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SOME RESULTS RELATING TO (p, q)-TH RELATIVE GOL'DBERG ORDER AND (p, q)-RELATIVE GOL'DBERG TYPE OF ENTIRE FUNCTIONS OF SEVERAL VARIABLES

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Abstract. In this paper we introduce the notions of (p, q)-th relative Gol'dberg order and (p, q)-th relative Gol'dberg type of entire functions of several complex variables where p, q are any positive integers. Then we study some growth properties of entire functions of several complex variables on the basis of their (p, q)-th relative Gol'dberg order and (p, q)-th relative Gol'dberg type.

1. Introduction and preliminaries

Let $\mathbb{C}^n$ and $\mathbb{R}^n$ respectively denote the complex and real n-spaces. Also let us indicate the point $(z_1, z_2, \cdots, z_n)$, $(m_1, m_2, \cdots, m_n)$ of $\mathbb{C}^n$ or I^n by their corresponding unsuffixed symbols z, m respectively where I denotes the set of non-negative integers. The modulus of z, denoted by $|z|$, is defined as $|z| = \left(|z_1|^2 + \cdots + |z_n|^2\right)^{\frac{1}{2}}$. If the coordinates of the vector m are non-negative integers, then z^m will denote $z_1^{m_1} \cdots z_n^{m_n}$ and $\|m\| = m_1 + \cdots + m_n$.

If $D \subseteq \mathbb{C}^n$ is an arbitrary bounded complex n-circular domain with center at the origin of coordinates then for any entire function $f(z)$ of n complex variables and $R > 0$, $M_{f,D}(R)$ may be defined as $M_{f,D}(R) = \sup_{z \in D_R} |f(z)|$ where a point $z \in D_R$ if and only if $\frac{z}{R} \in D$. If $f(z)$ is non-constant, then $M_{f,D}(R)$ is strictly increasing and its inverse $M_{f,D}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists such that $\lim_{R\to\infty} M_{f,D}^{-1}(R) = \infty$.

For $k \in \mathbb{N}$, we define $\exp^{[k]} R = \exp\left(\exp^{[k-1]} R\right)$ and $\log^{[k]} R = \log\left(\log^{[k-1]} R\right)$ where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} R = R$, $\log^{[-1]} R = \exp R$, $\exp^{[0]} R = R$ and $\exp^{[-1]} R = \log R$. Moreover we assume that throughout the present paper p, q, m and l always denote positive integers. Also throughout the paper the entire functions $f(z)$, $g(z)$ and $h(z)$ of n-complex variables will respectively stand for the entire functions $f(z)$, $g(z)$

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and $h(z)$ of n-complex variables with respect to any bounded complete n-circular domain D with center at origin in $\mathbb{C}^n$.

Considering this, the Gol'dberg order (respectively the Gol'dberg lower order) (see [7, 8]) of an entire function $f(z)$ of n-complex variables is given by

$$\rho_{f,D} = \overline{\lim_{R\to+\infty}} \frac{\log^{[2]} M_{f,D}(R)}{\log R} \text{ and } \lambda_{f,D} = \underline{\lim_{R\to+\infty}} \frac{\log^{[2]} M_{f,D}(R)}{\log R}.$$

It is well known that $\rho_{f,D}$ is independent of the choice of the domain D, and therefore we write ρ_f instead of $\rho_{f,D}$ (respectively λ_f instead of $\lambda_{f,D}$) (see [7, 8]).

An entire function of n-complex variables for which Gol'dberg order and Gol'dberg lower order are the same is said to be of regular Gol'dberg growth. Functions which are not of regular Gol'dberg growth are said to be of irregular Gol'dberg growth.

To compare the relative growth of two entire functions of n-complex variables having same nonzero finite Gol'dberg order, one may introduce the definition of Gol'dberg type and Gol'dberg lower type in the following manner:

Definition 1. *(see [7, 8]) The Gol'dberg type and Gol'dberg lower type respectively denoted by $\Delta_{f,D}$ and $\overline{\Delta}_{f,D}$ of an entire function $f(z)$ of n-complex variables are defined as follows:*

$$\Delta_{f,D} = \overline{\lim_{R\to+\infty}} \frac{\log M_{f,D}(R)}{(R)^{\rho_f}} \quad \text{and} \quad \overline{\Delta}_{f,D} = \underline{\lim_{R\to+\infty}} \frac{\log M_{f,D}(R)}{(R)^{\rho_f}},\ 0 < \rho_f < +\infty.$$

Analogously to determine the relative growth of two entire functions of n-complex variables having same nonzero finite Gol'dberg lower order, one may introduce the definition of Gol'dberg weak type in the following way:

Definition 2. *The Gol'dberg weak type denoted by $\tau_{f,D}$ of an entire function $f(z)$ of n-complex variables is defined as follows:*

$$\tau_{f,D} = \underline{\lim_{R\to+\infty}} \frac{\log M_{f,D}(R)}{(R)^{\lambda_f}},\ 0 < \lambda_f < +\infty.$$

Also one may define the growth indicator $\overline{\tau}_{f,D}$ in the following manner :

$$\overline{\tau}_{f,D} = \overline{\lim_{R\to+\infty}} \frac{\log M_{f,D}(R)}{(R)^{\lambda_f}},\ 0 < \lambda_f < +\infty.$$

Gol'dberg [8] has shown that Gol'dberg type depends on the domain D. Hence all the growth indicators defined in Definitions 1 and 2 depend on D.

However, extending the notion of Gol'dberg order, we introduce the definitions of (p,q)-th Gol'dberg order and (p,q)-th Gol'dberg lower order of an entire function $f(z)$ of n-complex variables which are as follows:

$$\rho_{f,D}(p,q) = \overline{\lim_{R\to+\infty}} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} R} = \overline{\lim_{R\to+\infty}} \frac{\log^{[p]} R}{\log^{[q]} M_{f,D}^{-1}(R)}$$

$$(\text{respectively } \lambda_{f,D}(p,q) = \underline{\lim_{R\to+\infty}} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} R} = \underline{\lim_{R\to+\infty}} \frac{\log^{[p]} R}{\log^{[q]} M_{f,D}^{-1}(R)}).$$

The above definition avoids the restriction $p \geq q$ of (p,q)-th Gol'dberg order (respectively (p,q)-th Gol'dberg lower order) of an entire function $f(z)$ of n-complex variables introduced by Datta et al. [4]. For $p = 2$ and $q = 1$, we respectively denote $\rho_{f,D}(2,1)$ and $\lambda_{f,D}(2,1)$ by $\rho_{f,D}$ and $\lambda_{f,D}$ which are classical growth indicators (see e.g. [7, 8]). Further in the line of Gol'dberg [7, 8], one can easily verify that $\rho_{f,D}(p,q)$ (respectively $\lambda_{f,D}(p,q)$) is independent of the choice of the domain D, and therefore one can write $\rho_f(p,q)$ (respectively $\lambda_f(p,q)$) instead of $\rho_{f,D}(p,q)$ (respectively $\lambda_{f,D}(p,q)$).

In this connection we state the following two definitions which will be needed in the sequel:

Definition 3. *An entire function $f(z)$ of n-complex variables is said to have index-pair (p,q) if $b<\rho_f(p,q)<\infty$ and $\rho_f(p-1,q-1)$ is not a nonzero finite number, where $b=1$ if $p=q$ and $b=0$ for otherwise. Moreover if $0<\rho_f(p,q)<\infty$, then*

$$\begin{cases} \rho_f(p-n,q)=\infty & \text{for} \quad n<p, \\ \rho_f(p,q-n)=0 & \text{for} \quad n<q, \\ \rho_f(p+n,q+n)=1 & \text{for} \quad n=1,2,\cdots \end{cases}.$$

Similarly for $0<\lambda_f(p,q)<\infty$, one can easily verify that

$$\begin{cases} \lambda_f(p-n,q)=\infty & \text{for} \quad n<p, \\ \lambda_f(p,q-n)=0 & \text{for} \quad n<q, \\ \lambda_f(p+n,q+n)=1 & \text{for} \quad n=1,2,\cdots \end{cases}.$$

An entire function $f(z)$ of n-complex variables of index-pair (p,q) is said to be of regular (p,q) Gol'dberg growth if its (p,q)-th Gol'dberg order coincides with its (p,q)-th Gol'dberg lower order, otherwise $f(z)$ is said to be of irregular (p,q) Gol'dberg growth.

To compare the relative growth of two entire functions having same nonzero finite (p,q)-Gol'dberg order, one may introduce the definition of (p,q)-Gol'dberg type and (p,q)-Gol'dberg lower type in the following manner:

Definition 4. *Let $f(z)$ be any entire functions of n-complex variables with $0<\rho_f(p,q)<+\infty$. The (p,q)-th Gol'dberg type and (p,q)-th Gol'dberg lower type respectively denoted by $\Delta_{f,D}(p,q)$ and $\overline{\Delta}_{f,D}(p,q)$ are defined as follows:*

$$\begin{aligned} \Delta_{f,D}(p,q) &= \overline{\lim_{R\to+\infty}} \frac{\log^{[p-1]} M_{f,D}(R)}{\left[\log^{[q-1]} R\right]^{\rho_f(p,q)}} \quad \text{and} \\ \overline{\Delta}_{f,D}(p,q) &= \underline{\lim_{R\to+\infty}} \frac{\log^{[p-1]} M_{f,D}(R)}{\left[\log^{[q-1]} R\right]^{\rho_f(p,q)}}. \end{aligned}$$

An entire function $f(z)$ of n-complex variables of index-pair (p,q) is said to be of perfectly regular (p,q)-Gol'dberg growth if its (p,q)-th Gol'dberg order coincides with its (p,q)-th Gol'dberg lower order as well as its (p,q)th Gol'dberg type coincides with its (p,q)th Gol'dberg lower type.

