(\gamma_j + 1)}$, that is,

$$\|\mathcal{C}_2(x, y) - \mathcal{C}_2(\bar{x}, \bar{y})\| \leq \sum_{j=1}^{n} \frac{\|L_{k_j}\|}{\Gamma(\gamma_j + 1)}\left(\|(x - \bar{x})\| + \|(y - \bar{y}\|\right)$$

for all $(x, y), (\bar{x}, \bar{y}) \in E$.

Hence, it follows that

$$\|\mathcal{C}(x, y) - \mathcal{C}(\bar{x}, \bar{y})\| \leq \left(\sum_{i=1}^{m} \frac{\|L_{h_i}\|}{\Gamma(\beta_i + 1)} + \sum_{j=1}^{n} \frac{\|L_{k_j}\|}{\Gamma(\gamma_j + 1)}\right)\left(\|(x - \bar{x})\| + \|(y - \bar{y}\|\right)$$

for all $(x, y), (\bar{x}, \bar{y}) \in E$, that is, $\mathcal{C}$ is a Lipschitzian with Lipschitz constant

$$L_C = \left(\sum_{i=1}^{m} \frac{\|L_{h_i}\|}{\Gamma(\beta_i + 1)} + \sum_{j=1}^{n} \frac{\|L_{k_j}\|}{\Gamma(\gamma_j + 1)}\right).$$

Step II: Now we show that $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$ is compact and continuous operator from S into E. For continuity of $\mathcal{B}$, let (x_n, y_n) be a sequence of points in S converging to a point $(x, y) \in S$. Then, by Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{B}_1(x_n, y_n)(t) &= \lim_{n\to\infty} \int_0^1 G_1(t,s)\phi(s, x_n(s), y_n(s))ds + (1-t)\frac{a}{f(0,a,c)} \\
&\quad + t \lim_{n\to\infty} \left(\frac{b - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta_i - 1}}{\Gamma(\beta_i)} h_i(s, x_n(s), y_n(s))ds}{f(1,b,d)} \right) \\
&= \int_0^1 G_1(t,s) \left[\lim_{n\to\infty} \phi(s, x_n(s), y_n(s)) \right] ds + (1-t)\frac{a}{f(0,a,c)} \\
&\quad + t \left(\frac{b - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta_i - 1}}{\Gamma(\beta_i)} \left[\lim_{n\to\infty} h_i(s, x_n(s), y_n(s)) \right] ds}{f(1,b,d)} \right) \\
&= \int_0^1 G_1(t,s)\phi(s, x(s), y(s))ds + (1-t)\frac{a}{f(0,a,c)} \\
&\quad + t \left(\frac{b - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta_i - 1}}{\Gamma(\beta_i)} h_i(s, x(s), y(s))ds}{f(1,b,d)} \right) \\
&= \mathcal{B}_1(x, y)(t)
\end{aligned}$$

for al $t \in [0, 1]$. Similarly, we prove

$$\lim_{n\to\infty} \mathcal{B}_2(x_n, y_n)(t) = \mathcal{B}_2(x, y)(t)$$

for all $t \in [0, 1]$. Hence $\mathcal{B}(x_n, y_n) = (\mathcal{B}_1(x_n, y_n); \mathcal{B}_2(x_n, y_n))$ converges to $\mathcal{B}(x, y)$ pointwise on $[0, 1]$.

Next we show that $\{\mathcal{B}(x_n, y_n)\}$ is equi-continuous sequence of functions in E. Choose $\tau_1, \tau_2 \in [0, 1]$ such that $\tau_1 < \tau_2$, then

$$\begin{aligned}
&\left| \mathcal{B}_1(x_n, y_n)(\tau_1) - \mathcal{B}_1(x_n, y_n)(\tau_2) \right| \\
&\quad \leq \left| \int_0^1 \left(G_1(\tau_1, s) - G_1(\tau_2, s) \right) \phi(s, x_n(s), y_(s)ds \right| \\
&\quad + |\tau_2 - \tau_1| \left(\frac{|b| + M_0 \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta_i - 1}}{\Gamma(\beta_i)} |h_i(s, x_n(s), y_n(s))| ds}{|f(1,b,d)|} \right) \\
&\quad + |\tau_2 - \tau_1| \frac{a}{f(0,a,c)} \\
&\quad \leq M_\phi \int_0^1 |G_1(\tau_1, s) - G_1(\tau_2, s)| ds \\
&\quad + |\tau_2 - \tau_1| \left(\frac{|b| + M_0 \sum_{i=1}^{m} M_{h_i} \int_0^1 \frac{(1-s)^{\beta_i - 1}}{\Gamma(\beta_i)} ds}{|f(1,b,d)|} \right)
\end{aligned}$$

