

EXISTENCE OF MILD SOLUTIONS FOR NON-PERIODIC COUPLED FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We obtain sufficient conditions for existence and uniqueness of mild solutions of coupled fractional differential equations using Schauder and Banach fixed point theorems. Two examples are introduced to explain the applicability of the obtained results.

1. INTRODUCTION

Fractional differential equations have gained considerable importance due to their varied applications in many problems of physics, chemistry, biology, applied sciences and engineering. Fractional-order differential equations are also regarded as a better tool for the description of hereditary properties of various materials and processes than the corresponding integer order differential equations, see [[1]-[6]].

The topic of boundary value problems of fractional differential equations supplemented with a variety of boundary conditions has attracted a significant attention in recent years. In particular, the literature on fractional order initial or boundary value problems involving nonlocal and integral boundary conditions is now much enriched, for instance, see [[7]-[12]], for some theoretical works on coupled systems of fractional-order differential equations, we refer the reader to a series of [[14]-[17]].

In fact, it can be seen that even if the (initial) boundary value problem of fractional order has a continuous right-hand side, the equivalence between differential and integral form of the problem can be lost [[18]-[19]]. Hence, it is desirable to search about the mild solution of such problems.

In this paper, we investigate the coupled system given by

$$\begin{cases} {}^C D^\alpha x(t) = f(t, x(t), {}^C D^\delta x(t), y(t), {}^C D^\gamma y(t)), \\ {}^C D^\beta y(t) = g(t, x(t), {}^C D^\delta x(t), y(t), {}^C D^\gamma y(t)), \end{cases} \quad (1.1)$$

for $t \in J = [0, 1]$, $1 < \alpha \leq 2$, $1 < \beta \leq 2$, $0 < \delta < 1$, and $0 < \gamma < 1$, $f, g : J \times \mathbb{R}^4 \rightarrow \mathbb{R}$ are continuous functions, together with non-periodic conditions

$$\begin{cases} x(1) = a_0 x(0), \quad x'(1) = a_1 x'(0), \\ y(1) = b_0 y(0), \quad y'(1) = b_1 y'(0), \end{cases} \quad (1.2)$$

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where $a_0, a_1, b_0, b_1 \in \mathbb{R}$, $a_0, a_1, b_0, b_1 \neq 1$, ${}^C D^i, i = \alpha, \beta, \delta, \gamma$ denote the Caputo fractional derivatives of order $i, i = \alpha, \beta, \delta, \gamma$ respectively.

2. PRELIMINARIES

Let us recall some basic definitions of fractional calculus (for more details see [1]-[5]) and we obtain a solution of the corresponding linear system of (1.1).

Definition 2.1. *The Riemann-Liouville fractional integral of order α for a continuous function $f : [0, \infty) \rightarrow \mathbb{R}$ is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \alpha > 0,$$

provided the integral exists.

Definition 2.2. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be n th-times continuously differentiable function. The Caputo derivative of fractional order α is defined as*

$${}^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $n-1 < \alpha < n$, $n = [\alpha] + 1$ such that $[\alpha]$ denotes the integer part of the real number α .

The interaction between Riemann-Liouville fractional integral and Caputo derivative is given by the next result.

Lemma 2.3. *If x has Caputo fractional derivative of order α , then*

$$I^\alpha {}^C D^\alpha x(t) = x(t) - \sum_{k=0}^{n-1} \frac{c_k t^k}{k!}, n-1 < \alpha < n,$$

for some constants $c_k, k = 0, 1, 2, \dots, n-1$.

Now we present an auxiliary lemma which plays key role to define a solution for the given problem (1.1).

Lemma 2.4. *Consider the following linear system*

$$\begin{cases} {}^C D_0^\alpha x(t) = f(t), \\ {}^C D_0^\beta y(t) = g(t), \end{cases} \quad (2.1)$$

for $t \in J$, $1 < \alpha \leq 2$, $1 < \beta \leq 2$, and $f, g : J \rightarrow \mathbb{R}$ are continuous functions, supplemented with non-periodic type boundary conditions (1.2). The mild solutions of (2.1) satisfy the following integral equations

$$\begin{aligned} x(t) &= \frac{1}{(a_0-1)(a_1-1)\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s) ds \\ &+ \frac{1}{(a_0-1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s) ds \\ &+ \frac{t}{(a_1-1)\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} y(t) &= \frac{1}{(b_0 - 1)(b_1 - 1)\Gamma(\beta - 1)} \int_0^1 (1 - s)^{\beta-2} g(s) ds \\ &+ \frac{1}{(b_0 - 1)\Gamma(\beta)} \int_0^1 (1 - s)^{\beta-1} g(s) ds \\ &+ \frac{t}{(b_1 - 1)\Gamma(\beta - 1)} \int_0^1 (1 - s)^{\beta-2} g(s) ds + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} g(s) ds. \end{aligned} \quad (2.3)$$

