Journal of Fractional Calculus and Applications
Vol. 10(2) July 2019, pp. 233-248
ISSN: 2090-5858.
http://math-frac.oreg/Journals/JFCA/

DYNAMICAL BEHAVIORS OF DISCRETE PREDATOR-PREY MODEL WITH HOLLING TYPE IV FUNCTIONAL RESPONSE

S. Z. RIDA, Y. GH. GOUDA, A. S. ZAKI

Abstract. In this paper, a discrete-time predator-prey model with Holling type IV functional response is investigated. Firstly, we introduced the local stability analysis of the model. Next, bifurcation theory and the center manifold theorem are used to study the bifurcation phenomena at the fixed points of the model. Bifurcation types (include flip and Neimark-Sacker) are addressed. Finally, numerical simulations are carried out to check the obtained theoretical results.

1. Introduction

It is well known that, the most popular model to investigate the dynamical behaviors of predator-prey model was made by Lotka [1] in (1925) and Volterra [2] in (1926). They have suggested a system of a pair of differential equations with first order to understand the evolution of the predator-prey. A predator-prey system with logistic growth and functional response $p(x)$ can be written in the following form:

$$\begin{aligned} \frac{dx}{dt} &= rx(t)(1-x(t)) - yp(x(t)), \\ \frac{dy}{dt} &= y(t)p(x(t)) - by(t), \end{aligned} \tag{1.1}$$

where $x(t)$ and $y(t)$ represent the prey and predator density at time t respectively. The prey grows logistically with intrinsic growth rate $r > 0$ in the absence of predator and $b > 0$ is the death rate of the predator population. The functional response represents the specific growth rate resulting of prey consumption per unit time. The functional response classified into three types are called Holling type I,II and III [3]. Furthermore it, some authors have also described a Holling type IV functional response which is humped and declines at high prey densities [4]. In [5], Sokol and Howell proposed a simplified Holling type IV function of the

2010 *Mathematics Subject Classification.* 31A35, 31C35,34K18.

Key words and phrases. Predator-Prey model, Discretization method, Fixed points, Local stability, Functional response, Bifurcations.

Submitted Nov. 5, 2018. Revised Nov. 27, 2018.

form $p(x) = \frac{mx}{a+x^2}$, where $m > 0$ denotes the maximal predation rate and $a > 0$ is half-saturation constant.

The model (1.1) with Holling type IV functional response becomes:

$$\begin{aligned} \frac{dx}{dt} &= rx(t)(1-x(t)) - \frac{mx(t)y(t)}{a+x^2(t)}, \\ \frac{dy}{dt} &= \frac{mx(t)y(t)}{a+x^2(t)} - by(t). \end{aligned} \tag{1.2}$$

Let

$$\bar{t} = mt, \bar{r} = \frac{r}{m}, \bar{b} = \frac{b}{m}.$$

The system (1.2) transformed to

$$\begin{aligned} \frac{dx}{dt} &= rx(1-x) - \frac{xy}{a+x^2}, \\ \frac{dy}{dt} &= \frac{xy}{a+x^2} - by. \end{aligned} \tag{1.3}$$

In fact, the mathematical modeling for some population dynamics becomes more convenient and realistic when it is approximated by difference equations especially if the populations have non-overlapping generations such as an annual plant or an insect population with one generation per year. Therefore, we apply the forward Euler scheme of system (1.3) to get the discrete-time predator-prey system as follow:

$$\begin{aligned} x_{n+1} &= x_n + \delta[rx_n(1-x_n) - \frac{x_n y_n}{a+x_n^2}], \\ y_{n+1} &= y_n + \delta[\frac{x_n y_n}{a+x_n^2} - by_n], \end{aligned} \tag{1.4}$$

and the mapping is given by the form:

$$\begin{aligned} x &= x + \delta[rx(1-x) - \frac{xy}{a+x^2}], \\ y &= y + \delta[\frac{xy}{a+x^2} - by], \end{aligned} \tag{1.5}$$

where δ is the step size.

In this work, our objectives are study the dynamical behaviors of system (1.5) and the sufficient condition to occurs a flip and Neimark-sacker bifurcations by using bifurcation theory and the center manifold theorem.

Many authors have studied the dynamical behaviors of predator-prey (see [6]-[19]). In [20], the forward Euler scheme was applied to a simple predator-prey model by Zhang et al.

The rest of this paper is organized as follows: in section (2), we study the local stability analysis of the fixed points. In section (3), we investigate local bifurcation analysis. In section (4), numerical simulation are carried out to confirm the theoretical results. Finally, we show our conclusions.

2. Existence of fixed points and their stability

In this section, we determine the existence of the fixed points of the system (1.5) and study their stability by calculating the eigenvalues of the Jacobian matrix of the system at each fixed point.
By simple calculations there exist three fixed points of the system (1.5):-

(1) $E_0(0,0)$ is the trivial equilibrium point.
(2) $E_1(1,0)$ is the axial fixed point.
(3) $E_2(x^*,y^*) = (x^*, r(1-x^*)(a+(x^*)^2))$ is the positive equilibrium point.

Now, we study the local stability of these fixed points. In deed, the local stability of the discrete-time system (1.5) is determine by calculating the eigenvalues of the Jacobian matrix.
The Jacobian matrix of the system (1.5) at its fixed point (x,y) can be written in the form:-

$$J(x,y) = \begin{pmatrix} 1+\delta(r-2rx-\frac{y-x^2y}{(a+x^2)^2}) & -\frac{\delta x}{a+x^2} \\ \delta\frac{y-x^2y}{(a+x^2)^2} & 1+\delta(\frac{x}{a+x^2}-b) \end{pmatrix} \tag{2.1}$$

To study the stability of fixed points of system (1.5), we need the following lemma to help us.
lemma 1. [16] Let $F(\lambda) = \lambda^2 + P\lambda + Q$. Suppose that $F(1) > 0$, λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then

(1) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$, $Q < 1$;
(2) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$;
(3) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $Q > 1$;
(4) $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $P \neq 0, 2$;
(5) λ_1 and λ_2 are complex and $|\lambda_1| = 1$and $|\lambda_2| = 1$ if and only if $P^2 - 4Q < 0$ and $Q = 1$.