$$
\begin{aligned}
&+ M_1 \, |\tau_2 - \tau_1| \\
&\to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}
$$

uniformly for all $n \in \mathbb{N}$ which implies that $\mathcal{B}_1(x_n, y_n) \to \mathcal{B}_1(x, y)$ uniformly and hence is uniformly continuous on E. Similarly we can prove that $\mathcal{B}_2$ is uniformly continuous. Thus $\mathcal{B}$ is uniformly continuous on E.

Next, we show that $\mathcal{B}$ is a compact operator on S. Let $(x, y) \in S$ be any point. Then, using (A4), we have

$$
\begin{aligned}
|\mathcal{B}_1(x,y)(t)| = \Bigg| &\int_0^1 G_1(t,s)\phi(s,x(s),y(s))ds \\
&+ t\left(\frac{b - \sum\limits_{i=1}^m \int_0^1 \frac{(1-s)^{\beta_i - 1}}{\Gamma(\beta_i)} h_i(s,x(s),y(s))ds}{f(1,b,d)} - \frac{a}{f(0,a,c)}\right)\Bigg| \\
\leq M_{G_1} &\int_0^1 |\phi(s,x(s),y(s))|\, ds \\
&+ \left|\frac{b - \sum\limits_{i=1}^m \int_0^1 \frac{(1-s)^{\beta_i - 1}}{\Gamma(\beta_i)} h_i(s,x(s),y(s))ds}{f(1,b,d)} - \frac{a}{f(0,a,c)}\right| \\
\leq M_{G_1} &M_\phi + M_1 + \frac{|b| + \sum\limits_{i=1}^m \frac{M_{h_i}}{\Gamma(\beta_{i+i})}}{|f(1,a,c)|}
\end{aligned}
$$

Taking the supremum over t in the above inequality, we obtain

$$
\big\|\mathcal{B}_1(x,y)\big\| \leq M_{G_1} M_\phi + M_1 + \frac{|b| + \sum\limits_{i=1}^m \frac{M_{h_i}}{\Gamma(\beta_{i+i})}}{|f(1,a,c)|}
$$

for all $(x, y) \in S$. Hence $\mathcal{B}_1$ is a uniformly bounded operator on S. Similarly we can show that $\mathcal{B}_2$ is also uniformly bounded operator on S. Hence $\mathcal{B}$ is a uniformly bounded operator on S. Next, let $(x, y) \in S$ be an arbitrary point and let $t, r \in J$. Then, we have

$$
\begin{aligned}
&\big|\mathcal{B}_1(x,y)(t) - \mathcal{B}_1(x,y)(r)\big| \\
&\quad \leq \left|\int_0^1 \left(G_1(t,s) - G_1(r,s)\right)\phi(s,x_n(s),y_(s)ds\right| \\
&\quad + |t-r|\left(\frac{|b| + \sum\limits_{i=1}^m \int_0^1 \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} |h_i(s,x_n(s),y_n(s))|\, ds}{|f(1,b,d)|}\right) \\
&\quad + |t-r| \frac{a}{f(0,a,c)} \\
&\quad \leq M_\phi \int_0^1 |G_1(t,s) - G_1(r,s)|\, ds
\end{aligned}
$$

$$
\begin{aligned}
&+ |t-r| \left(\frac{|b| + \sum\limits_{i=1}^{m} M_{h_i} \int_0^1 \frac{(1-s)^{\beta_i - 1}}{\Gamma(\beta_i)}\, ds}{|f(1,b,d)|} \right) \\
&+ M_1 \; |t-r| \\
&\to 0 \quad \text{as} \quad t \to r,
\end{aligned}
$$

uniformly for all $(x,y) \in S$. Similarly, we have

$$
|\mathcal{B}_2(x,y)(t) - \mathcal{B}_2(x,y)(r)| \to 0 \quad \text{as} \quad t \to r
$$

uniformly for all $(x,y) \in S$. Hence, it follows that

$$
|\mathcal{B}(x,y)(t) - \mathcal{B}(x,y)(r)| \to 0 \quad \text{as} \quad t \to r
$$

uniformly for all $(x,y) \in S$. Now, $\mathcal{B}(S)$ is uniformly bounded and equicontinuou subset of the Banach space E, it is compact subset of E in view of Arzelá-Ascoli theorem. Consequently, $\mathcal{B}$ is compact and continuous operator on S.