Proof. Taking the fractional integral to both sides of equation (2.1), we obtain

$$x(t) = c_0 + c_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) ds. \quad (2.4)$$

In the same way, we obtain

$$y(t) = d_0 + d_1 t + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} g(s) ds, \quad (2.5)$$

where $c_i, d_i \in \mathbb{R}, i = 0, 1$ are arbitrary constants. Applying the conditions (1.2), we get

$$\begin{aligned} c_0 &= \frac{1}{(a_0 - 1)(a_1 - 1)\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha-2} f(s) ds \\ &+ \frac{1}{(a_0 - 1)\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f(s) ds, \\ c_1 &= \frac{1}{(a_1 - 1)\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha-2} f(s) ds, \\ d_0 &= \frac{1}{(b_0 - 1)(b_1 - 1)\Gamma(\beta - 1)} \int_0^1 (1 - s)^{\beta-2} g(s) ds \\ &+ \frac{1}{(b_0 - 1)\Gamma(\beta)} \int_0^1 (1 - s)^{\beta-1} g(s) ds, \end{aligned}$$

and

$$d_1 = \frac{1}{(b_1 - 1)\Gamma(\beta - 1)} \int_0^1 (1 - s)^{\beta-2} g(s) ds.$$

Substituting the values c_0, c_1, d_0, d_1 in (2.4) and (2.5) we get (2.2) and (2.3). From other side, one can apply the Caputo fractional derivative to (2.2) and (2.3), to deduce (2.1). Moreover, the conditions (1.2) are satisfied in the integral solutions (2.2) and (2.3). This completes the proof. \square

3. MAIN RESULTS

Let $C(J, \mathbb{R})$ be the Banach space of all continuous functions defined on J . The space $X = \{x : x \in C(J, \mathbb{R}), {}^C D^\delta x \in C(J, \mathbb{R})\}$, equipped with the norm $\|x\|_X = \|x\| + \|{}^C D^\delta x\|$ is a Banach space. The space Y is defined similarly with the norm $\|y\|_Y = \|y\| + \|{}^C D^\gamma y\|$ which is also a Banach space. Clearly, the product space $X \times Y$ is Banach space with norm $\|(x, y)\|_{X \times Y} = \|x\|_X + \|y\|_Y$ for $(x, y) \in X \times Y$ (for more details on Banach spaces and fixed point theorems see [13]). Using Lemma 2.4, we define the operators $\mathcal{H}_1 : X \times Y \rightarrow X$, $\mathcal{H}_2 : X \times Y \rightarrow Y$, and $\mathcal{H} : X \times Y \rightarrow X \times Y$ by

$$\mathcal{H}(x, y)(t) := (\mathcal{H}_1(x, y)(t), \mathcal{H}_2(x, y)(t)),$$

where

$$\begin{aligned}
& \mathcal{H}_1(x, y)(t) \\
&= \frac{1}{(a_0 - 1)(a_1 - 1)\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha - 2} f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s)) ds \\
&+ \frac{1}{(a_0 - 1)\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha - 1} f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s)) ds \\
&+ \frac{t}{(a_1 - 1)\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha - 2} f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s)) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s)) ds, \tag{3.1}
\end{aligned}$$

and

$$\begin{aligned}
& \mathcal{H}_2(x, y)(t) \\
&= \frac{1}{(b_0 - 1)(b_1 - 1)\Gamma(\beta - 1)} \int_0^1 (1 - s)^{\beta - 2} g(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s)) ds \\
&+ \frac{1}{(b_0 - 1)\Gamma(\beta)} \int_0^1 (1 - s)^{\beta - 1} g(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s)) ds \\
&+ \frac{t}{(b_1 - 1)\Gamma(\beta - 1)} \int_0^1 (1 - s)^{\beta - 2} g(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s)) ds \\
&+ \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} g(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s)) ds. \tag{3.2}
\end{aligned}$$