Analogously to determine the relative growth of two entire functions of n-complex variables having same nonzero finite (p,q)-th Gol'dberg lower order, one may introduce the definition of (p,q)-th Gol'dberg weak type in the following way:

Definition 5. *Let $f(z)$ be any entire functions of n-complex variables with $0<\lambda_f(p,q)<+\infty$. The (p,q)- th Gol'dberg weak type denoted by $\tau_{f,D}(p,q)$ is defined* *as follows:*

$$\tau_{f,D}(p,q) = \varliminf_{R\to+\infty} \frac{\log^{[p-1]} M_{f,D}(R)}{\left[\log^{[q-1]} R\right]^{\lambda_f(p,q)}}.$$

Also one may define the growth indicator $\overline{\tau}_{f,D}(p,q)$ in the following manner :

$$\overline{\tau}_{f.D}(p,q) = \varlimsup_{R\to+\infty} \frac{\log^{[p-1]} M_f(R)}{\left[\log^{[q-1]} R\right]^{\lambda_f(p,q)}},\ 0 < \ \lambda_f(p,q) \ < +\infty.$$

Clearly $\Delta_{f,D}(2,1) = \Delta_{f,D}$ (respectively $\overline{\Delta}_{f,D}(2,1) = \overline{\Delta}_{f,D}$) and $\tau_{f,D}(2,1) = \tau_{f,D}$ (respectively $\overline{\tau}_{f,D}(2,1) = \overline{\tau}_{f,D}$).

Since Gol'dberg [8] has shown that Gol'dberg type depends on the domain D, therefore all the growth indicators defined in Definitions 4 and 5 depend on D.

For any two entire functions $f(z)$ and $g(z)$ of n-complex variables, Mondal et al. [9] introduced the concept of relative Gol'dberg order which is as follows:

$$\begin{aligned} \rho_{g,D}(f) &= \inf\{\mu > 0 : M_{f,D}(R) < M_{g,D}(R^{\mu}) \text{ for all } R > R_0(\mu) > 0\} \\ &= \varlimsup_{R\to+\infty} \frac{\log M_{g,D}^{-1} M_{f,D}(R)}{\log R}. \end{aligned}$$

In [9], Mandal and Roy also proved that the relative Gol'dberg order of $f(z)$ with respect to $g(z)$ is independent of the choice of the domain D. So the relative Gol'dberg order of $f(z)$ with respect to $g(z)$ may be denoted as $\rho_g(f)$ instead of $\rho_{g,D}(f)$.

Likewise, one can define the relative Gol'dberg lower order $\lambda_{g,D}(f)$ in the following manner:

$$\lambda_{g,D}(f) = \varliminf_{R\to+\infty} \frac{\log M_{g,D}^{-1} M_{f,D}(R)}{\log R}.$$

In the line of Mandal and Roy (see. [9]), one can also verify that $\lambda_{g,D}(f)$ is independent of the choice of the domain D, and therefore one can write $\lambda_g(f)$ instead of $\lambda_{g,D}(f)$.

In the case of relative Gol'dberg order, it therefore seems reasonable to define suitably the (p,q)-th relative Gol'dberg order of entire functions of n-complex variables. With this in view one may introduce the definition of (p,q)-th relative Gol'dberg order $\rho_{g,D}^{(p,q)}(f)$ of an entire function $f(z)$ of n-complex variables with respect to another entire function $g(z)$ of n-complex variables in the light of their index-pair.

Definition 6. *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pair (m,q) and (m,p), respectively. Then the (p,q)-th relative Gol'dberg order of $f(z)$ with respect to $g(z)$ is defined as*

$$\begin{aligned} \rho_{g,D}^{(p,q)}(f) &= \inf\left\{ \begin{array}{c} \mu > 0 : M_{f,D}(r) < M_{g,D}\left(\exp^{[p]}\left(\mu \log^{[q]} R\right)\right) \\ \textit{for all } R > R_0(\mu) > 0 \end{array} \right\} \\ &= \varlimsup_{R\to+\infty} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} R} = \varlimsup_{R\to+\infty} \frac{\log^{[p]} M_{g,D}^{-1}(R)}{\log^{[q]} M_{f,D}^{-1}(R)}. \end{aligned}$$

Similarly, the (p,q)*-th relative Gol'dberg lower order of* $f(z)$ *with respect to* $g(z)$ *is defined by:*

$$\lambda_{g,D}^{(p,q)}(f) = \varliminf_{R\to+\infty} \frac{\log^{[p]} M_{g.D}^{-1}(M_{f,D}(R))}{\log^{[q]} R} = \varliminf_{R\to+\infty} \frac{\log^{[p]} M_{g,D}^{-1}(R)}{\log^{[q]} M_{f,D}^{-1}(R)}.$$

Definition 6 avoids the restriction $p \geq q$ of Definition 1.3 of [1]. In view of Theorem 2.1 of [1] one can easily prove that $\rho_{g,D}^{(p,q)}(f)$ and $\lambda_{g,D}^{(p,q)}(f)$ are independent of the choice of the domain D, and therefore one can write $\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f)$ instead of $\rho_{g,D}^{(p,q)}(f)$ and $\lambda_{g,D}^{(p,q)}(f)$.

Now one may introduce the definition of relative index-pair of an entire function with respect to another entire function (both of n-complex variables) which is relevant in the sequel :

Definition 7. *Let* $f(z)$ *and* $g(z)$ *be any two entire functions of* n*-complex variables with index-pairs* (m,q) *and* (m,p) *respectively. Then the entire function* $f(z)$ *is said to have relative index-pair* (p,q) *with respect to* $g(z)$*, if* $b < \rho_g^{(p,q)}(f) < \infty$ *and* $\rho_g^{(p-1,q-1)}(f)$ *is not a nonzero finite number, where* $b = 1$ *if* $p = q = m$ *and* $b = 0$ *for otherwise. Moreover if* $0 < \rho_g^{(p,q)}(f) < \infty$*, then*

$$\begin{cases} \rho_g^{(p-n,q)}(f) = \infty & \textit{for} \quad n < p, \\ \rho_g^{(p,q-n)}(f) = 0 & \textit{for} \quad n < q, \\ \rho_g^{(p+n,q+n)}(f) \quad \textit{for} \quad n = 1, 2, \cdots & . \end{cases}$$

Similarly for $0 < \lambda_g^{(p,q)}(f) < \infty$*, one can easily verify that*

$$\begin{cases} \lambda_g^{(p-n,q)}(f) = \infty & \textit{for} \quad n < p, \\ \lambda_g^{(p,q-n)}(f) = 0 & \textit{for} \quad n < q, \\ \lambda_g^{(p+n,q+n)}(f) = 1 & \textit{for} \quad n = 1, 2, \cdots \quad . \end{cases}$$

Further an entire function $f(z)$ for which (p,q)-th relative Gol'dberg order and (p,q)-th relative Gol'dberg lower order with respect to entire function $g(z)$ are the same is called a function of regular (p,q) relative Gol'dberg growth with respect to $g(z)$. Otherwise, $f(z)$ is said to be irregular (p,q) relative Gol'dberg growth.with respect to $g(z)$.

Next we introduce the definition of (p,q)-th relative Gol'dberg type and (p,q)-th relative Gol'dberg lower type in order to compare the relative growth of two entire functions of n-complex variables having same nonzero finite (p,q)-th relative Gol'dberg order with respect to another entire function of n-complex variables.

Definition 8. *Let* $f(z)$ *and* $g(z)$ *be any two entire functions of* n*-complex variables with index-pair* (m,q) *and* (m,p)*, respectively. Also let* $0 < \rho_g^{(p,q)}(f) < +\infty$*. Then the* (p,q)*-th relative Gol'dberg type and* (p,q)*-th relative Gol'dberg lower type of*

$f(z)$ with respect to $g(z)$ are defined as

$$\Delta_{g,D}^{(p,q)}(f) = \overline{\lim_{R\to+\infty}} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{\left[\log^{[q-1]} R\right]^{\rho_g^{(p,q)}(f)}} \text{ and}$$

$$\overline{\Delta}_{g,D}^{(p,q)}(f) = \underline{\lim}_{R\to+\infty} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{\left[\log^{[q-1]} R\right]^{\rho_g^{(p,q)}(f)}}.$$

Analogously to determine the relative growth of two entire functions of n-complex variables having same nonzero finite $p,q)$-th relative Gol'dberg lower order with respect to another entire function of n-complex variables, one may introduce the definition of (p,q)-th relative Gol'dberg weak type in the following way:

Definition 9. *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pair (m,q) and (m,p), respectively. Also let $0 < \lambda_g^{(p,q)}(f) < +\infty$. Then (p,q)-th relative Gol'dberg weak type of $f(z)$ with respect to $g(z)$ denoted by $\tau_{g,D}^{(p,q)}(f)$ is defined as follows:*

$$\tau_{g,D}^{(p,q)}(f) = \underline{\lim}_{R\to+\infty} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{\left[\log^{[q-1]} R\right]^{\lambda_g^{(p,q)}(f)}}.$$

Similarly the growth indicator $\overline{\tau}_{g,D}^{(p,q)}(f)$ of an entire function $f(z)$ with respect to another entire function $g(z)$ both of n-complex variables in the following manner :

$$\overline{\tau}_{g,D}^{(p,q)}(f) = \overline{\lim_{R\to+\infty}} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{\left[\log^{[q-1]} R\right]^{\lambda_g^{(p,q)}(f)}},\ 0 < \lambda_{g,D}^{(p,q)}(f) < +\infty.$$

Therefore, for any two entire functions $f(z)$ and $g(z)$ both of n-complex variables, we note that

$$\rho_{g,D}^{(p,q)}(f) \neq \lambda_{g,D}^{(p,q)}(f), \Delta_{g,D}^{(p,q)}(f) > 0 \Rightarrow \overline{\tau}_{g,D}^{(p,q)}(f) = +\infty \text{ and}$$
$$\rho_{g,D}^{(p,q)}(f) \neq \lambda_{g,D}^{(p,q)}(f), \overline{\Delta}_{g,D}^{(p,q)}(f) > 0 \Rightarrow \tau_{g,D}^{(p,q)}(f) = +\infty.$$

Since Gol'dberg [8] has shown that Gol'dberg type depends on the domain D, all the growth indicators defined in Definitions 8 and 9 depend on D.