Step III: Now we prove the third condition (c) of Theorem 2.1 holds. Let for (x,y) and (u,v) be two elements in $E = X \times X$ such that

$$
(x,y) = \left(\mathcal{A}_1(x,y)\mathcal{B}_1(u,v) + \mathcal{C}_1(x,y)\, ,\, \mathcal{A}_2(x,y)\mathcal{B}_2(u,v) + \mathcal{C}_2(x,y) \right).
$$

Then, we have

$$
\begin{aligned}
|x(t)| &= |\mathcal{A}_1(x,y)(t)\mathcal{B}_1(u,v)(t) + \mathcal{C}_1(x,y)| \\
&\le |\mathcal{A}_1(x,y)(t) B_1(u,v)(t) + \mathcal{C}_1(x,y)| \\
&\le \Big[\, |f(t,x,y) - f(t,0,0)| + |f(t,0,0)| \,\Big] \left(\int_0^1 \Big| G_1(t,s)\phi(s,u(s),v(s)) \Big| ds \right. \\
&\quad \left. + \left| \frac{a}{f(0,a,c)} \right| + \left| \frac{|b| + \sum\limits_{i=1}^{m} \frac{M_{h_i}}{\Gamma(\beta_i+1)}}{|f(1,a,c)|} \right| \right) \\
&\quad + \sum_{i=1}^{m} I^{\beta_i} \, |h_i(t,x(t),y(t))| \\
&\le \Big[L_f \big(\|x\| + \|y\| \big) + F_0 \Big] \times \\
&\quad \times \left(M_{G_1} \, M_\phi + M_1 + \frac{|b| + \sum\limits_{i=1}^{m} \frac{M_{h_i}}{\Gamma(\beta_i+1)}}{|f(1,a,c)|} \right) \\
&\quad + \left(\sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\beta_i - 1}}{\Gamma(\beta_1)} \, |h_i(s,x(s),y(s) - h_i(s,0,0))| \, ds + H_0 \right) \\
&\le \Big[L_f \big(\|x\| + \|y\| \big) + F_0 \Big] \times \\
&\quad \times \left(M_{G_1} \, M_\phi + M_1 + \frac{|b| + \sum\limits_{i=1}^{m} \frac{M_{h_i}}{\Gamma(\beta_i+1)}}{|f(1,a,c)|} \right) \\
&\quad + \left(\sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\beta_i - 1}}{\Gamma(\beta_1)} \Big[L_{h_i} \big(\|x\| + \|y\| \big) + H_0 \Big] \right)
\end{aligned}
$$

$$\leq \Big[L_f\big(\|x\| + \|y\|\big) + F_0\Big]\times$$
$$\times \left(M_{G_1}\, M_\phi + M_1 + \frac{|b| + \sum\limits_{i=1}^{m} \dfrac{M_{h_i}}{\Gamma(\beta_i+1)}}{|f(1,a,c)|} \right)$$
$$+ \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+1)} \big[L_{h_i}\big(\|x\| + \|y\|\big) + H_0\big] \tag{3.16}$$

Taking the supremum in the above inequality (3.16), we obtain

$$\|x\| \leq \Big[L_f\big(\|x\| + \|y\|\big) + F_0\Big]\times$$
$$\times \left(M_{G_1}\, M_\phi + M_1 + \frac{|b| + \sum\limits_{i=1}^{m} \dfrac{M_{h_i}}{\Gamma(\beta_i+1)}}{|f(1,a,c)|} \right)$$
$$+ \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+1)} \big[L_{h_i}\big(\|x\| + \|y\|\big) + H_0\big]. \tag{3.17}$$