We recall some definitions of topological types for a fixed point (x,y). A fixed point (x,y) is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so the sink is locally asymptotically stable; it is called a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so the source is locally unstable; it is called a saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$); and (x,y) is called non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.
Now state the following three Propositions:
proposition 1. The fixed point $E_0(0,0)$ has the following topological properties

(1) $E_0(0,0)$ is saddle if $0 < \delta < \frac{2}{b}$.
(2) $E_0(0,0)$ is a source if $\delta > \frac{2}{b}$.
(3) $E_0(0,0)$ is a non-hyperbolic if $\delta = \frac{2}{b}$.

proposition 2. The fixed point $E_1(1,0)$ has the following topological properties

(1) $E_1(1,0)$ is sink if $0 < \delta < \frac{2}{r}$, $\frac{1}{2} < b < \frac{1}{2} + \frac{2}{\delta}$;
(2) $E_1(1,0)$ is saddle if $0 < \delta < \frac{2}{r}$, $\frac{1}{2} > b > \frac{1}{2} + \frac{2}{\delta}$ also for $\delta > \frac{2}{r}$, $\frac{1}{2} < b < \frac{1}{2} + \frac{2}{\delta}$;
(3) $E_1(1,0)$ is source if $\delta > \frac{2}{r}$ or $\frac{1}{2} > b > \frac{1}{2} + \frac{2}{\delta}$;
(4) $E_1(1,0)$ is a non-hyperbolic if $\delta = \frac{2}{r}$ or $b = \frac{1}{2}$ or $b = \frac{1}{2} + \frac{2}{\delta}$.

The characteristic equation of the Jacobian matrix J of the system (1.5) evaluated at the positive fixed point $E_2(x^*, y^*)$ can be written as

$$\lambda^2 - (2 + \delta G)\lambda + (1 + \delta G + \delta^2 H) = 0,$$

where

$$G = r(1 - 2x) + \frac{y(x^2 - 1)}{(1 + x^2)^2}, \qquad H = \frac{xy(1 - x^2)}{(1 + x^2)^3},$$

let

$$F(\lambda) = \lambda^2 - (2 + \delta G)\lambda + (1 + \delta G + \delta^2 H),$$

then,

$$F(1) = \delta^2 H > 0, \qquad F(-1) = 4 + 2\delta G + \delta^2 H,$$

Using Lemma 1 we obtain the local dynamics of the fixed point $E_2(x^*, y^*)$.
proposition 3. Let $E_2(x^*, y^*)$ be the positive fixed point of system (1.5):

- E_2 is a sink if one of the following conditions holds
  (1) $G = -2\sqrt{H}$ and $\delta < \frac{-G}{H}$,
  (2) $G < -2\sqrt{H}$ and $0 < \delta < \frac{-G-\sqrt{G^2-4H}}{H}$,
  so E_2 local asymptotic stable.
- E_2 is a source if one of the following conditions holds
  (1) $G = -2\sqrt{H}$ and $\delta > -\frac{G}{H}$,
  (2) $G < -2\sqrt{H}$ and $\delta > \frac{-G+\sqrt{G^2-4H}}{H}$,
- E_2 is a saddle if the following condition holds:

$$G < -2\sqrt{H}, \qquad and \qquad \frac{-G - \sqrt{G^2 - 4H}}{H} < \delta < \frac{-G + \sqrt{G^2 - 4H}}{H},$$

- E_2 is non-hyperbolic if one of the following conditions holds:
  (1) $G < -2\sqrt{H}$ and $\delta = \frac{-G\pm\sqrt{G^2-4H}}{H}$,
  (2) $-2\sqrt{H} < G < 0$ and $\delta = -\frac{G}{H}$.

3. Local bifurcations analysis

The main objective of this section is to investigate different types of bifurcations at the three fixed points of the discrete system (1.5).

3.1. Bifurcation of the fixed point $E_0(0, 0)$

. The Jacobian matrix at $E_0(0, 0)$ is given by:

$$J(E_0) = \begin{pmatrix} 1 + \delta r & 0 \\ 0 & 1 - \delta b \end{pmatrix} \tag{3.1}$$

has two eigenvalues $\lambda_1 = 1 + \delta r$ and $\lambda_2 = 1 - \delta b$. If $4 + 2\delta r = 0$, than $\lambda_1 = -1$, $\lambda_2 = 1 - \delta b$. One can check that the condition for a flip (period doubling) bifurcation is satisfied when $\delta = \frac{2}{b}$ as shown in the following lemma.
lemma 2. If $\delta = \frac{2}{b}$, the system (1.5) undergoes a flip bifurcation at E_0. Moreover, the stable periodic-2 point bifurcates from this fixed point.

proof. Let $\delta = \frac{2}{b}$, the two eigenvalues at E_0 become $\lambda_1 = 1 + \frac{2r}{b}$ and $\lambda_2 = -1$. Let $\mu = \delta - \frac{2}{b}$ such that parameter μ is a new and dependent variable, the system (1.5) is transformed into the following form

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ \mu_{n+1} \end{pmatrix} = \begin{pmatrix} 1+\frac{2b}{r} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ \mu_n \end{pmatrix} + \begin{pmatrix} (rx_n - rx_n^2 - \frac{x_n y_n}{1+x_n^2})\mu_n - \frac{2rx_n^2}{b} - \frac{2x_n y_n}{b(1+x_n^2)} \\ (\frac{x_n y_n}{1+x_n^2} - by_n)\mu_n + \frac{2x_n y_n}{b(1+x_n^2)} \\ 0 \end{pmatrix} \quad (3.2)$$

By applying the center manifold theorem to determine the dynamics behavior of the fixed point E_0. So the center manifold for the system (3.2), can be expressed as follow:

$$W^c(E_0) = \{(x, y, \mu) \in \mathbb{R}^3 | y = f(x, \mu), f(0,0) = Df(0,0), |x| < \epsilon, |\mu| < \delta\},$$

for ϵ, δ sufficiently small.
To compute the center manifold $W^c(E_0)$ we assume

$$y_n = f(x_n, \mu_n) = a_1 x_n^2 + a_2 x_n \mu_n + a_3 \mu_n^2 + o((|x_n| + |\mu_n|)^3), \quad (3.3)$$

where $O((|x| + |\mu|)^3)$ is the sum of all terms whose order is great than 2.
The center manifold must satisfy

$$f((1+\frac{2b}{r})x_n + (rx_n - rx_n^2 - \frac{x_n y_n}{1+x_n^2})\mu_n, \mu_n) = (-1)(a_1 x_n^2 + a_2 x_n \mu_n + a_3 \mu_n^2). \quad (3.4)$$

Substituting (3.2) and (3.3) into (3.4) and then equating coefficients of like powers , we get

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = 0.$$

Thus the map restricted to the center manifold is given by

$$f_1 : x_{n+1} = (-x_n + rx_n\mu_n + 2x_n^2 + O((|u| + |\mu_1|)^4). \quad (3.5)$$

Since

$$\begin{aligned} \alpha_1 &= \Big(2\frac{\partial^2 F_1}{\partial \mu_1 \partial u} + \frac{\partial F_1}{\partial \mu_1}\frac{\partial^2 F_1}{\partial u^2}\Big)_{(0,0)} = 2r \neq 0, \\ \alpha_2 &= \Big(\frac{1}{2}\Big(\frac{\partial^2 F_1}{\partial u^2}\Big)^2 + \frac{1}{3}\Big(\frac{\partial^3 F_1}{\partial u^3}\Big)\Big)_{(0,0)} = 8 \neq 0, \end{aligned}$$

Thus, system (1.5) undergoes a subcritical flip bifurcation at $E_0(0,0)$. This completes the proof.

