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# ON THE AWARENESS PROGRAMS OF THE EPIDEMIC OUTBREAKS FRACTIONAL MODEL

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Abstract. In this paper, a novel fractional model for the effect of awareness programs on the epidemic outbreaks is presented. This system is generalized to the standard awareness programs model by using fractional Caputo operator. Properties of the introduced system is studied analytically and numerically. Three numerical methods are introduced to solve the proposed model. These methods are the generalized Euler method, the predictor-corrector method, and the nonstandard finite difference method. Numerical simulations show that, the nonstandard finite difference method can be applied to solve such fractional differential equations simply and effectively.

## 1. Introduction

It is well known that a mathematical models has proven to be valuable in understating the dynamics of the spread of infectious diseases in a population ([1]). The classical models depend on the interactions between susceptibles and infected population. However, there are other factors, such as media and immigration of population etc., which affect the spread of infectious diseases. The awareness programs by media as posters, television messages, social media outlets (i.e., Twitter, Facebook) are used daily to inform the public on current health issues to reduce their chances of being infected. In ([2]), Misra et al. proposed a nonlinear mathematical model for the effects of awareness programs on the spread of infectious diseases and assumed the growth rate of awareness programs is proportional to the number of infective individuals. The model analysis showed that the spread of an infectious disease can be controlled by using awareness programs, but the disease remains endemic due to immigration. In ([3]), Lixia Zuo et al. proposed a nonlinear model with delay time and the effects of awareness programs driven by the media on the spread of an infection and assumed diseases spread due to the direct contact between susceptible and infective individuals only.

In this paper, we interest in fractional calculus because it is a new powerful tool which has been recently employed to model biological systems with non-linear

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behavior. The fractional order differential equations are generalizations of integer order differential equations. They are more suitable and accurate than integer order where the effects of previous values (memory) are important. The memory term ensures the history and its impact to the present and future ([4]-[6]).

The main aim of this work is to study numerically the approximate solutions of the fractional model. Three numerical methods are given to solve the proposed model, these method are the generalized Euler method (GEM), the predictor-corrector (PCM) method and the nonstandard finite difference method (NSFDM). Comparative studies are implemented between the proposed methods. To check the feasibility of our analysis, we use Matlab software ode45.

The outlines of this paper organized as follows: In section 2, we recall some definitions on fractional calculus. In section 3, the fractional order of model and analyze the stability of the equilibrium point. In section 4, three numerical methods are given to solve the proposed model are presented, moreover the properties of NSFDM method are discussed. Numerical simulations for the proposed model are given in Section 5. Finally, the conclusions are given in Section 6.

## 2. Preliminaries and Notations

In this section, we recall some important definitions of the fractional calculus used throughout the remaining sections of this paper. We present the Caputo fractional derivative of order $\alpha$, for more details see ([7, 8]). Let us consider the following fractional order differentiation equation:

$$ {}_0^C D_t^\alpha y(t) = f(t, y(t)), \;\; 0 < t \leq T, \quad 0 < \alpha \leq 1, \tag{1} $$

$$ y(0) = y_o. $$

Where ${}_0^C D_t^\alpha$ denotes the Caputo fractional derivative of $y(t)$, it is defined as follows:

$$ {}_0^C D_t^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n y(s)}{ds^n} ds. \tag{2} $$

Where $n-1 < \alpha \leq n, \;\; n \in N$, and $\Gamma(.)$ is the gamma function.
The Grünwald−Letinkov fractional derivative of order $\alpha$ of a time-function $y(t)$ at time $t$ is:

$$ D^\alpha y(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{[t/h]} (-1)^\alpha \binom{\alpha}{j} y(t - jh), \tag{3} $$

this formula can be discretized as

$$ D^\alpha y(t_n) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{n} w_j^{(\alpha)} y(t_{n-j}) \;\; , n = 1, 2, 3, ..., \tag{4} $$

where $[t]$ denotes to the integer part of $t$ and $h = \dfrac{t_n}{n}$ is the step-size. The Grünwald−Letinkov coefficients $w_j^{(\alpha)}$ can be calculated by the following formula:

$$ w_j^{(\alpha)} = (1 - \tfrac{1+\alpha}{j}) w_{j-1}^\alpha \text{ and } w_o^{(\alpha)} = 1 \;, \; j \geq 1. $$