If the entire functions $f(z)$ and $g(z)$ (both of n-complex variables) have the same index-pair $(p,1)$ where p is any positive integer, we get the definitions of relative Gol'dberg type as introduced by Roy [10] (respectively relative Gol'dberg lower type) and relative Gol'dberg weak type.

During the past decades, several authors (see [1, 2, 3, 4, 5, 6, 9, 10]) made close investigations on the properties of entire functions of several complex variables using different growth indicator such as Gol'dberg order, (p,q)-th Gol'dberg order etc. In this paper we wish to measure some properties of entire functions relative to another entire function of several complex variables. Actualluy in this paper we wish to study some relative growth properties of entire functions of n-complex variables using (p,q)-th relative Gol'dberg order, (p,q)-th relative Gol'dberg type and (p,q)-th relative Gol'dberg weak type. Through out the paper we consider that all the growth indicators are nonzero finite.

2. Results

In this section we state the main results of the paper.

Theorem 1. *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pair (m,q) and (m,p), respectively. Then*

$$\frac{\lambda_f(m,q)}{\rho_g(m,p)} \le \lambda_g^{(p,q)}(f) \le \min\left\{\frac{\lambda_f(m,q)}{\lambda_g(m,p)}, \frac{\rho_f(m,q)}{\rho_g(m,p)}\right\}$$
$$\le \max\left\{\frac{\lambda_f(m,q)}{\lambda_g(m,p)}, \frac{\rho_f(m,q)}{\rho_g(m,p)}\right\} \le \rho_g^{(p,q)}(f) \le \frac{\rho_f(m,q)}{\lambda_g(m,p)}.$$

Proof. From the definitions of $\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f)$ we get that

$$\log \rho_g^{(p,q)}(f) = \varlimsup_{R\to+\infty}\left[\log^{[p+1]} M_{g,D}^{-1}(R) - \log^{[q+1]} M_{f,D}^{-1}(R)\right], \tag{1}$$

and

$$\log \lambda_{g,D}^{(p,q)}(f) = \varliminf_{R\to+\infty}\left[\log^{[p+1]} M_{g,D}^{-1}(R) - \log^{[q+1]} M_{f,D}^{-1}(R)\right]. \tag{2}$$

Now from the definitions of $\rho_f(m,q)$ and $\lambda_f(m,q)$, it follows that

$$\log \rho_f(m,q) = \varlimsup_{R\to+\infty}\left[\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R)\right] \text{ and} \tag{3}$$
$$\log \lambda_f(m,q) = \varliminf_{R\to+\infty}\left[\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R)\right]. \tag{4}$$

Similarly, from the definitions of $\rho_g(m,p)$ and $\lambda_g(m,p)$, we obtain that

$$\log \rho_g(m,p) = \varlimsup_{R\to+\infty}\left[\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R)\right] \text{ and} \tag{5}$$
$$\log \lambda_g(m,p) = \varliminf_{R\to+\infty}\left[\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R)\right]. \tag{6}$$

Therefore from (2), (4) and (5), we get that

$$\log \lambda_g^{(p,q)}(f) =$$
$$\varliminf_{R\to+\infty}\left[\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) - \left(\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R)\right)\right]$$

$$i.e.,\ \log \lambda_g^{(p,q)}(f) \ge \left[\varliminf_{R\to+\infty}\left(\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R)\right)\right.$$
$$\left. - \varlimsup_{R\to+\infty}\left(\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R)\right)\right]$$

$$i.e.,\ \log \lambda_g^{(p,q)}(f) \ge \left(\log \lambda_f(m,q) - \log \rho_g(m,p)\right). \tag{7}$$

Similarly, from (1), (3) and (6), it follows that

$$\log \rho_g^{(p,q)}(f) =$$
$$\varlimsup_{R\to+\infty}\left[\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) - \left(\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R)\right)\right]$$

$$i.e.,\ \log \rho_g^{(p,q)}(f) \leq \left[\varlimsup_{R\to+\infty} \left(\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) \right) - \varliminf_{R\to+\infty} \left(\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right) \right]$$

$$i.e.,\ \log \rho_g^{(p,q)}(f) \leq \left(\log \rho_f(m,q) - \log \lambda_g(m,p) \right) . \tag{8}$$

Again, in view of $(2), (3), (4), (5)$ and (6), we obtain that

$$\log \lambda_g^{(p,q)}(f) = \varliminf_{R\to+\infty} \left[\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) - \left(\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right) \right]$$

$$i.e.,\ \log \lambda_g^{(p,q)}(f) \leq$$

$$\min \left[\varliminf_{R\to+\infty} \left(\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) \right) + \varlimsup_{R\to+\infty} - \left(\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right), \right.$$
$$\left. \varlimsup_{R\to+\infty} \left(\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) \right) + \varliminf_{R\to+\infty} - \left(\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right) \right]$$

$$i.e.,\ \log \lambda_g^{(p,q)}(f) \leq$$

$$\min \left[\varliminf_{R\to+\infty} \left(\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) \right) - \varliminf_{R\to+\infty} \left(\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right), \right.$$
$$\left. \varlimsup_{R\to+\infty} \left(\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) \right) - \varlimsup_{R\to+\infty} \left(\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right) \right]$$

$$i.e.,\ \log \lambda_g^{(p,q)}(f) \leq \min \left\{ \log \lambda_f(m,q) - \log \lambda_g(m,p), \log \rho_f(m,q) - \log \rho_g(m,p) \right\} . \tag{9}$$

Further from $(1), (3), (4), (5)$ and (6), it follows that

$$\log \rho_g^{(p,q)}(f) = \varlimsup_{R\to+\infty} \left[\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) - \left(\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right) \right]$$

$$i.e.,\ \log \rho_g^{(p,q)}(f) \geq$$

$$\max \left[\varliminf_{R\to+\infty} \left(\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) \right) + \varlimsup_{R\to+\infty} - \left(\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right), \right.$$
$$\left. \varlimsup_{R\to+\infty} \left(\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) \right) + \varliminf_{R\to+\infty} - \left(\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right) \right]$$

$$i.e.,\ \log \rho_g^{(p,q)}(f) \geq$$

$$\max \left[\varliminf_{R\to+\infty} \left(\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) \right) - \varliminf_{R\to+\infty} \left(\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right), \right.$$
$$\left. \varlimsup_{R\to+\infty} \left(\log^{[m+1]} R - \log^{[q+1]} M_{f,D}^{-1}(R) \right) - \varlimsup_{R\to+\infty} \left(\log^{[m+1]} R - \log^{[p+1]} M_{g,D}^{-1}(R) \right) \right]$$

$$i.e.,\ \log \rho_g^{(p,q)}(f) \geq \max \left\{ \log \lambda_f(m,q) - \log \lambda_g(m,p), \log \rho_f(m,q) - \log \rho_g(m,p) \right\} . \tag{10}$$

Thus the theorem follows from (7), (8), (9) and (10). □

In view of Theorem 1, one can easily verify the following corollaries:

Corollary 2. *Let $f(z)$ be an entire function of n-complex variables with index-pair (m,q) and let $g(z)$ be an entire function of n-complex variables with regular (m,p) Gol'dberg growth. Then*

$$\lambda_g^{(p,q)}(f) = \frac{\lambda_f(m,q)}{\rho_g(m,p)} \quad \text{and} \quad \rho_g^{(p,q)}(f) = \frac{\rho_f(m,q)}{\rho_g(m,p)}.$$

Moreover, if $\rho_f(m,q) = \rho_g(m,p)$, then

$$\rho_g^{(p,q)}(f) = \lambda_f^{(q,p)}(g) = 1.$$

Corollary 3. *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with regular (m,q) Gol'dberg growth and regular (m,p) Gol'dberg growth. Then*

$$\lambda_g^{(p,q)}(f) = \rho_g^{(p,q)}(f) = \frac{\rho_f(m,q)}{\rho_g(m,p)}.$$

Corollary 4. *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with regular (m,q) Gol'dberg growth and regular (m,p) Gol'dberg growth. Also suppose that $\rho_f(m,q) = \rho_g(m,p)$. Then*

$$\lambda_g^{(p,q)}(f) = \rho_g^{(p,q)}(f) = \lambda_f^{(q,p)}(g) = \rho_f^{(q,p)}(g) = 1.$$

Corollary 5. *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with regular (m,q) Gol'dberg growth and regular (m,p) Gol'dberg growth. Then*

$$\rho_g^{(p,q)}(f) \cdot \rho_f^{(q,p)}(g) = \lambda_g^{(p,q)}(f) \cdot \lambda_f^{(q,p)}(g) = 1.$$