We need the following assumptions:

H1: $f, g : J \times \mathbb{R}^4 \rightarrow \mathbb{R}$ are continuous functions and there exist positive real constants $L_1 > 0, L_2 > 0$ such that for all $t \in J$ and $x_i, y_i \in \mathbb{R}, i = 1, 2, 3, 4$, we have

$$\begin{aligned}
& |f(t, x_1, x_2, x_3, x_4) - f(t, y_1, y_2, y_3, y_4)| \\
&\leq L_1 (|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |x_4 - y_4|),
\end{aligned}$$

and

$$\begin{aligned}
& |g(t, x_1, x_2, x_3, x_4) - g(t, y_1, y_2, y_3, y_4)| \\
&\leq L_2 (|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |x_4 - y_4|).
\end{aligned}$$

H2: $f, g : J \times \mathbb{R}^4 \rightarrow \mathbb{R}$ are continuous functions and there exist real constants $m_i, n_i \geq 0, i = 1, 2, 3, 4$, and $m_0 > 0, n_0 > 0$ such that if $|x_i| \in \mathbb{R}, i = 1, 2, 3, 4$, we have

$$\begin{cases} |f(t, x_1, x_2, x_3, x_4)| \leq m_0 + m_1 |x_1| + m_2 |x_2| + m_3 |x_3| + m_4 |x_4|, \\ |g(t, x_1, x_2, x_3, x_4)| \leq n_0 + n_1 |x_1| + n_2 |x_2| + n_3 |x_3| + n_4 |x_4|. \end{cases}$$

For computation convenience, we introduce the notations:

$$\omega_1 = L_1 \nu_1, \quad \omega_2 = L_1 \nu_2, \quad \omega_3 = L_2 \nu_3, \quad \omega_4 = L_2 \nu_4, \tag{3.3}$$

$$\sigma_1 = N_1 \nu_1, \quad \sigma_2 = N_1 \nu_2, \quad \sigma_3 = N_2 \nu_3, \quad \sigma_4 = N_2 \nu_4, \tag{3.4}$$

where

$$\begin{aligned}\nu_1 &= \frac{1}{|a_0 - 1| |a_1 - 1| \Gamma(\alpha)} + \frac{1}{|a_0 - 1| \Gamma(\alpha + 1)} + \frac{1}{|a_1 - 1| \Gamma(\alpha)} + \frac{1}{\Gamma(\alpha + 1)}, \\ \nu_2 &= \frac{1}{|a_1 - 1| \Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}, \\ \nu_3 &= \frac{1}{|b_0 - 1| |b_1 - 1| \Gamma(\beta)} + \frac{1}{|b_0 - 1| \Gamma(\beta + 1)} + \frac{1}{|b_1 - 1| \Gamma(\beta)} + \frac{1}{\Gamma(\beta + 1)}, \\ \nu_4 &= \frac{1}{|b_1 - 1| \Gamma(\beta)} + \frac{1}{\Gamma(\beta)}, \\ N_1 &= \sup_{t \in [0,1]} |f(t, 0, 0, 0, 0)| < \infty, \quad N_2 = \sup_{t \in [0,1]} |g(t, 0, 0, 0, 0)| < \infty.\end{aligned}$$

Theorem 3.1. *Assume that (H1) hold and that*

$$\omega_1 + \frac{\omega_2}{\Gamma(2 - \delta)} < \frac{1}{2}, \quad \omega_3 + \frac{\omega_4}{\Gamma(2 - \gamma)} < \frac{1}{2},$$

then the boundary value problem (1.1) has a unique mild solution.

Proof. Let r be a positive real number such that

$$r \geq \max \left\{ \frac{\sigma_1 + \frac{\sigma_2}{\Gamma(2 - \delta)}}{\frac{1}{2} - \left(\omega_1 + \frac{\omega_2}{\Gamma(2 - \delta)} \right)}, \frac{\sigma_3 + \frac{\sigma_4}{\Gamma(2 - \gamma)}}{\frac{1}{2} - \left(\omega_3 + \frac{\omega_4}{\Gamma(2 - \gamma)} \right)} \right\},$$

where $\omega_1, \omega_2, \omega_3, \omega_4$ and $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are given by (3.3), and (3.4). Consider the closed ball $B_r = \{(x, y) \in X \times Y : \|(x, y)\|_{X \times Y} \leq r\}$, we show that $\mathcal{H}(B_r) \subset B_r$. Let $(x, y) \in B_r$, then