3.2. Bifurcation of the fixed point $E_1(1,0)$

. Now we study the bifurcation of non-trivial fixed point $E_1(1,0)$. In the following lemma, it will be shown that the system (1.5) undergoes a flip bifurcation at E_1. The Jacobian matrix at $E_1(1,0)$ is given by:

$$J(1,0) = \begin{pmatrix} 1-\delta r & -\frac{\delta}{2} \\ 0 & 1+\delta(\frac{1}{2} - b) \end{pmatrix}, \quad (3.6)$$

which has two eigenvalues $\lambda_1 = 1 - \delta r$ and $\lambda_2 = 1 + \delta(\frac{1}{2} - b)$. One can check that the condition for a flip bifurcation (period doubling) is satisfied when $\delta = \frac{2}{r}$ and

$b \neq \frac{1}{2}$, then $J(E_1)$ has two eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1 + \frac{1}{r}(1-2b)$.
lemma 3. If $\delta = \frac{2}{r}$ and $b \neq \frac{1}{2}$, the system (1.5) will undergo a flip (period doubling) bifurcation at E_1.

proof. If $\delta = \frac{2}{r}$ and $b \neq \frac{1}{2}$ the eigenvalues of Jacobian matrix at E_1 are $\lambda_1 = -1$ and $\lambda_2 = 1 + \frac{1}{r}(1-2b)$. Let $u = x - 1$, $v = y$, $\mu = \delta - \frac{2}{r}$, and let parameter μ be a new and dependent variable, then the system (1.5) becomes:

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \\ \mu_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & -\frac{1}{r} & 0 \\ 0 & 1-\frac{2b}{r} & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \\ \mu_n \end{pmatrix} + \begin{pmatrix} (-ru_n^2 - ru_n)\mu_n - 2u_n^2 - (\mu_n + \frac{2}{r})(\frac{u_n v_n}{1+u_n^2}) \\ (\mu_n + \frac{2}{r})(\frac{u_n v_n}{1+u_n^2}) - \mu_n b v_n \\ 0 \end{pmatrix} \tag{3.7}$$

Can construct an invertible matrix

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2r & 0 \\ 0 & 0 & 2(b-r) \end{pmatrix},$$

and use the translation

$$\begin{pmatrix} u \\ v \\ \mu \end{pmatrix} = T \begin{pmatrix} \zeta \\ \eta \\ \mu_1 \end{pmatrix},$$

then the system (3.7) becomes

$$\begin{pmatrix} \zeta \\ \eta \\ \mu_1 \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{1}{2r} & 0 \\ 0 & \frac{1}{2r} & 0 \\ 0 & 0 & \frac{1}{2(b-r)} \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \\ \mu_1 \end{pmatrix} + \begin{pmatrix} \phi(\zeta,\eta,\mu_1) \\ \psi(\zeta,\eta,\mu_1) \\ 0 \end{pmatrix}. \tag{3.8}$$

where

$$\phi(\zeta,\eta,\mu_1) = (-r\zeta^2 - r\zeta)\mu_1 - 2\zeta^2 - (\mu_1 + \frac{2}{r})(\frac{\zeta\eta}{1+\zeta^2}),$$

$$\psi(\zeta,\eta,\mu_1) = (\mu_1 + \frac{2}{r})(\frac{\zeta\eta}{1+\zeta^2}) - \mu_1 b\eta.$$

Then, the center manifold for (3.7), take the form:

$$W^c(E_1) = \{(\zeta,\eta,\mu_1) \in \mathbb{R}^3 | \eta = f(\zeta,\mu_1), f(0,0) = Df(0,0), |\zeta| < \epsilon, |\mu_1| < \hat{\delta}_1\},$$

for ϵ, $\hat{\delta}_1$ sufficiently small.
To compute the center manifold W^c we assume

$$\eta = f(\zeta,\mu_1) = a_1\zeta^2 + a_2\zeta\mu_1 + a_3{\mu_1}^2 + O((|\zeta| + |\mu_1|)^3). \tag{3.9}$$

The center manifold must satisfy
$f(\zeta + \phi(\zeta, f(\zeta,\mu_1),\mu_1),\mu_1) = \frac{1}{2r}(a_1\zeta^2 + a_2\zeta\mu_1 + a_3{\mu_1}^2) + \psi(\zeta, f(\zeta,\mu_1),\mu_1)$.
Substituting Eq. (3.9) into Eq. (3.10) and then equating coefficients of like powers in Eq. (3.10) , we get
$a_1 = 0$, $a_2 = 0$, $a_= 0$.
The system (3.8)is restricted to the center manifold, which is given by

$$f_2 : \zeta_{n+1} = \zeta - (r\zeta^2 - r\zeta)\mu_1 - 2\zeta^2. \tag{3.10}$$

Since

$$\gamma_1 = \Big(2\frac{\partial^2 F_2}{\partial\mu_1\partial\zeta} + \frac{\partial F_2}{\partial\mu_1}\frac{\partial^2 F_2}{\partial\zeta^2}\Big)_{(0,0)} = -2r \neq 0,$$

$$\gamma_2 = \Big(\frac{1}{2}\Big(\frac{\partial^2 F_2}{\partial\zeta^2}\Big)^2 + \frac{1}{3}\Big(\frac{\partial^3 F_2}{\partial\zeta^3}\Big)\Big)_{(0,0)} = (2r^2+8r+8) \neq 0.$$

So, system (1.5) undergoes a subcritical flip bifurcation at E_1.

3.3. Bifurcation of the fixed point $E_2(x^*, y^*)$

. In this section, we show that occur both flip and NeimarkSacker bifurcations in system (1.5) is studied at the interior fixed point E_2 where δ is taken as the bifurcation parameter. Firstly, we discuss the flip bifurcation of system (1.5) Let

$$FB_1 = \left\{(r,b,\delta) : \delta = \frac{-G+\sqrt{G^2-4H}}{H}, G < -2\sqrt{H}, r, b, \delta > 0\right\},$$

or

$$FB_2 = \left\{(r,b,\delta) : \delta = \frac{-G-\sqrt{G^2-4H}}{H}, G < -2\sqrt{H}, r, b, \delta > 0\right\}.$$

The fixed point (x^*, y^*) can undergo flip bifurcation when parameters vary in a small neighborhood of FB_1 or FB_2. Let

$$HB = \left\{(r,b,\delta) : \delta = \frac{-G}{H}, -2\sqrt{H} < G < 0, r, b, \delta > 0\right\}.$$

The fixed point $E_2(x^*, y^*)$ may undergo Neimark-Sacker bifurcation when parameters vary in a small neighborhood of HB. In the following analysis, we will study the flip bifurcation of the positive fixed point $E_2(x^*, y^*)$ if parameters vary in a small neighborhood of FB_1 (or FB_2), and the Neimark-Sacker bifurcation of $E_2(x^*, y^*)$ if parameters vary in a small neighborhood of HB.