## 3. Mathematical Model of Fractional Order

In this section, a fractional model of effects of awareness programs on the epidemic outbreaks is presented. The total population is divided into three classes, the susceptible population $X_m(t)$, the aware population $Y(t)$ and the infected population $X(t)$. One of the main assumptions of this model is that, at time $t$ the density of awareness programs driven by media is $M(t)$ using the fact $X(t)+X_m(t)+Y(t)=1$. The new parameters of the model are described in Table 1. It is important to notice that all the parameters here are depended on the fractional order $\alpha$. To make the system more consistent the reality, we must therefore make sure that the right-hand sides of these equations have the same dimensions. Therefore, we need to modify the right-hand sides to make the dimensions match, he most straightforward way of doing this is to put power $\alpha$ of each parameter in the right sides so that when $\alpha \to 1$ the system reduces to classical one. Due to the fact that the fractional-order play a vital role in biological systems with memory which gives more degree of freedom ([15]-[17]). The modified system of nonlinear fractional differential equations is given as follows:

$$ {}_0^C D_t^\alpha X(t) = \lambda^\alpha(1-X(t)-Y(t))M(t)-(\lambda_o^\alpha+d^\alpha)X(t)-\alpha_o^\alpha X(t)Y(t)+q^\alpha v^\alpha Y(t), \quad (5) $$

$$ {}_0^C D_t^\alpha Y(t) = \beta^\alpha(1-X(t)-Y(t))Y(t)-(v^\alpha+d^\alpha)Y(t)+\alpha_o^\alpha X(t)Y(t), \quad (6) $$

$$ {}_0^C D_t^\alpha M(t) = \mu^\alpha Y(t)-\mu_o^\alpha M(t). \quad (7) $$

Where ${}_0^C D_t^\alpha$ is the Caputo fractional derivative of order $0<\alpha\leq 1$. The region of attraction which is given by the set ([3]):

$$ \Omega = \left\{(X,Y,M)\in R_+^3,\ X\geq 0,\ Y\leq 1,\ 0\leq M<\mu^\alpha/\mu_o^\alpha\right\}. \quad (8) $$

If fractional order $\alpha=1$, then we get a nonlinear ordinary differential model.

### 3.1. The basic reproduction number.

In the following, the basic reproduction number $R_o$ will be discussed, where $R_o$ is defined as the expected number of secondary cases produced in a completely susceptible population by a typical aware population. It is well Known that $R_o=\rho(FV^{-1})$, for more details see ([14]), where $\rho$ is the spectral radius of the matrix $F$, and $V$.

$$ F = \frac{\partial H_i}{\partial x_j}, \quad (9) $$

and

$$ V = \frac{\partial G_i}{\partial x_j}. \quad (10) $$

Where $H$ is the rate of appearance of a new aware population in class $i$, $G$ is the rate of transfer of individuals out or into class $i$, and the order of aware variables $x_j=(x_1,x_2)=(Y,M)$.

Then

$$ \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} \beta^\alpha(1-X-Y)Y \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} (v^\alpha+d^\alpha)Y-\alpha_o^\alpha XY \\ -\mu^\alpha Y+\mu_o^\alpha M \end{pmatrix}. $$

TABLE 1. All symbols in the system and their definition ([3]).

| Symbol | Definition | Value |
|---|---|---|
| $t$ | Time | $t \geq 0$ |
| $q^\alpha$ | The rate of recovered people will become aware and join the aware susceptible class. | $0.85^\alpha$ |
| $p^\alpha$ | The rate remaining fraction will become unaware susceptible, where $p^\alpha + q^\alpha = 1$. | $0.15^\alpha$ |
| $\alpha_o^\alpha$ | The contact rate of aware susceptible with infective population. | $0.2^\alpha$ |
| $\beta^\alpha$ | The contact rate of unaware susceptible with infective population. | Assumed |
| $\lambda^\alpha$ | The dissemination rate of awareness among unaware susceptible class. | $0.08^\alpha$ |
| $\lambda_o^\alpha$ | The rate of transfer of aware susceptible to unaware class. | $0.02^\alpha$ |
| $v^\alpha$ | The recovery rate. | $0.43^\alpha$ |
| $d^\alpha$ | The Death rate. | $0.002^\alpha$ |
| $\mu^\alpha$ | The rate proportional to infection population. | $0.002^\alpha$ |
| $\mu_o^\alpha$ | The rate of inefficiency of programs. | $0.02^\alpha$ |

At disease-free equilibrium $E_o = (0, 0, 0)$:

$$F = \begin{pmatrix} \dfrac{\partial H_1}{\partial Y} & \dfrac{\partial H_1}{\partial M} \\ \dfrac{\partial H_2}{\partial Y} & \dfrac{\partial H_2}{\partial M} \end{pmatrix} = \begin{pmatrix} \beta^\alpha & 0 \\ 0 & 0 \end{pmatrix}, \tag{11}$$

$$V = \begin{pmatrix} \dfrac{\partial G_1}{\partial Y} & \dfrac{\partial G_1}{\partial M} \\ \dfrac{\partial G_2}{\partial Y} & \dfrac{\partial G_2}{\partial M} \end{pmatrix} = \begin{pmatrix} v^\alpha + d^\alpha & 0 \\ -\mu^\alpha & \mu_o^\alpha \end{pmatrix}. \tag{12}$$

The basic reproduction number $R_0$ for a system $(5) - (7)$ is

$$R_0 = \frac{\beta^\alpha}{v^\alpha + d^\alpha}. \tag{13}$$

.