$$\begin{aligned}& |f(t, x(t), {}^C D^\delta x(t), y(t), {}^C D^\gamma y(t))| \\ & \leq |f(t, x(t), {}^C D^\delta x(t), y(t), {}^C D^\gamma y(t)) - f(t, 0, 0, 0, 0)| \\ & \quad + |f(t, 0, 0, 0, 0)| \\ & \leq L_1 (|x(t)| + |{}^C D^\delta x(t)| + |y(t)| + |{}^C D^\gamma y(t)|) + N_1 \\ & \leq L_1 (\|x\|_X + \|y\|_Y) + N_1 \leq L_1 r + N_1,\end{aligned}$$

and similarly, we obtain

$$|g(t, x(t), {}^C D^\delta x(t), y(t), {}^C D^\gamma y(t))| \leq L_2 (\|x\|_X + \|y\|_Y) + N_2 \leq L_2 r + N_2.$$

Hence

$$\begin{aligned}
& |\mathcal{H}_1(x, y)(t)| \\
& \leq \frac{1}{|a_0 - 1||a_1 - 1|\Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} |f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s))| ds \\
& \quad + \frac{1}{|a_0 - 1|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s))| ds \\
& \quad + \frac{t}{|a_1 - 1|\Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} |f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s))| ds \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s))| ds \\
& \leq L_1 r \\
& \quad + N_1 \left(\frac{1}{|a_0 - 1||a_1 - 1|\Gamma(\alpha)} + \frac{1}{|a_0 - 1|\Gamma(\alpha + 1)} + \frac{t}{|a_1 - 1|\Gamma(\alpha)} + \frac{t^\alpha}{\Gamma(\alpha + 1)} \right) \\
& \leq \omega_1 r + \sigma_1.
\end{aligned}$$

The first order derivative of the operator \mathcal{H}_1 can be obtained as following:

$$\begin{aligned}
& \left| \mathcal{H}'_1(x, y)(t) \right| \\
& \leq \frac{1}{|a_1 - 1|\Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} |f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s))| ds \\
& \quad + \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t-s)^{\alpha-2} |f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s))| ds \\
& \leq \omega_2 r + \sigma_2,
\end{aligned}$$

which implies that

$$\begin{aligned}
|{}^C D^\delta \mathcal{H}_1(x, y)(t)| & \leq \int_0^t \frac{(t-s)^{-\delta}}{\Gamma(1-\delta)} \left| \mathcal{H}'_1(x, y)(s) \right| ds \\
& \leq \frac{t^{1-\delta}}{\Gamma(2-\delta)} (\omega_2 r + \sigma_2),
\end{aligned}$$

thus

$$\begin{aligned}
\|\mathcal{H}_1(x, y)\|_X & = \|\mathcal{H}_1(x, y)\| + \|{}^C D^\delta \mathcal{H}_1(x, y)\| \\
& \leq \left(\omega_1 + \frac{\omega_2}{\Gamma(2-\delta)} \right) r + \left(\sigma_1 + \frac{\sigma_2}{\Gamma(2-\delta)} \right) \\
& \leq \frac{r}{2}.
\end{aligned}$$

In the same way, we obtain

$$|\mathcal{H}_2(x, y)(t)| \leq \omega_3 r + \sigma_3, \quad \left| \mathcal{H}'_2(x, y)(t) \right| \leq \omega_4 r + \sigma_4,$$

and

$$\begin{aligned}
|{}^C D^\gamma \mathcal{H}_2(x, y)(t)| & \leq \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)} \left| \mathcal{H}'_2(x, y)(s) \right| ds \\
& \leq \frac{t^{1-\gamma}}{\Gamma(2-\gamma)} (\omega_4 r + \sigma_4).
\end{aligned}$$

In consequence, we get

$$\begin{aligned}\|\mathcal{H}_2(x, y)\|_Y &= \|\mathcal{H}_2(x, y)\| + \|^C D^\gamma \mathcal{H}(x, y)\| \\ &\leq \left(\omega_3 + \frac{\omega_4}{\Gamma(2-\gamma)}\right)r + \left(\sigma_3 + \frac{\sigma_4}{\Gamma(2-\gamma)}\right) \\ &\leq \frac{r}{2}.\end{aligned}$$