First, we discuss the flip bifurcation of system (1.5) at $E_2(x^*, y^*)$ when parameters vary in a small neighborhood of FB_1. By the same arguments can be applied to the other case FB_2. Taking parameters (r, b, δ_1) arbitrarily from FB_1, we consider system (1.5) with (r, b, δ_1), which is given by

$$\begin{cases} x \to x + \delta_1[rx(1-x) - \frac{xy}{1+x^2}], \\ y \to y + \delta_1[\frac{xy}{1+x^2} - by]. \end{cases} \tag{3.11}$$

The system (3.10) has a unique positive fixed point $E_2(x^*, y^*)$, has eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 3 + G\delta_1$ with $|\lambda_2| \neq 1$ by proposition 3.

Since $(r, b, \delta_1) \in FB_1$, taking δ^* as a bifurcation parameter, we consider a perturbation of (3.10) as follows:

$$\begin{cases} x \to x + (\delta_1+\delta^*)[rx(1-x) - \frac{xy}{1+x^2}], \\ y \to y + (\delta_1+\delta^*)[\frac{xy}{1+x^2} - by]. \end{cases} \tag{3.12}$$

where $|\delta^*| \ll 1$, which is a small perturbation parameter.

Let $u = x - x^*$, $v = y - y^*$. Then we transform the fixed point $E_2(x^*, y^*)$ of system

(3.12) into the origin. We have

$$\begin{cases} u \to & a_1u + a_2v + a_{13}u\delta^* + a_{23}v\delta^* + a_{12}uv + a_{11}u^2 + a_{112}u^2v + a_{113}u^2\delta^* \\ & +a_{111}u^3 + O((|u| + |v| + |\delta^*|)^4), \\ v \to & b_1u + b_2v + b_{13}u\delta^* + b_{23}v\delta^* + b_{12}uv + a_{11}u^2 + b_{112}u^2v + b_{113}u^2\delta^* \\ & +b_{111}u^3 + O((|u| + |v| + |\delta^*|)^4), \end{cases} \tag{3.13}$$

where

$$\begin{cases} a_1 = 1 + \delta[r(1-2x^*) - \frac{y^*(1-x^{*2})}{(1+x^{*2})^2}],\ a_2 = -\frac{\delta x^*}{1+x^{*2}},\ a_{13} = r(1-2x^*) - \frac{y^*(1-x^{*2})}{(1+x^{*2})^2},\ a_{23} = -\frac{x^*}{1+x^{*2}}, \\ a_{11} = \delta[-r + \frac{x^*y^*(3-x^{*2})}{(1+x^{*2})^3},\ a_{12} = -\frac{\delta(1-x^{*2})}{(1+x^{*2})^2},\ a_{111} = \frac{\delta}{2}[\frac{y^*(1-6x^{*2}+x^{*4})}{(1+x^{*4}}], \\ a_{112} = \delta[\frac{x^*(3-x^{*2})}{(1+x^{*2})^3}],\ a_{113} = \frac{x^*(3-x^{*2})}{(1+x^{*2})^3}, \\ b_1 = \frac{\delta y^*(1-x^{*2})}{(1+x^{*2})^2},\ b_2 = 1 + \delta[\frac{x^*}{1+x^{*2}} - b],\ b_{13} = \frac{y^*(1-x^{*2})}{(1+x^{*2})^2},\ b_{23} = \frac{x^*}{1+x^{*2}} - b, \\ b_{11} = \delta[\frac{x^*y^*(-3+x^{*2})}{(1+x^{*2})^3}],\ b_{12} = \frac{\delta(1-x^{*2})}{(1+x^{*2})^2},\ b_{123} = \frac{(1-x^{*2})}{(1+x^{*2})^2},\ b_{111} = \frac{\delta}{2}[\frac{y^*(-1+6x^{*2}-x^{*4})}{(1+x^{*2})^4}], \\ b_{112} = \delta[\frac{x^*(-3+x^{*2})}{(1+x^{*2})^3}],\ b_{113} = \frac{x^*y^*(-3+x^{*2})}{(1+x^{*2})^3}, \end{cases} \tag{3.14}$$

and $\delta = \delta_1$.
Can constructing an invertible matrix

$$T = \begin{pmatrix} a_2 & a_2 \\ -1-a_1 & \lambda_2 - a_1 \end{pmatrix},$$

and use the translation

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix},$$

then the system (3.13) becomes

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} f_1(\tilde{x}, \tilde{y}, \delta^*) \\ f_2(\tilde{x}, \tilde{y}, \delta^*) \end{pmatrix}, \tag{3.15}$$

where

$$\begin{aligned} f_1(\tilde{x}, \tilde{y}, \delta^*) &= \frac{a_{13}(\lambda_2 - a_1) - a_2b_{13}}{a_2(1+\lambda_2)}u\delta^* + \frac{a_{23}(\lambda_2 - a_1) - a_2b_{23}}{a_2(1+\lambda_2)}v\delta^* + \frac{a_{11}(\lambda_2 - a_1)}{a_2(1+\lambda_2)}u^2 \\ &+ \frac{a_{113}(\lambda_2 - a_1)}{a_2(1+\lambda_2)}u^2\delta^* - \frac{b_{12}}{(1+\lambda_2)}uv - \frac{b_{123}}{(1+\lambda_2)}uv\delta^* + \frac{a_{111}(\lambda_2 - a_1)}{a_2(1+\lambda_2)}u^3 \\ &+ O((|u| + |v| + |\delta^*|)^4), \\ f_2(\tilde{x}, \tilde{y}, \delta^*) &= \frac{a_{13}(1+a_1) + a_2b_{13}}{a_2(1+\lambda_2)}u\delta^* + \frac{a_{23}(1+a_1) + a_2b_{23}}{a_2(1+\lambda_2)}v\delta^* + \frac{a_{11}(1+a_1)}{a_2(1+\lambda_2)}u^2 \\ &+ \frac{a_{113}(1+a_1)}{a_2(1+\lambda_2)}u^2\delta^* + \frac{b_{12}}{(1+\lambda_2)}uv + \frac{b_{123}}{(1+\lambda_2)}uv\delta^* + \frac{a_{111}(1+a_1)}{a_2(1+\lambda_2)}u^3 \\ &+ O((|u| + |v| + |\delta^*|)^4), \end{aligned}$$

and

$$\begin{aligned} u &= a_2(\tilde{x}+\tilde{y}),\ v=-(1+a_1)\tilde{x}+(\lambda_2-a_1)\tilde{y},\\ uv &= a_2(-(1+a_1)\tilde{x}^2+(\lambda_2-2a_1-1)\tilde{x}\tilde{y}+a_2(\lambda_2-a_1)\tilde{y}^2),\\ u^2 &= a_2^2(\tilde{x}^2+\tilde{x}\tilde{y}+\tilde{y}^2),\\ u^3 &= a_2^3(\tilde{x}^3+3\tilde{x}^2\tilde{y}+3\tilde{x}\tilde{y}^2+\tilde{y}^3). \end{aligned}$$