3.2. **Equilibrium points and their asymptotic stability.** In the following, to discuss the local asymptotic stability for evaluating the equilibrium points, let us consider: ${}_0^C D_t^\alpha X = 0$, ${}_0^C D_t^\alpha Y = 0$ and ${}_0^C D_t^\alpha M = 0$. Then, from (1), we have

$$f_j(\bar{X}, \bar{Y}, \bar{M}) = 0, \quad j = 1, 2, 3,$$

where, $(\bar{X}, \bar{Y}, \bar{M})$ denotes any equilibrium point.

3.2.1. *Stability of disease-free equilibrium.* The system $(5) - (7)$ has a disease-free equilibrium when all the variables do not change with time (i.e., first derivative w.r.t time equal zero ) and the steady-state solutions of a model in the absence of infection or disease ([14]).

Let the right hand side of the system $(5)-(7)$ equal to zero and if $Y=0$ then the disease-free point is $E_o=(0,0,0)$. The Jacobian matrix $J$ of the system (5)-(7) evaluated at the equilibrium point is:

$$J(E)=\begin{pmatrix} -(\lambda^\alpha M+\alpha_0^\alpha Y+\lambda_o^\alpha+d^\alpha) & q^\alpha v^\alpha-\lambda^\alpha M-\alpha_o^\alpha X & \lambda^\alpha(1-X-Y) \\ 0 & (\alpha_0^\alpha-\beta^\alpha)X-2\beta^\alpha Y+\beta^\alpha-v^\alpha-d^\alpha & 0 \\ 0 & \mu^\alpha & -\mu_o^\alpha \end{pmatrix}, \tag{14}$$

such that the Jacobian matrix evaluated at the disease-free equilibrium point is

$$J(E_o)=\begin{pmatrix} -(\lambda_o^\alpha+d^\alpha) & q^\alpha v^\alpha & \lambda^\alpha \\ 0 & \beta^\alpha-v^\alpha-d^\alpha & 0 \\ 0 & \mu^\alpha & -\mu_o^\alpha \end{pmatrix}. \tag{15}$$

The characteristic equation associated with above matrix is $|J(E_o)-\eta I|=0$, we get eigenvalues $\eta$ of the matrix are:

$$\eta_1=-(\lambda_o^\alpha+d^\alpha)<0,\ \ \eta_2=\beta^\alpha-(v^\alpha+d^\alpha) \text{ and } \eta_3=-\mu_o^\alpha<0.$$

If $\beta^\alpha<v^\alpha+d^\alpha$, these eigenvalues are satisfied the condition $|arg\,\eta_i|>\dfrac{\alpha\pi}{2}$ [12]. Then the disease- free equilibrium point $E_0$ for the model (5)-(7) is local asymptotically stable.

3.2.2. *Stability of endemic equilibrium.* The endemic equilibrium for the system $(5)-(7)$ exist if at least one of the infected variables is not zero. [14]. Let the right hand side of the system $(5)-(7)$ equal to zero and $Y\neq 0$. Then the endemic equilibrium point $E^*=(X^*,Y^*,M^*)$ for system $(5)-(7)$ is given as follows:
from equation (5),

$$X^*=\frac{\beta^\alpha(1-Y^*)-v^\alpha-d^\alpha}{\beta^\alpha-\alpha_o^\alpha}, \tag{16}$$

from equation (7),

$$M^*=\frac{\mu^\alpha Y^*}{\mu_o^\alpha}, \tag{17}$$

then from equations (5), (16) and (17),

$$Y^*=\frac{-P_2\pm\sqrt{P_2^2-4P_1P_3}}{2P_1}. \tag{18}$$

where

$$P_1=\alpha_o^\alpha(\mu^\alpha\lambda^\alpha+\mu_o^\alpha\beta^\alpha),$$

$$P_2=(v^\alpha+d^\alpha-\beta^\alpha)(\mu^\alpha\lambda^\alpha+\mu_o^\alpha\alpha_o^\alpha)+q^\alpha v^\alpha\mu_o^\alpha(\beta^\alpha-\alpha_o^\alpha+\mu_o^\alpha\beta^\alpha(\beta^\alpha+d^\alpha)),$$

$$P_3=\mu_o(\lambda_o^\alpha+d^\alpha)(v^\alpha+d^\alpha-\beta^\alpha).$$

Fractional system has an endemic equilibrium if at least one of variables is not zero. Consider the values of parameters from Table 1, then the basic reproduction number is $R_o>1$. By analytical the endemic equilibrium point, $E^*=(X^*,Y^*,M^*)=(0.201,0.014,0.001)$, for $\beta=0.5$.