It follows that $\|\mathcal{H}(x, y)\|_{X \times Y} \leq r$, that is $\mathcal{H}(B_r) \subset B_r$. Next, we show that the operator \mathcal{H} is a contraction. For that, let $(x_1, y_1), (x_2, y_2) \in X \times Y$, then for any $t \in J$, we get

$$\begin{aligned}&|\mathcal{H}_1(x_1, y_1)(t) - \mathcal{H}_1(x_2, y_2)(t)| \\ &\leq \frac{1}{|a_0 - 1| |a_1 - 1| \Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} |f(s, x_1(s), {}^C D^\delta x_1(s), y_1(s), {}^C D^\gamma y_1(s)) \\ &\quad - f(s, x_2(s), {}^C D^\delta x_2(s), y_2(s), {}^C D^\gamma y_2(s))| ds \\ &\quad + \frac{1}{|a_0 - 1| \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f(s, x_1(s), {}^C D^\delta x_1(s), y_1(s), {}^C D^\gamma y_1(s)) \\ &\quad - f(s, x_2(s), {}^C D^\delta x_2(s), y_2(s), {}^C D^\gamma y_2(s))| ds \\ &\quad + \frac{t}{|a_1 - 1| \Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} |f(s, x_1(s), {}^C D^\delta x_1(s), y_1(s), {}^C D^\gamma y_1(s)) \\ &\quad - f(s, x_2(s), {}^C D^\delta x_2(s), y_2(s), {}^C D^\gamma y_2(s))| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x_1(s), {}^C D^\delta x_1(s), y_1(s), {}^C D^\gamma y_1(s)) \\ &\quad - f(s, x_2(s), {}^C D^\delta x_2(s), y_2(s), {}^C D^\gamma y_2(s))| ds \\ &\leq L_1 (\|x_1 - x_2\| + \|^C D^\delta x_1 - {}^C D^\delta x_2\| + \|y_1 - y_2\| + \|^C D^\gamma y_1 - {}^C D^\gamma y_2\|) \times \\ &\quad \left(\frac{1}{|a_0 - 1| |a_1 - 1| \Gamma(\alpha)} + \frac{1}{|a_0 - 1| \Gamma(\alpha + 1)} + \frac{t}{|a_1 - 1| \Gamma(\alpha)} + \frac{t^\alpha}{\Gamma(\alpha + 1)} \right) \\ &\leq \omega_1 (\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y).\end{aligned}$$

On the other hand, we have

$$\left| \mathcal{H}'_1(x_1, y_1)(t) - \mathcal{H}'_1(x_2, y_2)(t) \right| \leq \omega_2 (\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y),$$

and

$$\begin{aligned}&|^C D^\delta \mathcal{H}_1(x_1, y_1)(t) - {}^C D^\delta \mathcal{H}_1(x_2, y_2)(t)| \\ &\leq \int_0^t \frac{(t-s)^{-\delta}}{\Gamma(1-\delta)} \left| \mathcal{H}'_1(x_1, y_1)(s) - \mathcal{H}'_1(x_2, y_2)(s) \right| ds \\ &\leq \frac{t^{1-\delta}}{\Gamma(2-\delta)} \omega_2 (\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y).\end{aligned}$$

By the above inequalities, we get

$$\begin{aligned}&\|\mathcal{H}_1(x_1, y_1) - \mathcal{H}_1(x_2, y_2)\|_X \\ &= \|\mathcal{H}_1(x_1, y_1) - \mathcal{H}_1(x_2, y_2)\| + \|^C D^\delta \mathcal{H}_1(x_1, y_1) - {}^C D^\delta \mathcal{H}_1(x_2, y_2)\| \\ &\leq \left\{ \omega_1 + \frac{\omega_2}{\Gamma(2-\delta)} \right\} (\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y).\end{aligned}$$