Next, we determine the center manifold $W^c(0,0,0)$ of system (3.15) at the fixed point $(0,0)$ in a small neighborhood of δ^*. By the center manifold theorem, we know that there exists a center manifold

$$W^c(0,0,0)=\{(\tilde{x},\tilde{y},\delta^*)\in R^3,\tilde{y}=f(\tilde{x},\delta^*),f(0,0)=0,Df(0,0)=0\},$$

for $\tilde{x}$ and δ^* sufficiently small. We suppose that a center manifold take the form

$$\tilde{y}=f(\tilde{x},\delta^*)=c_1\tilde{x}^2+c_2\tilde{x}\delta^*+c_3\delta^{*2}+O((|\tilde{x}|+|\delta^*|)^3). \tag{3.16}$$

The center manifold must satisfy

$$h(-\tilde{x}+f_1(\tilde{x},f(\tilde{x},\delta^*),\delta^*),\delta^*)=\lambda_2 f(\tilde{x},\delta^*)+f_2(\tilde{x},f(\tilde{x},\delta^*),\delta^*). \tag{3.17}$$

By substituting from (3.16) into (3.17), and equating coefficients of like powers (3.17), we get that

$$\begin{aligned} c_1 &= \frac{(1+a_1)a_2}{1-\lambda_2^2}(a_{11}-b_{12}),\\ c_2 &= \frac{(1+a_1)[a_{23}(1+a_1)+a_2b_{23}]}{a_2(1+\lambda_2)^2}-\frac{a_{13}(1+a_1)+a_2b_{13}}{(1+\lambda_2)^2},\\ c_3 &= 0. \end{aligned}$$

Therefore, we consider the system (3.15) which is restricted to the center manifold $W_c(0,0,0)$:

$$f_3:\tilde{x}\to-\tilde{x}+g_1\tilde{x}^2+g_2\tilde{x}\delta^*+g_3\tilde{x}^2\delta^*+g_4\tilde{x}\delta^{*2}+g_5\tilde{x}^3+O((|\tilde{x}|+|\delta^*|)^4). \tag{3.18}$$

where

$$\begin{aligned} g_1 &= \frac{1}{\lambda_2+1}((\lambda_2-a_1)a_{11}a_2+(1+a_1)b_{12}a_2),\\ g_2 &= \frac{1}{a_2(\lambda_2+1)}(a_{13}a_2(\lambda_2-1)-b_{13}a_2{}^2-(1+a_1)(a_{23}(\lambda_2-a_1)-a_2b_{23})),\\ g_3 &= \frac{1}{\lambda_2+1}((\lambda_2-a_1)(a_{113}a_2+c_2a_2a_{11}+\frac{c_1}{a_2}a_{23}(\lambda_2-a_1)-c_1b_{23}+a_{13}c_1)+b_{123}a_2(1+a_1)-\\ &- b_{12}a_2c_2(\lambda_2-2a_1-1)-b_{13}a_2c_1),\\ g_4 &= \frac{1}{\lambda_2+1}((\lambda_2-a_1)(a_{13}c_2+\frac{c_2}{a_{23}}(\lambda_2-a_1)-c_2b_{23})-b_{13}a_2c_2),\\ g_5 &= \frac{1}{\lambda_2+1}((\lambda_2-a_1)(a_{11}a_2c_1+a_{111}a_2^2)-b_{12}a_2c_1(\lambda_2-2a_1-1)). \end{aligned}$$

In order to the system (3.17) undergo a flip bifurcation, we require that two discriminatory quantities γ_1 and γ_2 are not zero, where

$$\begin{aligned}\gamma_1 &= \Big(2\frac{\partial^2 f_3}{\partial\delta^*\partial\tilde{x}} + \frac{\partial f_3}{\partial\delta^*}\frac{\partial f_3}{\partial\tilde{x}}\Big)_{(0,0)} = 2g_2,\\ \gamma_2 &= \Big(\frac{1}{2}\Big(\frac{\partial^2 f_3}{\partial\tilde{x}^2}\Big)^2 + \frac{1}{3}\Big(\frac{\partial^3 f_3}{\partial\tilde{x}^3}\Big)\Big)_{(0,0)} = 2(g_5 + g_1^2).\end{aligned}$$

From the above analysis, we have the following result.

theorem 3. If $\gamma_2 \neq 0$, then the system (1.5) undergoes a flip bifurcation at the fixed point $E_2(x^*, y^*)$ when the parameter δ varies in a small neighborhood of δ_1. Moreover, if $\gamma_2 > 0$ (resp.$\gamma_2 < 0$), then the period-2 orbits that bifurcate from (x^*, y^*) are stable (resp., unstable).

Finally, we discuss the Neimark-Sacker bifurcation of $E_2(x^*, y^*)$ if parameters (r, b, δ_2) vary in a small neighborhood of HB. By taking parameters (r, b, δ_2) arbitrarily from HB, we consider the system (1.5) with (r, b, δ_2), which is described by

$$\begin{cases} x \to x + \delta_2[rx(1-x) - \frac{xy}{1+x^2}],\\ y \to y + \delta_2[\frac{xy}{1+x^2} - by].\end{cases} \tag{3.19}$$

System (3.19) has a unique positive fixed point $E_2(x^*, y^*)$. At parameters $(r, b, \delta_2) \in HB$, then $\delta_2 = -\frac{G}{H}$. Choosing δ^* as a bifurcation parameter, we consider a perturbation of the system (3.19) given as follows:

$$\begin{cases} x \to x + (\delta_2 + \bar{\delta}^*)[rx(1-x) - \frac{xy}{1+x^2}],\\ y \to y + (\delta_2 + \bar{\delta}^*)[\frac{xy}{1+x^2} - by].\end{cases} \tag{3.20}$$

Where $\delta^* \ll 1$, which is a small perturbation parameter.