Corollary 6. *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pair (m,q) and (m,p) respectively. If either $f(z)$ is not of regular (m,q) Gol'dberg growth or $g(z)$ is not of regular (m,p) Gol'dberg growth, then*

$$\lambda_g^{(p,q)}(f) \cdot \lambda_f^{(q,p)}(g) < 1 \ < \rho_g^{(p,q)}(f) \cdot \rho_f^{(q,p)}(g).$$

Corollary 7. *Let $f(z)$ be an entire function of n- complex variables with index-pair (m,q). Then for any entire function $g(z)$ of n-complex variables*

$$\begin{aligned} (i)\ \ \lambda_g^{(p,q)}(f) &= \infty \text{ when } \rho_g(m,p) = 0, \\ (ii)\ \ \rho_g^{(p,q)}(f) &= \infty \text{ when } \lambda_g(m,p) = 0, \\ (iii)\ \ \lambda_g^{(p,q)}(f) &= 0 \text{ when } \rho_g(m,p) = \infty \end{aligned}$$

and

$$(iv)\ \ \rho_g^{(p,q)}(f) = 0 \text{ when } \lambda_g(m,p) = \infty.$$

Corollary 8. *Let $g(z)$ be an entire function of n- complex variables with index-pair (m,p). Then for any entire function $f(z)$ of n-complex variables*

$$\begin{aligned} (i)\ \ \rho_g^{(p,q)}(f) &= 0 \text{ when } \rho_f(m,q) = 0, \\ (ii)\ \ \lambda_g^{(p,q)}(f) &= 0 \text{ when } \lambda_f(m,q) = 0, \\ (iii)\ \ \rho_g^{(p,q)}(f) &= \infty \text{ when } \rho_f(m,q) = \infty, \end{aligned}$$

and

$$(iv)\ \ \lambda_g^{(p,q)}(f) = \infty \text{ when } \lambda_f(m,q) = \infty.$$

Remark 1. *From the conclusion of Theorem 1, one may write* $\rho_g^{(p,q)}(f) = \frac{\rho_f(m,q)}{\rho_g(m,p)}$ *and* $\lambda_g^{(p,q)}(f) = \frac{\lambda_f(m,q)}{\lambda_g(m,p)}$ *when* $g(z)$ *is an entire function of* n*-complex variables with regular* (m,p) *Gol'dberg growth. Similarly* $\rho_g^{(p,q)}(f) = \frac{\lambda_f(m,q)}{\lambda_g(m,p)}$ *and* $\lambda_g^{(p,q)}(f) = \frac{\rho_f(m,q)}{\rho_g(m,p)}$ *when* $f(z)$ *is an entire function of* n*-complex variables with regular* (m,q) *Gol'dberg growth.*

When $f(z)$ and $g(z)$ are any two entire functions of n-complex variables with index-pair (m,q) and (l,p) respectively, where $m \neq l$, the next definition enables us to study their relative order.

Definition 10. *Let* $f(z)$ *and* $g(z)$ *be any two entire functions of* n*-complex variables with index-pair* (m,q) *and* (l,p) *respectively. If* $m > l$*, then the* $(p+m-l,q)$*-th relative Gol'dberg order and* $(p+m-l,q)$*-th relative Gol'dberg lower order) of* $f(z)$ *with respect to* $g(z)$ *are defined as*

$$(i)\ \ \rho_g^{(p+m-l,q)}(f) = \overline{\lim_{R\to+\infty}} \frac{\log^{[p+m-l]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} R} \text{ and}$$

$$\lambda_g^{(p+m-l,q)}(f) = \underline{\lim_{R\to+\infty}} \frac{\log^{[p+m-l]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} R}.$$

If $m < l$*, then the* $(p,q+l-m)$*-th relative Gol'dberg order (resp.* $(p,q+l-m)$*-th relative Gol'dberg lower order) of* $f(z)$ *with respect to* $g(z)$ *is defined as*

$$(ii)\ \ \rho_g^{(p,q+l-m)}(f) = \overline{\lim_{R\to+\infty}} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q+l-m]} R} \text{ and}$$

$$\lambda_g^{(p,q+l-m)}(f) = \underline{\lim_{R\to+\infty}} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q+l-m]} R}.$$

The following result is easy to check.

Theorem 9. *Under the hypothesis of Definition 10, for* $m > l$:

$$(i)\ \ \rho_g^{(p+m-l,q)}(f) = \overline{\lim_{R\to+\infty}} \frac{\log^{[m]} M_{f,D}(R)}{\log^{[q]} R},$$

$$\lambda_g^{(p+m-l,q)}(f) = \underline{\lim_{R\to+\infty}} \frac{\log^{[m]} M_{f,D}(R)}{\log^{[q]} R}.$$

and for $m < l$:

$$(ii)\ \ \rho_g^{(p,q+l-m)}(f) = \overline{\lim_{R\to+\infty}} \frac{\log^{[p]} R}{\log^{[n]} M_{g,D}(R)},$$

$$\lambda_g^{(p,q+l-m)}(f) = \underline{\lim_{R\to+\infty}} \frac{\log^{[p]} R}{\log^{[n]} M_{g,D}(R)}.$$

In the next theorem we intend to find out (p,q)-th relative Gol'dberg order (respectively (p,q)-th relative Gol'dberg lower order) of an entire function $f(z)$ with respect to another entire function $g(z)$ (both $f(z)$ and $g(z)$ are of n-complex variables) when (m,q)-th relative Gol'dberg order (respectively (m,q)-th relative Gol'dberg lower order) of $f(z)$ and (m,p)-th relative Gol'dberg order (respectively

(m,p)-th relative Gol'dberg lower order) of $g(z)$ with respect to another entire function $h(z)$ ($h(z)$ is also of n- complex variables) are given.

Theorem 10. *Let $f(z)$, $g(z)$ and $h(z)$ be any three entire functions of n-complex variables. If (m,q)-th relative Gol'dberg order (respectively (m,q)-th relative Gol'dberg lower order) of $f(z)$ with respect to $h(z)$ and (m,p)-th relative Gol'dberg order (respectively (m,p)-th relative Gol'dberg lower order) of $g(z)$ with respect to $h(z)$ are respectively denoted by $\rho_h^{(m,q)}(f)$ $\left(\text{respectively } \lambda_h^{(m,q)}(f)\right)$ and $\rho_h^{(m,p)}(g)$ $\left(\text{respectively } \lambda_h^{(m,p)}(g)\right)$, then*

$$\frac{\lambda_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \leq \lambda_g^{(p,q)}(f) \leq \min\left\{\frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}\right\}$$
$$\leq \max\left\{\frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}\right\} \leq \rho_g^{(p,q)}(f) \leq \frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}.$$

The conclusion of the above theorem can be carried out after applying the same technique of Theorem 1 and therefore its proof is omitted.

In view of Theorem 10, one can easily verify the following corollaries:

Corollary 11. *Let $f(z)$, $g(z)$ and $h(z)$ be any three entire functions of n-complex variables. Also let $f(z)$ be an entire function with regular relative (m,q) Gol'dberg growth with respect to entire function $h(z)$ and $g(z)$ be entire having relative index-pair (m,p) with respect to entire function $h(z)$. Then*

$$\lambda_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \text{ and } \rho_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}.$$

In addition, if $\rho_h^{(m,q)}(f) = \rho_h^{(m,p)}(g)$, then

$$\lambda_g^{(p,q)}(f) = \rho_f^{(q,p)}(g) = 1.$$

Corollary 12. *Let $f(z)$, $g(z)$ and $h(z)$ be any three entire functions of n-complex variables. Also let $f(z)$ be an entire function with relative index-pair (m,q) with respect to entire function $h(z)$ and $g(z)$ be entire of regular (m,p) relative Gol'dberg growth with respect to entire function $h(z)$. Then*

$$\lambda_g^{(p,q)}(f) = \frac{\lambda_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \text{ and } \rho_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}.$$

In addition, if $\rho_h^{(m,q)}(f) = \rho_h^{(m,p)}(g)$, then

$$\rho_g^{(p,q)}(f) = \lambda_f^{(q,p)}(g) = 1.$$

Corollary 13. *Let $f(z)$, $g(z)$ and $h(z)$ be any three entire functions of n-complex variables. Also let $f(z)$ and $g(z)$ be any two entire functions with regular (m,q) relative Gol'dberg growth and regular (m,p) relative Gol'dberg growth with respect to entire function $h(z)$. Then*

$$\lambda_g^{(p,q)}(f) = \rho_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}.$$

Corollary 14. *Let $f(z)$, $g(z)$ and $h(z)$ be any three entire functions of n-complex variables. Also let $f(z)$ and $g(z)$ be any two entire functions with regular (m,q) relative Gol'dberg growth and regular (m,p) relative Gol'dberg growth with respect to entire function $h(z)$. Then*

$$\lambda_g^{(p,q)}(f) = \rho_g^{(p,q)}(f) = \lambda_f^{(q,p)}(g) = \rho_f^{(q,p)}(g) = 1$$

if $\rho_h^{(m,q)}(f) = \rho_h^{(m,p)}(g)$.