The Jacobian matrix in (14) at $E^*$ as follows:

$$J(E^*) = \begin{pmatrix} -(\lambda^\alpha M^* + \alpha_0^\alpha Y^* + \lambda_o^\alpha + d^\alpha) - \eta & q^\alpha v^\alpha - \lambda^\alpha M^* - \alpha_o^\alpha X^* & \lambda^\alpha(1 - X^* - Y^*) \\ 0 & (\alpha_0^\alpha - \beta^\alpha)X^* - 2\beta^\alpha Y^* + \beta^\alpha - v^\alpha - d^\alpha - \eta & 0 \\ 0 & \mu^\alpha & -\mu_o^\alpha - \eta \end{pmatrix}. \tag{19}$$

The eigenvalues of matrix in (19) are:
$\eta_1 = -(\lambda^\alpha M^* + \alpha_0^\alpha Y^* + \lambda_o^\alpha + d^\alpha) < 0$, $\eta_2 = (\alpha_0^\alpha - \beta^\alpha)X^* - 2\beta^\alpha Y^* + \beta^\alpha - v^\alpha - d^\alpha$ and $\eta_3 = -\mu_o^\alpha < 0$. If $\beta^\alpha + (\alpha_0^\alpha - \beta^\alpha)X^* < 2\beta^\alpha Y^* + v^\alpha + d^\alpha$, these eigenvalues are satisfied the condition $|arg\,\eta_i| > \dfrac{\alpha\pi}{2}$ [12]. Then the endemic equilibrium point, $E^*$ is local asymptoically stable.

## 4. Numerical Methods

The aim in this section is to solve the fractional model (5)-(7) numerically. Three numerical methods will be consider here, GEM, PCM and NSFDM. In the following, we will give a brief summary on these methods:

4.1. **GEM.** The Euler algorithm has been extended to study the fractional nonlinear differential equations, where the derivative is defined by the Caputo fractional derivative, for more details see ([7]).

$$y_{n+1} = y_n + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_n, y_n). \tag{20}$$

4.2. **Estimation of the local truncation error.** The local truncation error (LTE) of GEM is the claimed error in a single execution step of the proposed method. It is the difference between the numerical solution $y_1$, and the exact solution at time $t_1 = t_0 + h$, after one step. The numerical solution is given from (20) by:

$$y_1 = y_0 + \frac{(\varphi(h))^\alpha}{\Gamma(\alpha+1)} f(y_0, t_0). \tag{21}$$

For the exact solution, we assume that $y(t)$, ${}_a^c D_t^\alpha y(t)$ and ${}_a^c D_t^{2\alpha} y(t)$ are continuous on $[0, a]$, using the generalized Taylor's formula [25] to expand $y(t)$ about $t = t_0 = 0$,

$$y(t) = \sum_{i=0}^{n} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} {}_a^c D^{i\alpha} y(0) + \frac{{}_a^c D^{(n+1)} y(\mu) t^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)},$$

$$0 \le \mu \le t, \quad \forall t \in (0, a],$$

Now, for each value $t$ there is a value $c_1$ such that:

$$y(t) = y(t_0) + \frac{{}_a^c D_t^\alpha y(t_0)}{\Gamma(\alpha+1)} t^\alpha + \frac{{}_a^c D_t^\alpha y(c_1)}{\Gamma(2\alpha+1)} t^{2\alpha} + .... \tag{22}$$

When ${}_a^c D_t^\alpha y(t_0) = f(y(t_0), t_0)$ and $h = t_1$ are substituted into (22) we have:

$$y(t_1) = y(t_0) + \frac{\xi(y(t_0), t_0)}{\Gamma(\alpha+1)} (h)^\alpha + \frac{{}_a^c D_t^{2\alpha} y(c_1)}{\Gamma(2\alpha+1)} (h)^{2\alpha} + .... \tag{23}$$

The LTE of GEM is given by the difference between (21) and (23) as follows:

$$LTE = y(t_1) - y_1 = \frac{{}_a^c D_t^{2\alpha} y(c_1)}{\Gamma(2\alpha+1)} \varphi(h)^{2\alpha} + O(\varphi(h))^{3\alpha}. \tag{24}$$

4.3. **PCM.** PCM is used to approximate than one previous point to determine the approximation at the next point. It is a combination of an explicit method to predict and an implicit method to improve the prediction corrector.

We get $y^P_{n+1}$ (predictor) by using the forward Euler method, then we get $y^C_{n+1}$ (corrector) by using the trapezoidal rule, for more details see ([20]).

Then the discrete form of the Caputo fractional derivative

$$y^P_{n+1} = \sum_{j=0}^{n} \frac{t^j_{n+1}}{j!} y^{(j)}_o + \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j), \tag{25}$$

$$y^C_{n+1} = \sum_{j=0}^{n} \frac{t^j_{n+1}}{j!} y^{(j)}_o + \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_j) + a_{n+1,n+1} f(t_{n+1}, y^P_{n+1}), \tag{26}$$

where

$$b_{j,n+1} = \frac{(n-j+1)^\alpha - (n-j)^\alpha}{\Gamma(\alpha+1)},$$

$$a_{j,n+1} = \frac{\Delta t^\alpha}{\Gamma(\alpha+2)} \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & j = 0, \\ (n-j+2)^{\alpha+1} - 2(n-j+1)^{\alpha+1} + (n-j)^{\alpha+1}, & 1 \le j \le n, \\ 1, & j = n+1. \end{cases} \tag{27}$$

**Remark** The error analysis of the fractional PCM is investigated in ([7], [26]).

4.4. **NSFDM.** The technique of the NSFDM was firstly proposed by Mickens to solve numerically the ordinary differential equations, [10, 11]. We construct the NSFDM scheme to obtain numerical solutions of the epidemic model of fractional differential equations, (5)-(7).