Let $u = x - x^*$, $v = y - y^*$. Then we transform the fixed point $E_2(x^*, y^*)$ of the system (3.20) into the origin. We have

$$\begin{cases} u \to & a_1u + a_2v + a_{11}u^2 + a_{12}uv + a_{111}u^3 + a_{112}u^2v + O((|u| + |v|)^4),\\ v \to & b_1u + b_2v + b_{11}u^2 + b_{12}uv + b_{111}u^3 + b_{112}u^2v + O((|u| + |v|)^4),\end{cases} \tag{3.21}$$

where $a_1, a_2, a_{11}, a_{12}, a_{111}, a_{112}$ and $b_1, b_2, b_{11}, b_{12}, b_{111}, b_{112}$ are given in (3.14) by substituting δ_1 for $\delta_2 + \bar{\delta}^*$. So, the characteristic equation of the system (3.21) at $(u, v) = (0, 0)$ is given by

$$\lambda^2 + p(\bar{\delta}^*)\lambda + q(\bar{\delta}^*) = 0,$$

where

$$\begin{aligned} p(\bar{\delta}^*) &= -2 - G(\bar{\delta}^* + \delta_2),\\ q(\bar{\delta}^*) &= 1 + G(\bar{\delta}^* + \delta_2) + H(\bar{\delta}^* + \delta_2)^2.\end{aligned}$$

Since parameters $(r, b, \delta_2) \in H_B$, the eigenvalues of $(0, 0)$ are a pair of complex conjugate numbers λ and $\bar{\lambda}$ with modulus 1 by proposition 3 , where

$$\lambda, \bar{\lambda} = -\frac{p(\bar{\delta}^*)}{2} \pm \frac{i}{2}\sqrt{4q(\bar{\delta}^*) - p^2(\bar{\delta}^*)},$$

and so

$$|\lambda|_{\bar{\delta}^*=0} = \sqrt{q(0)} = 1, \qquad l = \frac{d|\lambda|}{d\bar{\delta}^*}|_{\bar{\delta}^*=0} = -\frac{G}{2} \neq 0.$$

This implies that when $\bar{\delta}^* = 0$, $\lambda^m, \bar{\lambda}^m \neq 1 (m = 1, 2, 3, 4)$, which is require to $p(0) \neq -2, 0, 1, 2$. We note that $(r, b, \delta_2) \in HB$. Thus, $p(0) \neq -2, 2$. We need to require that $p(0) \neq 0, 1$, which leads to

$$G^2 \neq 2H, 3H. \tag{3.22}$$

Next, we study the normal form of the system (3.21) at $\bar{\delta}^* = 0$.
Let $\bar{\delta}^* = 0$, $\alpha = 1 + \frac{G\delta_2}{2}$, $\beta = \frac{\delta_2}{2}\sqrt{4H - G^2}$.

$$T = \begin{pmatrix} a_2 & 0 \\ \alpha - a_1 & -\beta \end{pmatrix},$$

and use the translation

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix},$$

then the system (3.21) becomes

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \to \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \bar{f}_1(\tilde{x}, \tilde{y}) \\ \bar{f}_2(\tilde{x}, \tilde{y}) \end{pmatrix}, \tag{3.23}$$

where

$$\begin{aligned} \bar{f}_1(\tilde{x}, \tilde{y}) &= \frac{a_{11}}{a_2}u^2 + \frac{a_{111}}{a_2}u^3 + O((|\tilde{x}| + |\tilde{y}|)^4), \\ \bar{f}_2(\tilde{x}, \tilde{y}) &= \frac{(\alpha - a_1)a_{11}}{a_2\beta}u^2 - \frac{b_{12}}{\beta}uv + \frac{(\alpha - a_1)a_{111}}{a_2\beta}u^3 + O((|\tilde{x}| + |\tilde{y}|)^4). \end{aligned}$$

and

$$\begin{aligned} uv &= a_2(\alpha - a_1)\tilde{x}^2 - a_2\beta\tilde{x}\tilde{y}, \\ u^2 &= {a_2}^2\tilde{x}^2, \ u^3 = a_2^3\tilde{x}^3. \end{aligned}$$

Therefore,

$$\begin{aligned} \bar{f}_{1\tilde{x}\tilde{x}} &= 2a_{11}a_2, \ \bar{f}_{1\tilde{x}\tilde{x}\tilde{x}} = 6a_{111}a_2^2, \\ \bar{f}_{1\tilde{x}\tilde{y}} &= \bar{f}_{1\tilde{y}\tilde{y}} = \bar{f}_{1\tilde{x}\tilde{x}\tilde{y}} = \bar{f}_{1\tilde{x}\tilde{y}\tilde{y}} = \bar{f}_{1\tilde{y}\tilde{y}\tilde{y}} = 0, \\ \bar{f}_{2\tilde{x}\tilde{x}} &= \frac{2a_2(\alpha - a_1)}{\beta}(a_{11} - b_{12}), \ \bar{f}_{2\tilde{x}\tilde{y}} = a_2b_{12}, \\ \bar{f}_{2\tilde{x}\tilde{x}\tilde{x}} &= \frac{6a_2^2(\alpha - a_1)a_{111}}{\beta}, \\ \bar{f}_{2\tilde{y}\tilde{y}} &= \bar{f}_{2\tilde{x}\tilde{x}\tilde{y}} = \bar{f}_{2\tilde{x}\tilde{y}\tilde{y}} = \bar{f}_{2\tilde{y}\tilde{y}\tilde{y}} = 0. \end{aligned}$$

at point $(0, 0)$.
To the system (3.23) undergo Neimark-Sacker bifurcation, we implies that the following discriminatory quantity is not zero:

$$\theta = \Big[- Re\Big(\frac{(1 - 2\lambda)\bar{\lambda}^2}{1 - \lambda}L_{11}L_{12}\Big) - \frac{1}{2}|L_{11}|^2 - |L_{21}|^2 + Re(\bar{\lambda}L_{22})\Big]_{\bar{\delta}^*=0},$$