Corollary 15. *Let $f(z)$, $g(z)$ and $h(z)$ be any three entire functions of n-complex variables. Also let $f(z)$ and $g(z)$ be any two entire functions with relative index-pairs (m,q) and (m,p) respectively with respect to entire function $h(z)$ and either $f(z)$ is not of regular (m,q) relative Gol'dberg growth or $g(z)$ is not of regular (m,p) relative Gol'dberg growth with respect to $h(z)$, then*

$$\rho_g^{(p,q)}(f) \,.\rho_f^{(q,p)}(g) \geq 1.$$

If $f(z)$ and $g(z)$ are both of regular (m,q) relative Gol'dberg growth and regular (m,p) relative Gol'dberg growth with respect to entire function $h(z)$, then

$$\rho_g^{(p,q)}(f) \,.\rho_f^{(q,p)}(g) = 1.$$

Corollary 16. *Let $f(z)$, $g(z)$ and $h(z)$ be any three entire functions of n-complex variables. Also let $f(z)$ and $g(z)$ be any two entire functions with relative index-pairs (m,q) and (m,p) with respect to entire function $h(z)$ and either $f(z)$ is not of regular (m,q) relative Gol'dberg growth or $g(z)$ is not of regular (m,p) relative Gol'dberg growth with respect to $h(z)$, then*

$$\lambda_g^{(p,q)}(f) \,.\lambda_f^{(q,p)}(g) \leq 1.$$

If $f(z)$ and $g(z)$ are both of regular (m,q) relative Gol'dberg growth and regular (m,p) relative Gol'dberg growth with respect to entire function $h(z)$, then

$$\lambda_g^{(p,q)}(f) \,.\lambda_f^{(q,p)}(g) = 1.$$

Corollary 17. *Let $f(z)$, $g(z)$ and $h(z)$ be any three entire functions of n-complex variables. Also let $f(z)$ be an entire function with relative index-pair (m,q) with respect to $h(z)$, Then*

$$\begin{aligned} (i)\ \ \lambda_g^{(p,q)}(f) &= \infty \text{ when } \rho_h^{(m,p)}(g) = 0, \\ (ii)\ \ \rho_g^{(p,q)}(f) &= \infty \text{ when } \lambda_h^{(m,p)}(g) = 0, \\ (iii)\ \ \lambda_g^{(p,q)}(f) &= 0 \text{ when } \rho_h^{(m,p)}(g) = \infty \end{aligned}$$

and

$$(iv)\ \ \rho_g^{(p,q)}(f) = 0 \text{ when } \lambda_h^{(m,p)}(g) = \infty.$$

Corollary 18. *Let $f(z)$, $g(z)$ and $h(z)$ be any three entire functions of n-complex variables. Also let $g(z)$ be an entire function with relative index-pair (m,p) with respect to $h(z)$, Then*

$$\begin{aligned} (i)\ \ \rho_g^{(p,q)}(f) &= 0 \text{ when } \rho_h^{(m,q)}(f) = 0, \\ (ii)\ \ \lambda_g^{(p,q)}(f) &= 0 \text{ when } \lambda_h^{(m,q)}(f) = 0, \\ (iii)\ \ \rho_g^{(p,q)}(f) &= \infty \text{ when } \rho_h^{(m,q)}(f) = \infty \end{aligned}$$

and

$$(iv)\ \lambda_g^{(p,q)}(f)=\infty \text{ when } \lambda_h^{(m,q)}(f)=\infty.$$

Remark 2. *Under the same conditions of Theorem 10, one may write* $\rho_g^{(p,q)}(f)=\frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}$ *and* $\lambda_g^{(p,q)}(f)=\frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}$ *when* $\lambda_h^{(m,p)}(g)=\rho_h^{(m,p)}(g)$. *Similarly* $\rho_g^{(p,q)}(f)=\frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}$ *and* $\lambda_g^{(p,q)}(f)=\frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}$ *when* $\lambda_h^{(m,q)}(f)=\rho_h^{(m,q)}(f)$.

Next we prove our theorem based on (p,q)-th relative Gol'dberg type and (p,q)-th relative Gol'dberg weak type of entire functions of n-complex variables.

Theorem 19. *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pair (m,q) and (m,p), respectively. Then*

$$\max\left\{\left[\frac{\overline{\Delta}_{f,D}(m,q)}{\tau_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}},\left[\frac{\Delta_{f,D}(m,q)}{\overline{\tau}_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}\right\}\leq \Delta_{g,D}^{(p,q)}(f)$$
$$\leq\left[\frac{\Delta_{f,D}(m,q)}{\overline{\Delta}_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}.$$

Proof. From the definitions of $\Delta_{f,D}(m,q)$ and $\overline{\Delta}_{f,D}(m,q)$, we have for all sufficiently large values of R that

$$M_{f,D}(R) \leq \exp^{[m-1]}\left((\Delta_{f,D}(m,q)+\varepsilon)\left[\log^{[q-1]}R\right]^{\rho_f(m,q)}\right), \tag{11}$$
$$M_{f,D}(R) \geq \exp^{[m-1]}\left((\overline{\Delta}_{f,D}(m,q)-\varepsilon)\left[\log^{[q-1]}R\right]^{\rho_f(m,q)}\right) \tag{12}$$

and also for a sequence of values of R tending to infinity, we get that

$$M_{f,D}(R) \geq \exp^{[m-1]}\left((\Delta_{f,D}(m,q)-\varepsilon)\left[\log^{[q-1]}R\right]^{\rho_f(m,q)}\right), \tag{13}$$
$$M_{f,D}(R) \leq \exp^{[m-1]}\left((\overline{\Delta}_{f,D}(m,q)+\varepsilon)\left[\log^{[q-1]}R\right]^{\rho_f(m,q)}\right). \tag{14}$$

Similarly from the definitions of $\Delta_{g,D}(m,p)$ and $\overline{\Delta}_{g,D}(m,p)$, it follows for all sufficiently large values of R that

$$M_{g,D}(R) \leq \exp^{[m-1]}\left((\Delta_{g,D}(m,p)+\varepsilon)\left[\log^{[p-1]}R\right]^{\rho_g(m,p)}\right)$$
$$i.e.,\ R \leq M_{g,D}^{-1}\left(\exp^{[m-1]}\left[(\Delta_{g,D}(m,p)+\varepsilon)\left[\log^{[p-1]}R\right]^{\rho_g(m,p)}\right]\right)$$
$$i.e.,\ M_{g,D}^{-1}(R) \geq \left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}R}{(\Delta_{g,D}(m,p)+\varepsilon)}\right)^{\frac{1}{\rho_g(m,p)}}\right) \text{ and} \tag{15}$$

$$M_{g,D}^{-1}(R)\leq\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}R}{(\overline{\Delta}_{g,D}(m,p)-\varepsilon)}\right)^{\frac{1}{\rho_g(m,p)}}\right). \tag{16}$$

Also for a sequence of values of R tending to infinity, we obtain that

$$M_{g,D}^{-1}(R) \leq \left(\exp^{[p-1]} \left(\frac{\log^{[m-1]} R}{(\Delta_{g,D}(m,p) - \varepsilon)} \right)^{\frac{1}{\rho_g(m,p)}} \right) \text{ and } \tag{17}$$

$$M_{g,D}^{-1}(R) \geq \left(\exp^{[p-1]} \left(\frac{\log^{[m-1]} R}{(\overline{\Delta}_{g,D}(m,p) + \varepsilon)} \right)^{\frac{1}{\rho_g(m,p)}} \right). \tag{18}$$

From the definitions of $\overline{\tau}_{f,D}(m,q)$ and $\tau_{f,D}(m,q)$, we have for all sufficiently large values of R that

$$M_{f.D}(R) \leq \exp^{[m-1]} \left((\overline{\tau}_{f,D}(m,q) + \varepsilon) \left[\log^{[q-1]} R \right]^{\lambda_f(m,q)} \right), \tag{19}$$

$$M_{f,D}(R) \geq \exp^{[m-1]} \left((\tau_{f,D}(m,q) - \varepsilon) \left[\log^{[q-1]} R \right]^{\lambda_f(m,q)} \right) \tag{20}$$

and also for a sequence of values of R tending to infinity, we get that

$$M_{f,D}(R) \geq \exp^{[m-1]} \left((\overline{\tau}_{f.D}(m,q) - \varepsilon) \left[\log^{[q-1]} R \right]^{\lambda_f(m,q)} \right), \tag{21}$$

$$M_{f,D}(R) \leq \exp^{[m-1]} \left((\tau_{f,D}(m,q) + \varepsilon) \left[\log^{[q-1]} R \right]^{\lambda_f(m,q)} \right). \tag{22}$$

Similarly from the definitions of $\overline{\tau}_{g,D}(m,p)$ and $\tau_{g,D}(m,p)$, it follows for all sufficiently large values of R that

$$M_{g,D}(R) \leq \exp^{[m-1]} \left((\overline{\tau}_{g,D}(m,p) + \varepsilon) \left[\log^{[p-1]} R \right]^{\lambda_g(m,p)} \right)$$

$$i.e.,\ R \leq M_{g,D}^{-1} \left(\exp^{[m-1]} \left((\overline{\tau}_{g,D}(m,p) + \varepsilon) \left[\log^{[p-1]} R \right]^{\lambda_g(m,p)} \right) \right)$$

$$i.e.,\ M_{g,D}^{-1}(R) \geq \left(\exp^{[p-1]} \left(\frac{\log^{[m-1]} R}{(\overline{\tau}_{g,D}(m,p) + \varepsilon)} \right)^{\frac{1}{\lambda_g(m,p)}} \right) \text{ and } \tag{23}$$

$$M_{g,D}^{-1}(R) \leq \left(\exp^{[p-1]} \left(\frac{\log^{[m-1]} R}{(\tau_{g,D}(m,p) - \varepsilon)} \right)^{\frac{1}{\lambda_g(m,p)}} \right). \tag{24}$$

Also for a sequence of values of R tending to infinity, we obtain that

$$M_{g,D}^{-1}(R) \leq \left(\exp^{[p-1]} \left(\frac{\log^{[m-1]} R}{(\overline{\tau}_{g,D}(m,p) - \varepsilon)} \right)^{\frac{1}{\lambda_g(m,p)}} \right) \text{ and } \tag{25}$$

$$M_{g,D}^{-1}(R) \geq \left(\exp^{[p-1]} \left(\frac{\log^{[m-1]} R}{(\tau_{g,D}(m,p) + \varepsilon)} \right)^{\frac{1}{\lambda_g(m,p)}} \right). \tag{26}$$