A scheme is called NSFDM discretization if at least one of the following conditions is satisfied:

• The nonlocal approximation is used such as

$$yx = \begin{cases} y_n x_{n+1}, \\ y_{n+1} x_{n-1}, \\ y_{n-1} x_n. \end{cases}$$

• Discretization of a derivative is not traditional and use a nonnegative function,

$$\varphi(h) = h + O(h^2), \ \ 0 < \varphi < 1, \text{ for all } h > 0,$$

where the function $\varphi(h)$ is a continuous of step size.

The nonstandard differences approximation of Caputo derivative is given by the Grünwald−Letinkov's definition [13]:

$${}^C_0 D^\alpha_t y(t) = \frac{1}{(\phi(\Delta t))^\alpha} (y_{n+1} - \sum_{i=1}^{n+1} w_i y_{n+1-i} - C_{n+1} y_o), \tag{28}$$

where $h = \Delta t = \dfrac{t_n}{n}$, $\ n = 0, 1, 2, 3, \cdots, N_n$, denotes $N_n$ is a natural number.

The Grünwald−Letinkov coefficients define as follows [13]:

$$\omega_i = (1 - \frac{1+\alpha}{i}) \omega^\alpha_{i-1}, \ \ \omega_1 = \alpha, \tag{29}$$

$$C_i = \frac{i^{-\alpha}}{\Gamma(1-\alpha)}, \quad C_1 = \frac{1}{\Gamma(1-\alpha)}, \quad i = 1, 2, 3, \cdots, n+1. \tag{30}$$

**Lemma** Assume that $0 < \alpha < 1$, then the coefficients $w_i$ and $C_i$ satisfy for $i \geq 1$ the properties.

$$0 < w_{i+1} < w_i < \cdots < w_1 = \alpha < 1,$$
$$0 < C_{i+1} < C_i < \cdots < C_1 = \frac{1}{\Gamma(1-\alpha)}.$$

4.4.1. *NSFDM for fractional differential equations.* We construct NSFDM for obtaining explicit discretization of equations (5)-(7). Using nonstandard technique and equation (28). The new system is described as follows:

$$X_{n+1} - \sum_{i=1}^{n+1} w_i X_{n+1-i} - C_{n+1} X_o = (\phi(h))^\alpha (\lambda^\alpha (1 - X_{n+1} - Y_n) M_n - (\lambda_o^\alpha + d^\alpha) X_{n+1} - \alpha_o^\alpha X_{n+1} Y_n + q^\alpha v^\alpha Y_n), \tag{31}$$

$$Y_{n+1} - \sum_{i=1}^{n+1} w_i Y_{n+1-i} - C_{n+1} Y_o = ((\phi(h))^\alpha)(\beta^\alpha (1 - X_n - Y_n) Y_{n+1} - (v^\alpha + d^\alpha) Y_{n+1} + \alpha_o^\alpha X_n Y_{n+1}), \tag{32}$$

$$M_{n+1} - \sum_{i=1}^{n+1} w_i M_{n+1-i} - C_{n+1} M_o = ((\phi(h))^\alpha)(\mu^\alpha Y_n - \mu_o^\alpha M_{n+1}). \tag{33}$$

The explicit form expressions for $X_{n+1}$ , $Y_{n+1}$ and $M_{n+1}$:

$$\begin{aligned} X_{n+1} &= \frac{\sum_{i=1}^{n+1} \omega_i X_{n+1-i} + C_{n+1} X_o + (\phi(h))^\alpha (\lambda^\alpha (1 - Y_n) M_n + q^\alpha v^\alpha Y_n)}{1 + (\phi(h))^\alpha (\lambda^\alpha M_n + \lambda_o^\alpha + d^\alpha + \alpha_0^\alpha Y_n)}, \\ Y_{n+1} &= \frac{\sum_{i=1}^{n+1} w_i Y_{n+1-i} + C_{n+1} Y_o}{1 + (\phi(h))^\alpha (v^\alpha + d^\alpha - \beta^\alpha (1 - X_n - Y_n) - \alpha_o^\alpha X_n)}, \\ M_{n+1} &= \frac{\sum_{i=1}^{n+1} w_1 M_{n+1-i} + C_{n+1} M_o + (\phi(h))^\alpha \mu^\alpha Y_n}{1 + \mu_o^\alpha (\phi(h))^\alpha}. \end{aligned} \tag{34}$$

4.4.2. *Properties of the solutions of the proposed model.* In the following, the properties of a numerical scheme that given by (34) are analyzed [15].

- Positivity and boundedness of NSFDM scheme:

**Theorem** [Positivity] Suppose that $X_o \geq 0$, $Y_o \geq 0$, $M_o \geq 0.$, then $X_n \geq 0,\ Y_n \geq 0,\ M_n \geq 0$. It is satisfied for all $n = 1, 2, \cdots$.