Similarly, proceeding with the analogous arguments, we obtain

$$\|y\| \leq \Big[L_g\big(\|x\| + \|y\|\big) + G_0\Big]\times$$
$$\times \left(M_{G_2}\, M_\psi + M_2 + \frac{|b| + \sum\limits_{j=1}^{n} \dfrac{M_{k_j}}{\Gamma(\gamma_j+1)}}{|f(1,a,c)|} \right)$$
$$+ \sum_{j=1}^{n} \frac{1}{\Gamma(\gamma_j+1)} \big[L_{k_j}\big(\|x\| + \|y\|\big) + K_0\big] \tag{3.18}$$

Adding the inequalities (3.17) and (3.18), we obtain

$$\|x\| + \|y\| \leq \Big[L_f\big(\|x\| + \|y\|\big) + F_0\Big]\times$$
$$\times \left(M_{G_1}\, M_\phi + M_1 + \frac{|b| + \sum\limits_{i=1}^{m} \dfrac{M_{h_i}}{\Gamma(\beta_i+1)}}{|f(1,a,c)|} \right)$$
$$+ \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+1)} \big[L_{h_i}\big(\|x\| + \|y\|\big) + H_0\big]$$
$$+ \Big[L_g\big(\|x\| + \|y\|\big) + G_0\Big]\times$$
$$\times \left(M_{G_2}\, M_\psi + M_2 + \frac{|b| + \sum\limits_{j=1}^{n} \dfrac{M_{k_j}}{\Gamma(\gamma_j+1)}}{|f(1,a,c)|} \right)$$
$$+ \sum_{j=1}^{n} \frac{1}{\Gamma(\gamma_j+1)} \big[L_{k_j}\big(\|x\| + \|y\|\big) + K_0\big]$$
$$\leq \Big[L_f\big(\|x\| + \|y\|\big) + F_0\Big]\Big(M_{G_1}\, M_\phi + A\Big)$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+1)}\big[L_{h_i}\big(\|x\|+\|y\|\big)+H_0\big]\\
&+\Big[L_g\big(\|x\|+\|y\|\big)+G_0\Big]\Big(M_{G_2}\,M_\psi+B\Big)\\
&+\sum_{j=1}^{n}\frac{1}{\Gamma(\gamma_j+1)}\big[L_{k_j}\big(\|x\|+\|y\|\big)+K_0\big]\\
&\le\frac{F_0\Big(M_{G_1}\,M_\phi+A\Big)+G_0\Big(M_{G_2}\,M_\psi+B\Big)+\sum_{i=1}^{m}\frac{M_{\phi_i}}{\Gamma(\beta_{i+1})}+\sum_{j=1}^{n}\frac{M_{\psi_j}}{\Gamma(\gamma_{j+1})}}{1-\Omega}\\
&\le\rho,
\end{aligned}
$$

As $\|(x,y)\|=\|x\|+\|y\|$, we have that $\|(x,y)\|\le\rho$ and so the hypothesis (c) of Theorem 2.1 is satisfied.

Step IV: Finally, we have

$$
\begin{aligned}
M_B=\|\mathcal{B}(S)\|&=\sup\big\{\|\mathcal{B}(x,y)\|:(x,y)\in S\big\}\\
&=\sup\big\{\ \|\mathcal{B}_1(x,y)\|+\|\mathcal{B}_2(x,y)\|:(x,y)\in S\ \big\}\\
&\le A+M_{G_1}\,M_\phi+B+M_{G_2}\,M_\psi.
\end{aligned}
$$

From above estimate, we obtain

$$
\begin{aligned}
L_{\mathcal{A}}M_{\mathcal{B}}+L_{\mathcal{C}}&\le\big(L_f+L_g\big)\Big(A+M_{G_1}\,M_\phi+B+M_{G_2}\,M_\psi)\Big)\\
&\qquad+\sum_{i=1}^{m}\frac{\|L_{h_i}\|}{\Gamma(\beta_i+1)}+\sum_{j=1}^{n}\frac{\big\|L_{k_j}\big\|}{\Gamma(\gamma_j+1)}\\
&<1
\end{aligned}
$$

and so the hypothesis(d) of Theorem 2.1 is satisfied.

Thus, the operator $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ satisfy all the conditions of Theorem 2.1 and so, the operator equation $\mathcal{A}(x,y)\mathcal{B}(x,y)+\mathcal{C}(x,y)=(x,y)$ has a solution in S. Consequently, the coupled hybrid system of fractional differential equations (1.3) has a mild coupled solution defined on J. This completes the proof. □

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