Now from (13) and in view of (23), we get for a sequence of values of R tending to infinity that

$$M_{g,D}^{-1}(M_{f,D}(R)) \geq M_{g,D}^{-1} \left(\exp^{[m-1]} \left((\Delta_{f,D}(m,q) - \varepsilon) \left[\log^{[q-1]} R \right]^{\rho_f(m,q)} \right) \right)$$

$i.e.,\ M_{g,D}^{-1}(M_{f,D}(R)) \geq$

$$\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left(\left(\Delta_{f,D}(m,q)-\varepsilon\right)\left[\log^{[q-1]}R\right]^{\rho_f(m,q)}\right)}{\left(\overline{\tau}_{g,D}(m,p)+\varepsilon\right)}\right)^{\frac{1}{\lambda_g(m,p)}}\right)$$

$$i.e.,\ \log^{[p-1]}M_{g,D}^{-1}\left(M_{f,D}(R)\right)\geq\left[\frac{\left(\Delta_{f,D}(m,q)-\varepsilon\right)}{\left(\overline{\tau}_{g,D}(m,p)+\varepsilon\right)}\right]^{\frac{1}{\lambda_g(m,p)}}\cdot\left[\log^{[q-1]}R\right]^{\frac{\rho_f(m,q)}{\lambda_g(m,p)}}.$$

Since in view of Theorem 1, $\frac{\rho_f(m,q)}{\lambda_g(m,p)}\geq\rho_g^{(p,q)}(f)$ and as $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\overline{\lim_{R\to+\infty}}\frac{\log^{[p-1]}M_{g,D}^{-1}\left(M_{f,D}(R)\right)}{\left[\log^{[q-1]}R\right]^{\rho_g^{(p,q)}(f)}}\geq\left[\frac{\Delta_{f,D}(m,q)}{\overline{\tau}_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}$$

$$i.e.,\ \Delta_{g,D}^{(p,q)}(f)\geq\left[\frac{\Delta_{f,D}(m,q)}{\overline{\tau}_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}. \tag{27}$$

Similarly from (12) and in view of (26), it follows for a sequence of values of R tending to infinity that

$$M_{g,D}^{-1}\left(M_{f,D}(R)\right)\geq M_{g,D}^{-1}\left(\exp^{[m-1]}\left(\left(\overline{\Delta}_{f,D}(m,q)-\varepsilon\right)\left[\log^{[q-1]}R\right]^{\rho_f(m,q)}\right)\right)$$

$i.e.,\ M_{g,D}^{-1}\left(M_{f,D}(R)\right)\geq$

$$\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\overline{\Delta}_{f,D}(m,q)-\varepsilon\right)\left[\log^{[q-1]}R\right]^{\rho_f(m,q)}\right]}{\left(\tau_{g,D}(m,p)+\varepsilon\right)}\right)\right)^{\frac{1}{\lambda_g(m,p)}}$$

$$i.e.,\ \log^{[p-1]}M_{g,D}^{-1}\left(M_{f,D}(R)\right)\geq\left[\frac{\left(\overline{\Delta}_{f,D}(m,q)-\varepsilon\right)}{\left(\tau_{g,D}(m,p)+\varepsilon\right)}\right]^{\frac{1}{\lambda_g(m,p)}}\cdot\left[\log^{[q-1]}R\right]^{\frac{\rho_f(m,q)}{\lambda_g(m,p)}}.$$

In view of Theorem 1, it follows that $\frac{\rho_f(m,q)}{\lambda_g(m,p)}\geq\rho_g^{(p,q)}(f)$. Also $\varepsilon\,(>0)$ is arbitrary, so we get from above that

$$\overline{\lim_{R\to+\infty}}\frac{\log^{[p-1]}M_{g,D}^{-1}\left(M_{f,D}(R)\right)}{\left[\log^{[q-1]}R\right]^{\rho_g^{(p,q)}(f)}}\geq\left[\frac{\overline{\Delta}_{f,D}(m,q)}{\tau_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}$$

$$i.e.,\ \Delta_{g,D}^{(p,q)}(f)\geq\left[\frac{\overline{\Delta}_{f,D}(m,q)}{\tau_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}. \tag{28}$$

Again in view of (16), we have from (11) for all sufficiently large values of R that

$$M_{g,D}^{-1}\left(M_{f,D}(R)\right)\leq M_{g,D}^{-1}\left(\exp^{[m-1]}\left(\left(\Delta_{f,D}(m,q)+\varepsilon\right)\left[\log^{[q-1]}R\right]^{\rho_f(m,q)}\right)\right)$$

$i.e.,\ M_{g,D}^{-1}\left(M_{f,D}(R)\right)\leq$

$$\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\Delta_{f,D}\left(m,q\right)+\varepsilon\right)\left[\log^{[q-1]}R\right]^{\rho_f(m,q)}\right]}{\left(\overline{\Delta}_{g,D}\left(m,p\right)-\varepsilon\right)}\right)\right)^{\frac{1}{\rho_g(m,p)}}$$

$$i.e.,\ \log^{[p-1]}M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right)\leq\left[\frac{\left(\Delta_{f,D}\left(m,q\right)+\varepsilon\right)}{\left(\overline{\Delta}_{g,D}\left(m,p\right)-\varepsilon\right)}\right]^{\frac{1}{\rho_g(m,p)}}\cdot\left[\log^{[q-1]}R\right]^{\frac{\rho_f(m,q)}{\rho_g(m,p)}}. \tag{29}$$

In view of Theorem 1, it follows that $\frac{\rho_f(m,q)}{\rho_g(m,p)}\leq\rho_g^{(p,q)}\left(f\right)$. Since $\varepsilon\left(>0\right)$ is arbitrary, we get from (29) that

$$\varlimsup_{R\rightarrow+\infty}\frac{\log^{[p-1]}M_{g,D}^{-1}\left(M_{f,D}\left(R\right)\right)}{\left[\log^{[q-1]}R\right]^{\rho_g^{(p,q)}(f)}}\leq\left[\frac{\Delta_{f,D}\left(m,q\right)}{\overline{\Delta}_{g,D}\left(m,p\right)}\right]^{\frac{1}{\rho_g(m,p)}}$$

$$i.e.,\ \Delta_{g,D}^{(p,q)}\left(f\right)\leq\left[\frac{\Delta_{f,D}\left(m,q\right)}{\overline{\Delta}_{g,D}\left(m,p\right)}\right]^{\frac{1}{\rho_g(m,p)}}. \tag{30}$$

Thus the theorem follows from (27), (28) and (30). □

The conclusion of the following corollary can be carried out from (16) and (19); (19) and (24) respectively after applying the same technique of Theorem 19 and with the help of Theorem 1. Therefore its proof is omitted.

Corollary 20. *Let $f\left(z\right)$ and $g\left(z\right)$ be any two entire functions of n-complex variables with index-pair $\left(m,q\right)$ and $\left(m,p\right)$, respectively. Then*

$$\Delta_{g,D}^{(p,q)}\left(f\right)\leq\min\left\{\left[\frac{\overline{\tau}_{f,D}\left(m,q\right)}{\tau_{g,D}\left(m,p\right)}\right]^{\frac{1}{\lambda_g(m,p)}},\left[\frac{\overline{\tau}_{f,D}\left(m,q\right)}{\overline{\Delta}_{g,D}\left(m,p\right)}\right]^{\frac{1}{\rho_g(m,p)}}\right\}.$$

Similarly in the line of Theorem 19 and with the help of Theorem 1, one may easily carry out the following theorem from pairwise inequalities numbers (20) and (23); (17) and (19); (16) and (22) respectively and therefore its proofs is omitted:

Theorem 21. *Let $f\left(z\right)$ and $g\left(z\right)$ be any two entire functions of n-complex variables with index-pair $\left(m,q\right)$ and $\left(m,p\right)$, respectively. Then*

$$\left[\frac{\tau_{f,D}\left(m,q\right)}{\overline{\tau}_{g,D}\left(m,p\right)}\right]^{\frac{1}{\lambda_g(m,p)}}\leq\tau_{g,D}^{(p,q)}\left(f\right)\leq$$
$$\min\left\{\left[\frac{\tau_{f,D}\left(m,q\right)}{\overline{\Delta}_{g,D}\left(m,p\right)}\right]^{\frac{1}{\rho_g(m,p)}},\left[\frac{\overline{\tau}_{f,D}\left(m,q\right)}{\Delta_{g,D}\left(m,p\right)}\right]^{\frac{1}{\rho_g(m,p)}}\right\}.$$

Corollary 22. *Let $f\left(z\right)$ and $g\left(z\right)$ be any two entire functions of n-complex variables with index-pair $\left(m,q\right)$ and $\left(m,p\right)$, respectively. Then*

$$\tau_{g,D}^{(p,q)}\left(f\right)\geq\max\left\{\left[\frac{\overline{\Delta}_{f,D}\left(m,q\right)}{\Delta_{g,D}\left(m,p\right)}\right]^{\frac{1}{\rho_g(m,p)}},\left[\frac{\overline{\Delta}_{f,D}\left(m,q\right)}{\overline{\tau}_{g,D}\left(m,p\right)}\right]^{\frac{1}{\lambda_g(m,p)}}\right\}.$$

With the help of Theorem 1, the conclusion of the above corollary can be carried out from (12), (15) and (12), (23) respectively after applying the same technique of Theorem 19 and therefore its proof is omitted.