Similarly, we find that

$$\begin{aligned} |\mathcal{H}_2(x_1, y_1)(t) - \mathcal{H}_2(x_2, y_2)(t)| &\leq \omega_3 (\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y), \\ |\mathcal{H}'_2(x_1, y_1)(t) - \mathcal{H}'_2(x_2, y_2)(t)| &\leq \omega_4 (\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y), \end{aligned}$$

and

$$\begin{aligned} &|{}^C D^\gamma \mathcal{H}_2(x_1, y_1)(t) - {}^C D^\gamma \mathcal{H}_2(x_2, y_2)(t)| \\ &\leq \frac{t^{1-\gamma}}{\Gamma(2-\gamma)} \omega_4 (\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y), \end{aligned}$$

hence

$$\begin{aligned} &\|\mathcal{H}_2(x_1, y_1) - \mathcal{H}_2(x_2, y_2)\|_Y \\ &= \|\mathcal{H}_2(x_1, y_1) - \mathcal{H}_2(x_2, y_2)\| + \|{}^C D^\gamma \mathcal{H}_2(x_1, y_1) - {}^C D^\gamma \mathcal{H}_2(x_2, y_2)\| \\ &\leq \left\{ \omega_3 + \frac{\omega_4}{\Gamma(2-\gamma)} \right\} (\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} &\|\mathcal{H}(x_1, y_1) - \mathcal{H}(x_2, y_2)\|_{X \times Y} \\ &\leq \left\{ \omega_1 + \omega_3 + \frac{\omega_2}{\Gamma(2-\delta)} + \frac{\omega_4}{\Gamma(2-\gamma)} \right\} (\|(x_1, y_1) - (x_2, y_2)\|_{X \times Y}). \end{aligned}$$

This shows that the operator \mathcal{H} is a contraction in view of the assumption $\omega_1 + \omega_3 + \frac{\omega_2}{\Gamma(2-\delta)} + \frac{\omega_4}{\Gamma(2-\gamma)} < 1$. Hence, it follows by contraction mapping principle that the operator \mathcal{H} has a unique fixed point, which corresponds to the unique mild solution of the problem (1). This completes the proof. \square

Theorem 3.2. (Leray-Schauder) *Let $\mathcal{H} : E \rightarrow E$ be a completely continuous operator and $\mathcal{E}(\mathcal{H}) = \{x \in E : x = \lambda \mathcal{H}(x), 0 < \lambda < 1\}$. Then, either the set $\mathcal{E}(\mathcal{H})$ is unbounded, or \mathcal{H} has at least one fixed point.*

To facilitate the proof, we introduce the following notations:

$$\phi = v_1 m_0 + v_3 n_0 + \frac{v_2 m_0}{\Gamma(2-\delta)} + \frac{v_4 n_0}{\Gamma(2-\gamma)}, \quad (3.5)$$

$$\begin{aligned} \psi &= v_1 \max\{m_1, m_2\} + v_3 \max\{n_1, n_2\} + \frac{v_2}{\Gamma(2-\delta)} \max\{m_1, m_2\} \\ &\quad + \frac{v_4}{\Gamma(2-\gamma)} \max\{n_1, n_2\}, \end{aligned} \quad (3.6)$$

$$\begin{aligned} \chi &= v_1 \max\{m_3, m_4\} + v_3 \max\{n_3, n_4\} + \frac{v_2}{\Gamma(2-\delta)} \max\{m_3, m_4\} \\ &\quad + \frac{v_4}{\Gamma(2-\gamma)} \max\{n_3, n_4\}. \end{aligned} \quad (3.7)$$

Theorem 3.3. *Assume that the condition (H2) hold. In addition it is assumed that $\max\{\psi, \chi\} < 1$ where ψ, χ are defined by (3.6) and (3.7) respectively. Then, there exist at least one mild solution for the problem (1.1) on J .*

Proof. In the first step, we show that the operator $\mathcal{H} : X \times Y \rightarrow X \times Y$ is completely continuous. By continuity of the functions f and g , it is obvious that the operator \mathcal{H} is continuous. Let $\Omega \subset X \times Y$ be bounded, then in virtue of (H2), there exist positive constants M_1 and M_2 such that $|f(t, x(t), {}^C D^\delta x(t), y(t), {}^C D^\gamma y(t))| \leq$

M_1 , and $|g(t, x(t), {}^C D^\delta x(t), y(t), {}^C D^\gamma y(t))| \leq M_2$, for $(x, y) \in \Omega$. Then for any $(x, y) \in \Omega$, we have

$$\begin{aligned} & |\mathcal{H}_1(x, y)(t)| \\ & \leq \frac{1}{|a_0 - 1| |a_1 - 1| \Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} |f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s))| ds \\ & \quad + \frac{1}{|a_0 - 1| \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s))| ds \\ & \quad + \frac{t}{|a_1 - 1| \Gamma(\alpha - 1)} \int_0^1 (1-s)^{\alpha-2} |f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s))| ds \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s))| ds \\ & \leq M_1 \left(\frac{1}{|a_0 - 1| |a_1 - 1| \Gamma(\alpha)} + \frac{1}{|a_0 - 1| \Gamma(\alpha + 1)} + \frac{t}{|a_1 - 1| \Gamma(\alpha)} + \frac{t^\alpha}{\Gamma(\alpha + 1)} \right) \\ & \leq M_1 v_1, \end{aligned}$$