**Proof.** By induction, for $n = 0$, we have from a system (34).

$$\begin{aligned} X_1 &= \frac{w_1 X_o + C_1 X_o + (\phi(h))^\alpha(\lambda^\alpha(1-Y_o)M_0 + q^\alpha v^\alpha Y_o)}{1+(\phi(h))^\alpha(\lambda^\alpha M_o + \lambda_o^\alpha + d^\alpha + \alpha_o^\alpha Y_o)} \geq 0, \\ Y_1 &= \frac{w_1 Y_o + C_1 Y_o}{1+((\phi(h))^\alpha)(v^\alpha + d^\alpha - \beta^\alpha(1-X_o-Y_o) - \alpha_o^\alpha X_o)} \geq 0, \\ M_1 &= \frac{w_1 M_o + C_1 M_o + (\phi(h))^\alpha \mu^\alpha Y_o}{1+\mu_o^\alpha(\phi(h))^\alpha} \geq 0. \end{aligned} \tag{35}$$

**Notice:** All parameters are positive. We suppose that $X_n \geq 0,\ Y_n \geq 0,\ M_n \geq 0$. for all $n < n+1$. Thus for $n+1$.

$$\begin{aligned} X_{n+1} &= \frac{\sum_{i=1}^{n+1}\omega_i X_{n+1-i} + C_{n+1}X_o + (\phi(h))^\alpha(\lambda^\alpha(1-Y_n)M_n + q^\alpha v^\alpha Y_n)}{1+(\phi(h))^\alpha(\lambda^\alpha M_n + \lambda_o^\alpha + d^\alpha + \alpha_0^\alpha Y_n)} \geq 0, \\ Y_{n+1} &= \frac{\sum_{i=1}^{n+1} w_i Y_{n+1-i} + C_{n+1}Y_o}{1+((\phi(h))^\alpha)(v^\alpha + d^\alpha - \beta^\alpha(1-X_n-Y_n) - \alpha_o^\alpha X_n)} \geq 0, \\ M_{n+1} &= \frac{\sum_{i=1}^{n+1} w_i M_{n+1-i} + C_{n+1}M_o + (\phi(h))^\alpha \mu^\alpha Y_n}{1+\mu_o^\alpha(\phi(h))^\alpha} \geq 0. \end{aligned} \tag{36}$$

**Theorem** [Boundedness] Suppose that initial conditions are $X_{mo} = 0, X_o = 0, Y_o = 1$, where $X(t) + X_m(t) + Y(t) = 1$, then $X_n, Y_n, M_n$ are bounded for all $n = 1, 2, ....$

**Proof** Multiplying each equation in system (34) by its denominator give:

$$\begin{aligned} X_{n+1}(1+(\phi(h))^\alpha(\lambda_o^\alpha+d^\alpha))+Y_{n+1}(1+(\phi(h))^\alpha(v^\alpha+d^\alpha-\beta^\alpha)) &= \sum_{i=1}^{n+1}\omega_i(X_{n+1-i}+Y_{n+1-i}) \\ &\quad + C_{n+1}(X_o+Y_o). \end{aligned} \tag{37}$$

$$M_{n+1}(1+\mu_o((\phi(h))^\alpha)) = \sum_{i=1}^{n+1} w_i M_{n+1-i} + C_{n+1}M_o. \tag{38}$$

By induction, using Lemma 4.4, it follows that for $n = 0$:

$$\begin{aligned} X_1(1+(\phi(h))^\alpha(\lambda_o^\alpha+d^\alpha))+Y_1(1+(\phi(h))^\alpha(v^\alpha+d^\alpha-\beta^\alpha)) &= \omega_1(X_o+Y_o)+C_1(X_o+Y_o) \\ &= \omega_1 + C_1 = K_1. \end{aligned} \tag{39}$$

$$M_1(1+\mu_o((\phi(h))^\alpha)) = w_1 M_o + C_1 M_o \simeq 0. \tag{40}$$

So, we have

$$X_1 \leq \frac{K_1}{(1+(\phi(h))^\alpha(\lambda_o^\alpha+d^\alpha))}\ ,\quad Y_1 \leq \frac{K_1}{(1+(\phi(h))^\alpha(v^\alpha+d^\alpha-\beta^\alpha))}. \tag{41}$$

i.e,

$$X_1 \leq K_1, \quad Y_1 \leq K_1, \quad M_1 \simeq 0.$$

For $n = 1$, we have:

$$\begin{aligned} X_2(1+(\phi(h))^\alpha(\lambda_o^\alpha+d^\alpha))+Y_2 \quad & (1+(\phi(h))^\alpha(v^\alpha+d^\alpha-\beta^\alpha)) = \omega_2(X_o+Y_o)+\omega_1(X_1+Y_1) \\ & +C_2(X_o+Y_o) \leq \omega_1(2K_1)+w_1+C_1, \\ & = K_1(2\alpha+1) = K_2, \end{aligned} \tag{42}$$

$$M_2(1+\mu_o((\phi(h))^\alpha)) = w_2M_o + w_1M_1 + C_1M_o \simeq 0. \tag{43}$$

$$\text{So } X_2 \leq K_2, \ \ Y_2 \leq K_2, \ \ M_2 \simeq 0..$$

For $n = 2$, we have:

$$\begin{aligned} X_3(1+(\phi(h))^\alpha(\lambda_o^\alpha+d^\alpha))+Y_3(1+(\phi(h))^\alpha \quad & (v^\alpha+d^\alpha-\beta^\alpha)) = \omega_3(X_o+Y_o)+\omega_2(X_1+Y_1) \\ & +\omega_1(X_2+Y_2)+C_3(X_o+Y_o) \\ & \leq \omega_1+w_1(2K_1)+w_1(2K_2)+w_1+C_1, \\ & = K_2(2\alpha+1) = K_3. \end{aligned} \tag{44}$$

$$M_3(1+\mu_o(\phi(h))^\alpha) \simeq 0. \tag{45}$$

$$\text{So } X_3 \leq K_3, \ \ Y_3 \leq K_3, \quad M_3 \simeq 0.$$

Now, we suppose that

$$X_n(1+(\phi(h))^\alpha(\lambda_o^\alpha+d^\alpha))+Y_n(1+(\phi(h))^\alpha(v^\alpha+d^\alpha-\beta^\alpha)) \leq K_{n-1}(2\alpha+1) = K_n. \tag{46}$$

$$M_n \simeq 0. \tag{47}$$

i.e.

$$X_n \leq K_n, \ \ Y_n \leq K_n, \quad M_n \simeq 0.$$

So $M_{n+1} \simeq 0$ is closer to zero as increasing $n$, then $M_{n+1}$, it is bounded.

To proof $X_{n+1} \leq K_{n+1}, \ \ Y_{n+1} \leq K_{n+1}$.

$$\begin{aligned} X_n(1+(\phi(h))^\alpha(\lambda_o^\alpha+d^\alpha))+Y_n(1 \quad & + \quad (\phi(h))^\alpha(v^\alpha+d^\alpha-\beta^\alpha)) = \omega_{n+1}(X_o+Y_o)+\omega_n(X_1+Y_1)+... \\ & + \quad \omega_2(X_{n-1}+Y_{n-1})+w_1(X_n+Y_n)+C_{n+1}(X_o+Y_o) \\ & \leq \quad \omega_1+w_1(2K_1)+...+w_1(2K_{n-1})+w_1(2K_n)+C_1, \\ & = \quad K_n(2\alpha+1) = K_{n+1}. \end{aligned} \tag{48}$$

**Remark** Concerning the convergence of the method, we refer to ([22]-[24]) and the references cited there in.

## 5. Numerical Simulations

In this section, we use the proposed NSFDM (34) to simulate the solution of (5)-(7). In order to test stability properties of the schemes, we compared the proposed method with GEM and PCM at different values of the parameter $\beta^\alpha$. Throughout this section, some of simulations are performed with initial conditions $(X_0, Y_0, M_0) = (0.38, 0.3, 0.09)$, and different values of $\alpha$. The values of the parameters are taken from Table 1 and we used $\phi(h) = 1 - e^{-h}$. We consider the solutions which obtained by Matlab Ode45 as the exact solution. Figure 1, shows

the behaviour of the approximation solutions $X(t), Y(t)$ and $M(t)$ at $\alpha = 1$ using different methods are convergent to the endemic equilibrium when $R_o > 1$. Figure 2, shows the NSFDM is more stable than GEM and PCM. Figure 3, show how the solutions of system (5)-(7) obtained by the proposed NSFDM change when $\alpha$ takes different values. Table 2, reports the convergence behaviour of the solutions when $R_o < 1$ of following numerical methods: NSFDM, PCM and GEM. Table 3, reports the convergence behaviour of the solutions when $R_o > 1$ using the same methods in Table 2. We can conclude from these tables that NSFDM is convergent for large $h$ while PCM and the GEM converge only when $h$ is small. So, NSFDM can save the computational time.

TABLE 2. Results for different time step sizes $h$, $\beta = 0.35$ and $R_o < 1$.

| $h$ | GEM | PCM | NSFDM |
|---|---|---|---|
| 0.01 | Convergence | Convergence | Convergence |
| 0.05 | Convergence | Convergence | Convergence |
| 0.1 | Convergence | Convergence | Convergence |
| 0.5 | Convergence | Convergence | Convergence |
| 1 | Convergence | Convergence | Convergence |
| 20 | Divergence | Divergence | Convergence |

TABLE 3. Results for different time step sizes $h$, $\beta = 0.5$ and $R_o > 1$.

| $h$ | GEM | PCM | NSFDM |
|---|---|---|---|
| 0.01 | Convergence | Convergence | Convergence |
| 0.05 | Convergence | Convergence | Convergence |
| 0.1 | Convergence | Convergence | Convergence |
| 0.5 | Convergence | Convergence | Convergence |
| 1 | Convergence | Convergence | Convergence |
| 20 | Divergence | Divergence | Convergence |