Theorem 23. *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pair (m,q) and (m,p), respectively. Then*

$$\left[\frac{\overline{\Delta}_{f,D}(m,q)}{\overline{\tau}_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}} \leq \overline{\Delta}^{(p,q)}_{g,D}(f) \leq$$

$$\min\left\{\left[\frac{\overline{\Delta}_{f,D}(m,q)}{\overline{\Delta}_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}, \left[\frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}\right\}.$$

Proof. From (12) and in view of (23), we get for all sufficiently large values of R that

$$M^{-1}_{g,D}(M_{f,D}(R)) \geq M^{-1}_{g}\left(\exp^{[m-1]}\left(\left(\overline{\Delta}_{f,D}(m,q)-\varepsilon\right)\left[\log^{[q-1]}R\right]^{\rho_f(m,q)}\right)\right)$$

i.e., $_{g,D}^{-1}(M_{f,D}(R)) \geq$

$$\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\overline{\Delta}_{f,D}(m,q)-\varepsilon\right)\left[\log^{[q-1]}R\right]^{\rho_f(m,q)}\right]}{\left(\overline{\tau}_{g,D}(m,p)+\varepsilon\right)}\right)^{\frac{1}{\lambda_g(m,p)}}\right)$$

$$i.e.,\ \log^{[p-1]}M^{-1}_{g,D}(M_{f,D}(R)) \geq \left[\frac{\left(\overline{\Delta}_{f,D}(m,q)-\varepsilon\right)}{\left(\overline{\tau}_{g,D}(m,p)+\varepsilon\right)}\right]^{\frac{1}{\lambda_g(m,p)}}\cdot\left[\log^{[q-1]}R\right]^{\frac{\rho_f(m,q)}{\lambda_g(m,p)}}.$$

Now in view of Theorem 1, it follows that $\frac{\rho_f(m,q)}{\lambda_g(m,p)} \geq \rho^{(p,q)}_g(f)$. Since $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\underline{\lim}_{R\to+\infty}\frac{\log^{[p-1]}M^{-1}_{g,D}(M_{f,D}(R))}{\left[\log^{[q-1]}R\right]^{\rho^{(p,q)}_g(f)}} \geq \left[\frac{\overline{\Delta}_{f,D}(m,q)}{\overline{\tau}_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}$$

$$i.e.,\ \overline{\Delta}^{(p,q)}_{g,D}(f) \geq \left[\frac{\overline{\Delta}_{f,D}(m,q)}{\overline{\tau}_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}. \qquad (31)$$

Further in view of (17), we get from (11) for a sequence of values of R tending to infinity that

$$M^{-1}_{g,D}(M_{f,D}(R)) \leq M^{-1}_{g,D}\left(\exp^{[m-1]}\left(\left(\Delta_{f,D}(m,q)+\varepsilon\right)\left[\log^{[q-1]}R\right]^{\rho_f(m,q)}\right)\right)$$

i.e., $M^{-1}_{g,D}(M_{f,D}(R)) \leq$

$$\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\Delta_{f,D}(m,q)+\varepsilon\right)\left[\log^{[q-1]}R\right]^{\rho_f(m,q)}\right]}{\left(\Delta_{g,D}(m,p)-\varepsilon\right)}\right)^{\frac{1}{\rho_g(m,p)}}\right)$$

$$i.e.,\ \log^{[p-1]}M^{-1}_{g,D}(M_{f,D}(R)) \leq \left[\frac{\left(\Delta_{f,D}(m,q)+\varepsilon\right)}{\left(\Delta_{g,D}(m,p)-\varepsilon\right)}\right]^{\frac{1}{\rho_g(m,p)}}\cdot\left[\log^{[q-1]}R\right]^{\frac{\rho_f(m,q)}{\rho_g(m,p)}}. \qquad (32)$$

Again in view of Theorem 1, $\frac{\rho_f(m,q)}{\rho_g(m,p)} \leq \rho_g^{(p,q)}(f)$ and $\varepsilon (> 0)$ is arbitrary, therefore we get from (32) that

$$\varliminf_{R\to+\infty} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{\left[\log^{[q-1]} R\right]^{\rho_g^{(p,q)}(f)}} \leq \left[\frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}$$

$$i.e.,\ \overline{\Delta}_{g,D}^{(p,q)}(f) \leq \left[\frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}. \tag{33}$$

Likewise from (14) and in view of (16), it follows for a sequence of values of R tending to infinity that

$$M_{g,D}^{-1}(M_{f,D}(R)) \leq M_{g,D}^{-1}\left(\exp^{[m-1]}\left(\left(\overline{\Delta}_{f,D}(m,q)+\varepsilon\right)\left[\log^{[q-1]} R\right]^{\rho_f(m,q)}\right)\right)$$

$i.e.,\ M_{g,D}^{-1}(M_{f,D}(R)) \leq$

$$\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left[\left(\overline{\Delta}_{f,D}(m,q)+\varepsilon\right)\left[\log^{[q-1]} R\right]^{\rho_f(m,q)}\right]}{\left(\overline{\Delta}_{g,D}(m,p)-\varepsilon\right)}\right)^{\frac{1}{\rho_g(m,p)}}\right)$$

$$i.e.,\ \log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R)) \leq \left[\frac{\left(\overline{\Delta}_{f,D}(m,q)+\varepsilon\right)}{\left(\overline{\Delta}_{g,D}(m,p)-\varepsilon\right)}\right]^{\frac{1}{\rho_g(m,p)}} \cdot \left[\log^{[q-1]} R\right]^{\frac{\rho_f(m,q)}{\rho_g(m,p)}}. \tag{34}$$

Analogously, we get from (34) that

$$\varliminf_{r\to\infty} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{\left[\log^{[q-1]} R\right]^{\rho_g^{(p,q)}(f)}} \leq \left[\frac{\overline{\Delta}_{f,D}(m,q)}{\overline{\Delta}_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}$$

$$i.e.,\ \overline{\Delta}_{g,D}^{(p,q)}(f) \leq \left[\frac{\overline{\Delta}_{f,D}(m,q)}{\overline{\Delta}_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}, \tag{35}$$

since in view of Theorem 1, $\frac{\rho_f(m,q)}{\rho_g(m,p)} \leq \rho_g^{(p,q)}(f)$ and $\varepsilon (> 0)$ is arbitrary.

Thus the theorem follows from (31), (33) and (35). □

Corollary 24. *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pair (m,q) and (m,p), respectively. Then*

$$\overline{\Delta}_{g,D}^{(p,q)}(f) \leq \min\left\{\left[\frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}, \left[\frac{\overline{\tau}_{f,D}(m,q)}{\overline{\tau}_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}},\right.$$
$$\left.\left[\frac{\overline{\tau}_{f,D}(m,q)}{\sigma_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}, \left[\frac{\tau_{f,D}(m,q)}{\overline{\sigma}_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}\right\}.$$

The conclusion of the above corollary can be carried out from pairwise inequalities no (16) and (22); (17) and (19); (22) and (24); (19) and (25) respectively after applying the same technique of Theorem 23 and with the help of Theorem 1. Therefore its proof is omitted.

Similarly in the line of Theorem 19 and with the help of Theorem 1, one may easily carry out the following theorem from pairwise inequalities no (21) and (23) ; (20) and (26); (16) and (19) respectively and therefore its proofs is omitted:

Theorem 25. *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pair (m,q) and (m,p), respectively. Then*

$$\max\left\{\left[\frac{\overline{\tau}_{f,D}(m,q)}{\overline{\tau}_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}},\left[\frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}\right\}\leq\overline{\tau}^{(p,q)}_{g,D}(f)$$
$$\leq\left[\frac{\overline{\tau}_{f,D}(m,q)}{\overline{\Delta}_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}.$$

Corollary 26. *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pair (m,q) and (m,p), respectively. Then*

$$\overline{\tau}^{(p,q)}_{g,D}(f)\geq\max\left\{\left[\frac{\overline{\Delta}_{f,D}(m,q)}{\overline{\Delta}_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}},\left[\frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}},\right.$$
$$\left.\left[\frac{\Delta_{f,D}(m,q)}{\overline{\tau}_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}},\left[\frac{\overline{\Delta}_{f,D}(m,q)}{\tau_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}\right\}.$$

The conclusion of the above corollary can be carried out from pairwise inequalities no (13) and (15) ; (12) and (18) ; (13) and (23); (12) and (26) respectively after applying the same technique of Theorem 23 and with the help of Theorem 1. Therefore its proof is omitted.

Now we state the following theorems without their proofs as because they can be derived easily using the same technique or with some easy reasoning with the help of Remark 1 and therefore left to the readers.