and

$$|\mathcal{H}'_1(x, y)(t)| \leq M_1 \left(\frac{1}{|a_1 - 1| \Gamma(\alpha)} + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) \leq M_1 v_2,$$

which implies that

$$\begin{aligned} |{}^C D^\delta \mathcal{H}(x, y)(t)| & \leq \int_0^t \frac{(t-s)^{-\delta}}{\Gamma(1-\delta)} |\mathcal{H}'_1(x, y)(s)| ds \\ & \leq \frac{M_1 v_2 t^{1-\delta}}{\Gamma(2-\delta)} \leq \frac{M_1 v_2}{\Gamma(2-\delta)}. \end{aligned}$$

Thus

$$\|\mathcal{H}_1(x, y)\|_X = \|\mathcal{H}_1(x, y)\| + \|{}^C D^\delta \mathcal{H}_1(x, y)\| \leq M_1 v_1 + \frac{M_1 v_2}{\Gamma(2-\delta)}.$$

Similarly, one can obtain that

$$|\mathcal{H}_2(x, y)(t)| \leq M_2 v_3, \quad |\mathcal{H}'_2(x, y)(t)| \leq M_2 v_4,$$

and

$$|{}^C D^\gamma \mathcal{H}_2(x, y)(t)| \leq \frac{t^{1-\gamma}}{\Gamma(2-\gamma)} M_2 v_4.$$

Hence

$$\|\mathcal{H}_2(x, y)\|_Y = \|\mathcal{H}_2(x, y)\| + \|{}^C D^\gamma \mathcal{H}_2(x, y)\| \leq M_2 v_3 + \frac{M_2 v_4}{\Gamma(2-\gamma)}.$$

Therefore

$$\begin{aligned} \|\mathcal{H}(x, y)\|_{X \times Y} & \leq \|\mathcal{H}_1(x, y)\|_X + \|\mathcal{H}_2(x, y)\|_Y \\ & \leq M_1 \left\{ v_1 + \frac{v_2}{\Gamma(2-\delta)} \right\} + M_2 \left\{ v_3 + \frac{v_4}{\Gamma(2-\gamma)} \right\}. \end{aligned}$$

Thus, it follows that the operator \mathcal{H} is uniformly bounded. Next, we show that the operator \mathcal{H} is equicontinuous, let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, we have

$$\begin{aligned} & |\mathcal{H}_1(x, y)(t_2) - \mathcal{H}_1(x, y)(t_1)| \\ & \leq \frac{(t_2 - t_1)}{|a_1 - 1|\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha - 2} \\ & \quad \times |f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s))| ds \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1}| \\ & \quad \times |f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s))| ds \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} |f(s, x(s), {}^C D^\delta x(s), y(s), {}^C D^\gamma y(s))| ds \\ & \leq \frac{M_1(t_2 - t_1)}{|a_1 - 1|\Gamma(\alpha)} + \frac{M_1(2(t_2 - t_1)^\alpha + |t_2^\alpha - t_1^\alpha|)}{\Gamma(\alpha + 1)}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & |{}^C D^\delta \mathcal{H}_1(x, y)(t_2) - {}^C D^\delta \mathcal{H}_1(x, y)(t_1)| \\ & \leq \left| \int_0^{t_1} \frac{(t_2 - s)^{-\delta} - (t_1 - s)^{-\delta}}{\Gamma(1 - \delta)} |\mathcal{H}'_1(x, y)(s)| ds \right| \\ & \quad + \int_{t_1}^{t_2} \frac{(t_2 - s)^{-\delta}}{\Gamma(1 - \delta)} |\mathcal{H}'_1(x, y)(s)| ds \\ & \leq \frac{M_1 v_2 \left(2(t_2 - t_1)^{-\delta + 1} + |t_2^{-\delta + 1} - t_1^{-\delta + 1}| \right)}{\Gamma(2 - \delta)}. \end{aligned}$$