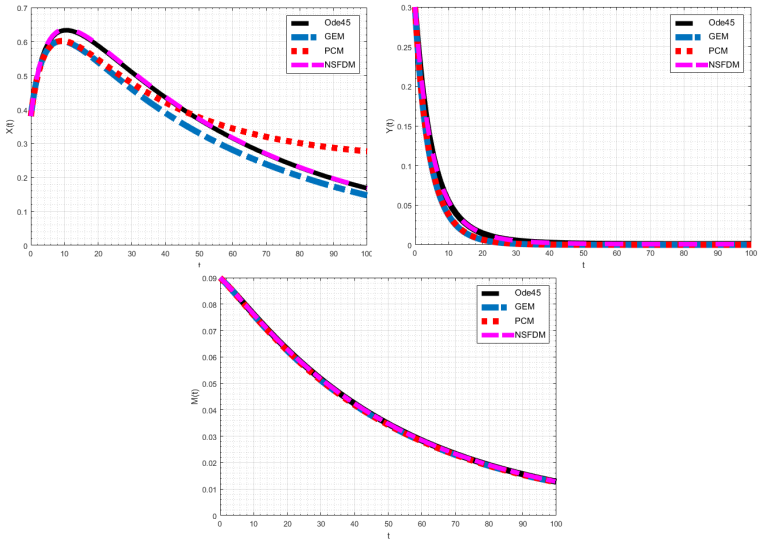


FIGURE 1. Behavior of the approximation solutions $X(t), Y(t)$ and $M(t)$ when, $R_o > 1$ and $\alpha = 1$ using different methods.

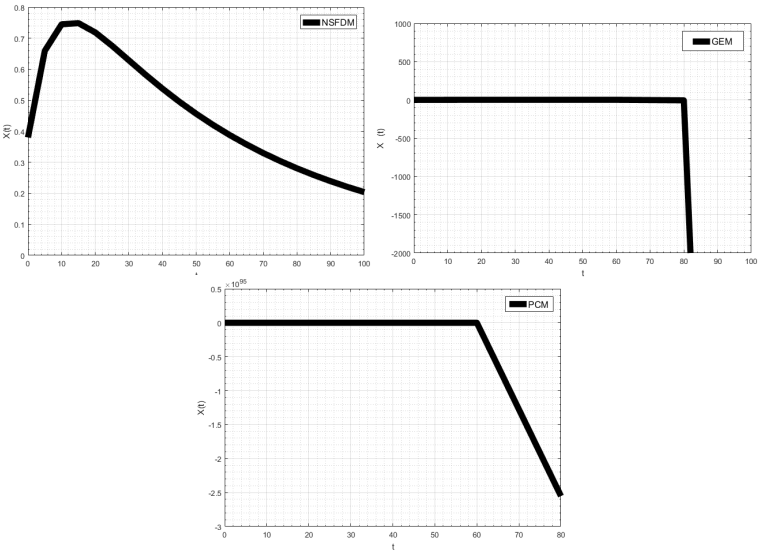


FIGURE 2. Behavior of the approximate solutions of infection population when $h = 20$ by using different methods.

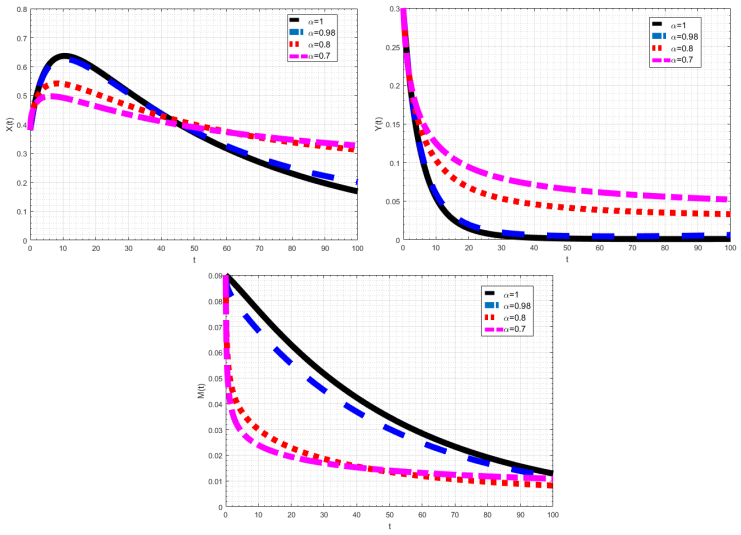


FIGURE 3. The approximation solutions $X(t), Y(t)$ and $M(t)$ in case $R_o > 1$, $\beta^\alpha = 0.5^\alpha$ and different values of $\alpha$ using NSFDM.

## 6. Conclusions

In this article, a fractional model for the effects of awareness programs on the epidemic outbreaks is presented. It is concluded that the proposed fraction order model is more suitable to describe the biological phenomena with memory than the integer order model and can include easily the memory effects presented in many real world phenomena. Three numerical methods are given to solve numerically the proposed model. NSFDM is constructed such that the numerical solutions have the same properties of the analytical solutions. Moreover, NSFDM can decrease the calculation's time since it is able to use larger time steps and it provides good approximations for the proposed model and it is more stable than other methods.

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