Theorem 27. *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pair (m,q) and (m,p), respectively. Also let $g(z)$ be of regular (m,p) Gol'dberg growth. Then*

$$\left[\frac{\overline{\Delta}_{f,D}(m,q)}{\Delta_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}\leq\overline{\Delta}^{(p,q)}_{g,D}(f)\leq$$
$$\min\left\{\left[\frac{\overline{\Delta}_{f,D}(m,q)}{\overline{\Delta}_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}},\left[\frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}\right\}\leq$$
$$\max\left\{\left[\frac{\overline{\Delta}_{f,D}(m,q)}{\overline{\Delta}_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}},\left[\frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}\right\}\leq$$
$$\Delta^{(p,q)}_{g,D}(f)\leq\left[\frac{\Delta_{f,D}(m,q)}{\overline{\Delta}_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}$$

and

$$\left[\frac{\tau_{f,D}(m,q)}{\overline{\tau}_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}} \leq \tau_{g,D}^{(p,q)}(f) \leq$$
$$\min\left\{\left[\frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}, \left[\frac{\overline{\tau}_{f,D}(m,q)}{\overline{\tau}_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}\right\} \leq$$
$$\max\left\{\left[\frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}, \left[\frac{\overline{\tau}_{f,D}(m,q)}{\overline{\tau}_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}\right\} \leq$$
$$\overline{\tau}_{g,D}^{(p,q)}(f) \leq \left[\frac{\overline{\tau}_{f}(m,q)}{\tau_{g}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}.$$

Theorem 28. *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pair (m,q) and (m,p), respectively. Also let $f(z)$ be of regular (m,q) Gol'dberg growth. Then*

$$\left[\frac{\overline{\Delta}_{f,D}(m,q)}{\Delta_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}} \leq \tau_{g,D}^{(p,q)}(f) \leq$$
$$\min\left\{\left[\frac{\overline{\Delta}_{f,D}(m,q)}{\overline{\Delta}_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}, \left[\frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}\right\} \leq$$
$$\max\left\{\left[\frac{\overline{\Delta}_{f,D}(m,q)}{\overline{\Delta}_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}, \left[\frac{\Delta_{f,D}(m,q)}{\Delta_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}\right\} \leq$$
$$\overline{\tau}_{g,D}^{(p,q)}(f) \leq \left[\frac{\Delta_{f,D}(m,q)}{\overline{\Delta}_{g,D}(m,p)}\right]^{\frac{1}{\rho_g(m,p)}}$$

and

$$\left[\frac{\tau_{f,D}(m,q)}{\overline{\tau}_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}} \leq \overline{\Delta}_{g,D}^{(p,q)}(f) \leq$$
$$\min\left\{\left[\frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}, \left[\frac{\overline{\tau}_{f,D}(m,q)}{\overline{\tau}_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}\right\} \leq$$
$$\max\left\{\left[\frac{\tau_{f,D}(m,q)}{\tau_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}, \left[\frac{\overline{\tau}_{f,D}(m,q)}{\overline{\tau}_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}\right\} \leq$$
$$\Delta_{g,D}^{(p,q)}(f) \leq \left[\frac{\overline{\tau}_{f,D}(m,q)}{\tau_{g,D}(m,p)}\right]^{\frac{1}{\lambda_g(m,p)}}.$$

In the next theorems we intend to find out (p,q)-th relative Gol'dberg type (respectively (p,q)-th relative Gol'dberg lower type, (p,q)-th relative Gol'dberg weak type) of an entire function $f(z)$ with respect to another entire function $g(z)$ (both $f(z)$ and $g(z)$ are of n-complex variables) when (m,q)-th relative Gol'dberg type (respectively (m,q)-th relative Gol'dberg lower type, (m,q)-th relative Gol'dberg weak type) of $f(z)$ and (m,p)-th relative Gol'dberg type (respectively (m,p)-th relative Gol'dberg lower type, (m,p)-th relative Gol'dberg weak type) of $g(z)$ with respect to another entire function $h(z)$ ($h(z)$ is also of n- complex variables) are given. Basically we state the theorems without their proofs as those can easily be carried out after applying the same technique our previous discussion and with the help of Theorem 10 and Remark 2.

Theorem 29. *Let $f(z)$, $g(z)$ and $h(z)$ be any three entire functions of n-complex variables. Also let $f(z)$ and $g(z)$ be entire functions with relative index-pairs (m,q) and (m,p) with respect to $h(z)$ respectively. If $\lambda_h^{(m,p)}(g) = \rho_h^{(m,p)}(g)$, then*

$$\left[\frac{\overline{\sigma}_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}} \leq \overline{\sigma}_{g,D}^{(p,q)}(f) \leq$$

$$\min\left\{\left[\frac{\overline{\sigma}_{h,D}^{(m,q)}(f)}{\overline{\sigma}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\sigma_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\} \leq$$

$$\max\left\{\left[\frac{\overline{\sigma}_{h,D}^{(m,q)}(f)}{\overline{\sigma}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\sigma_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\} \leq$$

$$\sigma_{g,D}^{(p,q)}(f) \leq \left[\frac{\sigma_{h,D}^{(m,q)}(f)}{\overline{\sigma}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

and

$$\left[\frac{\tau_{h,,D}^{(m,q)}(f)}{\overline{\tau}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}} \leq \tau_{g,D}^{(p,q)}(f) \leq$$

$$\min\left\{\left[\frac{\tau_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left[\frac{\overline{\tau}_{h,D}^{(m,q)}(f)}{\overline{\tau}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\} \leq$$

$$\max\left\{\left[\frac{\tau_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left[\frac{\overline{\tau}_{h,D}^{(m,q)}(f)}{\overline{\tau}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\} \leq$$

$$\overline{\tau}_{g,D}^{(p,q)}(f) \leq \left[\frac{\overline{\tau}_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}.$$

Theorem 30. *Let $f(z)$, $g(z)$ and $h(z)$ be any three entire functions of n-complex variables. Also let $f(z)$ and $g(z)$ be entire functions with relative index-pairs (m,q) and (m,p) with respect to $h(z)$ respectively. If $f(z)$ is of regular (m,q) relative*

Gol'dberg growth with respect to entire function $h(z)$, *then*

$$\left[\frac{\overline{\sigma}_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}} \leq \tau_{g,D}^{(p,q)}(f) \leq$$
$$\min\left\{\left[\frac{\overline{\sigma}_{h,D}^{(m,q)}(f)}{\overline{\sigma}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\sigma_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\} \leq$$
$$\max\left\{\left[\frac{\overline{\sigma}_{h,D}^{(m,q)}(f)}{\overline{\sigma}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\sigma_{h,D}^{(m,q)}(f)}{\sigma_{h,D,}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\} \leq$$
$$\overline{\tau}_{g,D}^{(p,q)}(f) \leq \left[\frac{\sigma_{h,D}^{(m,q)}(f)}{\overline{\sigma}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

and

$$\left[\frac{\tau_{h,D}^{(m,q)}(f)}{\overline{\tau}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}} \leq \overline{\sigma}_{g,D}^{(p,q)}(f) \leq$$
$$\min\left\{\left[\frac{\tau_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left[\frac{\overline{\tau}_{h,D}^{(m,q)}(f)}{\overline{\tau}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\} \leq$$
$$\max\left\{\left[\frac{\tau_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left[\frac{\overline{\tau}_{h,D}^{(m,q)}(f)}{\overline{\tau}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\} \leq$$
$$\sigma_{g,D}^{(p,q)}(f) \leq \left[\frac{\overline{\tau}_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}.$$

Theorem 31. *Let* $f(z)$, $g(z)$ *and* $h(z)$ *be any three entire functions of* n*-complex variables. Also let* $f(z)$ *and* $g(z)$ *be entire functions with relative index-pairs* (m,q) *and* (m,p) *with respect to* $h(z)$ *respectively. Then*

$$\max\left\{\left[\frac{\overline{\sigma}_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left[\frac{\sigma_{h,D}^{(m,q)}(f)}{\overline{\tau}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\} \leq \sigma_{g,D}^{(p,q)}(f)$$
$$\leq \min\left\{\left[\frac{\overline{\tau}_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left[\frac{\sigma_{h,D}^{(m,q)}(f)}{\overline{\sigma}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\overline{\tau}_{h,D}^{(m,q)}(f)}{\overline{\sigma}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}$$

and

$$\left[\frac{\overline{\sigma}_{h,D}^{(m,q)}(f)}{\overline{\tau}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}} \leq \overline{\sigma}_{g,D}^{(p,q)}(f)$$

$$\leq \min\left\{\begin{array}{c} \left[\frac{\overline{\sigma}_{h,D}^{(m,q)}(f)}{\overline{\sigma}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\sigma_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\tau_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \\ \left[\frac{\overline{\tau}_{h,D}^{(m,q)}(f)}{\overline{\tau}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left[\frac{\overline{\tau}_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\tau_{h,D}^{(m,q)}(f)}{\overline{\sigma}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}} \end{array}\right\}.$$

Theorem 32. *Let $f(z)$, $g(z)$ and $h(z)$ be any three entire functions of n-complex variables. Also let $f(z)$ and $g(z)$ be entire functions with relative index-pairs (m,q) and (m,p) with respect to $h(z)$ respectively. Then*

$$\max\left\{\begin{array}{c} \left[\frac{\overline{\tau}_{h,D}^{(m,q)}(f)}{\overline{\tau}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left[\frac{\tau_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left[\frac{\overline{\sigma}_{h,D}^{(m,q)}(f)}{\overline{\sigma}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \\ \left[\frac{\sigma_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\sigma_{h,D}^{(m,q)}(f)}{\overline{\tau}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left[\frac{\overline{\sigma}_{h,D}^{(m,q)}(f)}{\tau_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}} \end{array}\right\}$$

$$\leq \overline{\tau}_{g,D}^{(p,q)}(f) \leq \left[\frac{\tau_{h,D}^{(m,q)}(f)}{\overline{\sigma}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

and

$$\max\left\{\left[\frac{\overline{\sigma}_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\tau_{h,D}^{(m,q)}(f)}{\overline{\tau}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left[\frac{\overline{\sigma}_{h,D}^{(m,q)}(f)}{\overline{\tau}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\} \leq \tau_{g,D}^{(p,q)}(f)$$

$$\leq \min\left\{\left[\frac{\tau_{h,D}^{(m,q)}(f)}{\overline{\sigma}_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left[\frac{\overline{\tau}_{h,D}^{(m,q)}(f)}{\sigma_{h,D}^{(m,p)}(g)}\right]^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}.$$

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