where

$$\begin{aligned}
L_{11} &= \frac{1}{4}((\bar{f}_{1\tilde{x}\tilde{x}} + \bar{f}_{1\tilde{y}\tilde{y}}) + i(\bar{f}_{2\tilde{x}\tilde{x}} + \bar{f}_{2\tilde{y}\tilde{y}})),\\
L_{12} &= \frac{1}{8}((\bar{f}_{1\tilde{x}\tilde{x}} - \bar{f}_{1\tilde{y}\tilde{y}} + 2\bar{f}_{2\tilde{x}\tilde{y}}) + i(\bar{f}_{2\tilde{x}\tilde{x}} - \bar{f}_{2\tilde{y}\tilde{y}} - 2\bar{f}_{1\tilde{x}\tilde{y}})),\\
L_{21} &= \frac{1}{8}((\bar{f}_{1\tilde{x}\tilde{x}} - \bar{f}_{1\tilde{y}\tilde{y}} - 2\bar{f}_{2\tilde{x}\tilde{y}}) + i(\bar{f}_{2\tilde{x}\tilde{x}} - \bar{f}_{2\tilde{y}\tilde{y}} + 2\bar{f}_{1\tilde{x}\tilde{y}})),\\
L_{22} &= \frac{1}{16}((\bar{f}_{1\tilde{x}\tilde{x}\tilde{x}} + \bar{f}_{1\tilde{x}\tilde{y}\tilde{y}} + \bar{f}_{2\tilde{x}\tilde{x}\tilde{y}} + \bar{f}_{2\tilde{y}\tilde{y}\tilde{y}}) + i(\bar{f}_{2\tilde{x}\tilde{x}\tilde{x}} + \bar{f}_{2\tilde{x}\tilde{y}\tilde{y}} - \bar{f}_{1\tilde{x}\tilde{x}\tilde{y}} - \bar{f}_{1\tilde{y}\tilde{y}\tilde{y}})),
\end{aligned}$$

From the above analysis, we deduce the following theorem.

theorem 3. If the condition (3.22) holds and $\theta \neq 0$, then system (1.5) undergoes Neimark Sacker bifurcation at the fixed point $E_2(x^*, y^*)$ when the parameter δ varies in a small neighborhood of δ_2. Moreover, if $\theta < 0$ (resp., $\theta > 0$), then an attracting (resp., repelling) invariant closed curve bifurcates from the fixed point for $\delta > \delta_2$ (resp., $\delta < \delta_2$).

4. Numerical simulation

In this section, we present some numerical simulation results to confirm our analytical results and to obtain more complex dynamics of the system (1.5) by presenting bifurcation diagram, phase plane for specific parameter values. We consider the numerical simulation of system (1.5) is discussed in the different cases as follows:

Case1. Let b and r be fixed and δ vary from 0 to 1.5. We take the initial value $(x_0, y_0) = (0.9, 0.5)$ in all numerical simulation. The bifurcation diagram in the (δ, x) plane is illustrated in Fig. 1.

In Fig. 2, various phase plane diagrams are showed for $b = 0.28$, $r = 2.82$, and different δ. So the bifurcation diagram of the fixed point of $E_1(1, 0)$ is verify by lemma 2. In Fig. 3 , various phase plane diagrams are illustrated for $b = 0.28$, $r = 2.82$ and different δ, we get. Fig. 3 (a) shows that this fixed point is stable while part (b) shows the dynamics of the fixed point E_1 before Neimark-Sacker bifurcation. Part (c) shows the dynamics of E_1 after NeimarkSacker bifurcation while Part (d) shows that increasing δ resulting in E_1 loses stability and a closed invariant curve is created.

Also, in Fig.3 (e), (f), and (g) the chaotic behavior occurs at $\delta = 1.59, 1.64, 1.68$ respectively. Finally in (h) shows the strange attractor for $\delta = 1.7$.

case 2. Let $b = 0.66$ and $r = 3.8$ be fixed and $\delta = 0.8$. Fig. 3 the bifurcation diagram of system (1.5) in the (δ, x) plane.

The Maximum Lyapunov exponents corresponding to Fig. 3 computed in Fig. 4.

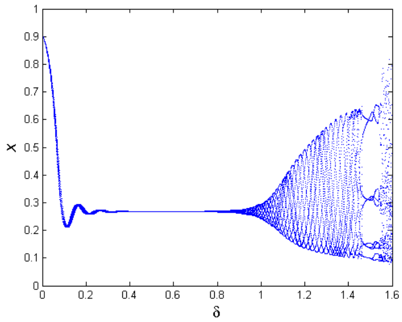

FIGURE 1. Bifurcation of system (1.5) in (δ, x) plane for $b = 0.28$, $r = 2.82$.

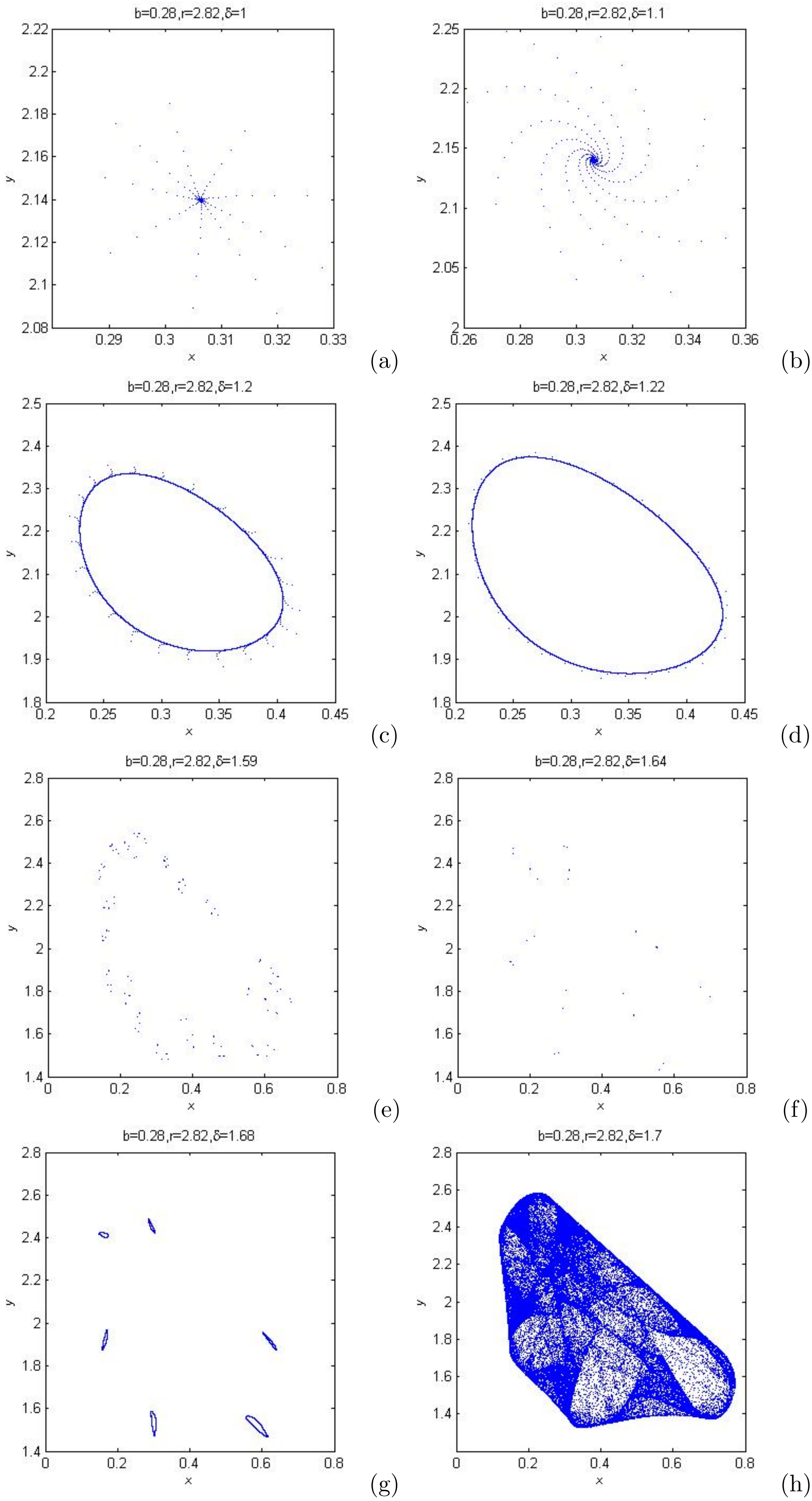

FIGURE 2. Phase plane for system (1.5) with $b = 0.28$, $r = 2.82$, and different δ.

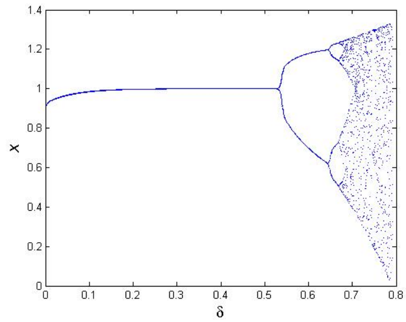

Figure 3. Bifurcation of system (1.5) in (δ, x) plane for $b = 0.66$, $r = 3.8$.

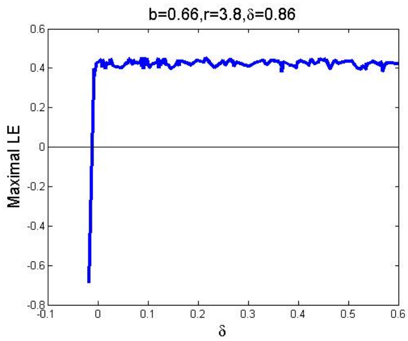

Figure 4. Bifurcation of system (1.5) in $(\delta, MaximalLE)$ plane for $b = 0.66$, $r = 3.8$.

5. Conclusion

In this paper, we have investigated the dynamical behaviors of a discrete-time predator-prey model with Holling type IV functional response. Also, we have introduced the sufficient conditions to occur both flip and Neimark-Sacker bifurcations. Finally, numerical simulations are implemented to confirm our theoretical analysis. These results show that the predator-prey system with Holling type IV functional response have rich dynamics.

References

[1] A. J. Lotka, Elements of physical biology. Baltimore: Williams and Wilkins Co.; 1925.
[2] V. Volterra, Variazioni e fluttuazioni del numero dindividui in specie animali conviventi. Mem R Accad Naz dei Lincei 1926;2(VI).
[3] C. S. Holling, The functional response of predator to prey density and its role in mimicry and population regulation, Mem. Entomol. Soc. Can. 45(1965), 1-60.
[4] H. I. Freedman, Deterministic Mathematical Models in Population Ecology. New York: Marcel Dekker, (1980).

[5] W. Sokol and J. A. Howell, Kinetics of phenol oxidation by washed cells, Biotechnol. Bioeng. 23(1980), 2039-2049.

[6] S. M. Salman, A. M. Yousef, A. A. Elsadany, Chaos, Stability bifurcation analysis and chaos control of a discrete predator-prey system with square root functional response. Journal Of Solitons And Fractals 93(2016), 20-31.

[7] S.Z. Rida, M. Khalil, Hany A. Hosham And Soheir Gadella, Predator-Prey Fractional-Order Dynamical System With Both The Populations Affected By Diseases. Journal Of Fractional Calculus And Applications, Vol. 5(3s) No. 13, Pp. 1-11, 2014.

[8] H. N. Agiza, E. M. Elabbasy, E. K. Metwally and A. A. Elsadany, Chaotic dynamics of a discrete prey-predator model with Holling type II, Nonlinear analysis: Real world applications 10 (2009), pp.116-129.

[9] M. A. Aziz-Alaoui and M. Daher Okiye, Boundedness and global stability for a predator-prey model with modified Leslie-gower and Holling-type II schemes, 16 (2003), pp.1069-1075.

[10] S. Wiggins, Introduction to applied nonlinear dynamical systems and chaos. Texts in applied mathematics, vol. 2. New York: Springer; 1990.

[11] J. Guckenheimer and P. Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields. New York: Springer-Verlag; 1983.

[12] J. C. Huang and D. M. Xiao, Analyses of bifurcations and stability in a predator-prey system with Holling type-IV functional response. Acta Math Appl Sin 2004; 20(1):16778.

[13] H. I. Freedman, Deterministic mathematical models in population ecology. New York: Marcel Dekker; 1980.

[14] D. P. Hu and H. J. Cao, Bifurcation and chaos in a discrete-time predatorprey system of Holling and Leslie type. Commun Nonlinear Sci Numer Simul 2015; 22:702-715.

[15] H.N. Agiza, E.M. Elabbssy, H.A. EL-Metwally and A.A. Elsadany, Chaotic dynamics of a discrete prey-predator model with Holling type II, Nonlinear Anal.: Real World Appl. 10 (2009) 116-129.

[16] D. Jana, Chaotic dynamics of a discrete predator-prey system with prey refuge, Appl. Math. Comput. 224 (2013) 848865.

[17] Y. H. Wan, Computation of the stability condition for the Hopf bifurcation of dieomorphism on R2, SIAM. J. Appl. Math. 34 (1978) 167-175.

[18] Z. He and X. Lai, Bifurcation and chaotic behavior of a discrete-time predatorprey system, Nonlinear Anal. Real World Appl. 12 (1) (2011) 403-417.

[19] Z. Hu, Z. Teng and L. Zhang, Stability and bifurcation analysis of a discrete predator-prey model with nonmonotonic functional response, Nonlinear Anal.

[20] Y. Zhang, Q. Zhang, L. Zhao and C. Yang, Dynamical behaviors and chaos control in a discrete functional response model, Chaos Solitons Fract. 4 (2007) 1318-1327.

S. Z. Rida
Faculty of Science, South Valley University, Qena, Egypt
E-mail address: szagloul@yahoo.com

Y. Gh. Gouda
Faculty of Science, Aswan University, Aswan, Egypt
E-mail address: yas.gouda@yahoo.com

A. S. Zaki
Faculty of Science, Aswan University, Aswan, Egypt
E-mail address: zasmaasabry@yahoo.com