Hence, we have $|\mathcal{H}_1(x, y)(t_2) - \mathcal{H}_1(x, y)(t_1)| \rightarrow 0$, and $|{}^C D^\delta \mathcal{H}_1(x, y)(t_2) - {}^C D^\delta \mathcal{H}_1(x, y)(t_1)| \rightarrow 0$ independent of x and y as $t_2 \rightarrow t_1$. In the same way, we can obtain

$$|\mathcal{H}_2(x, y)(t_2) - \mathcal{H}_2(x, y)(t_1)| \leq \frac{M_2(t_2 - t_1)}{|b_1 - 1|\Gamma(\beta)} + \frac{M_2 \left(2(t_2 - t_1)^\beta + |t_2^\beta - t_1^\beta| \right)}{\Gamma(\beta + 1)},$$

and

$$|{}^C D^\gamma \mathcal{H}_2(x, y)(t_2) - {}^C D^\gamma \mathcal{H}_2(x, y)(t_1)| \leq \frac{M_2 v_4 \left(2(t_2 - t_1)^{-\gamma + 1} + |t_2^{-\gamma + 1} - t_1^{-\gamma + 1}| \right)}{\Gamma(2 - \gamma)}.$$

Thus, we have $|\mathcal{H}_2(x, y)(t_2) - \mathcal{H}_2(x, y)(t_1)| \rightarrow 0$, and $|{}^C D^\gamma \mathcal{H}_2(x, y)(t_2) - {}^C D^\gamma \mathcal{H}_2(x, y)(t_1)| \rightarrow 0$ independent of x and y as $t_2 \rightarrow t_1$. Therefore the operator $\mathcal{H}(x, y)$ is equicontinuous. We infer that the operator $\mathcal{H}(x, y)$ is completely continuous by Arzela-Ascoli theorem. Finally, let $\mathcal{E} = \{(x, y) \in X \times Y : (x, y) = \lambda \mathcal{H}(x, y), 0 \leq \lambda < 1\}$ be bounded. For any $t \in J$, we have $x(t) = \lambda \mathcal{H}_1(x, y)(t)$ and $y(t) = \lambda \mathcal{H}_2(x, y)(t)$. Then,

$$\begin{aligned} |x(t)| & \leq v_1 \{m_0 + m_1 |x(t)| + m_2 |{}^C D^\delta x(t)| + m_3 |y(t)| + m_4 |{}^C D^\gamma y(t)|\} \\ & \leq v_1 \{m_0 + \max\{m_1, m_2\} \|x\|_X + \max\{m_3, m_4\} \|y\|_Y\}, \end{aligned}$$

and

$$|x'(t)| \leq v_2 \{m_0 + \max\{m_1, m_2\} \|x\|_X + \max\{m_3, m_4\} \|y\|_Y\}.$$

Hence

$$|{}^C D^\delta x(t)| \leq \frac{t^{1-\delta} v_2}{\Gamma(2-\delta)} \{m_0 + \max\{m_1, m_2\} \|x\|_X + \max\{m_3, m_4\} \|y\|_Y\}.$$

In view of the above estimates, we get

$$\begin{aligned} \|x\|_X &\leq v_1 \{m_0 + \max\{m_1, m_2\} \|x\|_X + \max\{m_3, m_4\} \|y\|_Y\} \\ &\quad + \frac{v_2}{\Gamma(2-\delta)} \{m_0 + \max\{m_1, m_2\} \|x\|_X + \max\{m_3, m_4\} \|y\|_Y\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |y(t)| &\leq v_3 \{n_0 + n_1 |x(t)| + n_2 |{}^C D^\delta x(t)| + n_3 |y(t)| + n_4 |{}^C D^\gamma y(t)|\} \\ &\leq v_3 \{n_0 + \max\{n_1, n_2\} \|x\|_X + \max\{n_3, n_4\} \|y\|_Y\}, \end{aligned}$$

and

$$|y'(t)| \leq v_3 \{n_0 + \max\{n_1, n_2\} \|x\|_X + \max\{n_3, n_4\} \|y\|_Y\}.$$

Hence

$$|{}^C D^\gamma y(t)| \leq \frac{t^{1-\gamma} v_4}{\Gamma(2-\gamma)} \{n_0 + \max\{n_1, n_2\} \|x\|_X + \max\{n_3, n_4\} \|y\|_Y\}.$$

In view of the above estimates, we get

$$\begin{aligned} \|y\|_Y &\leq v_3 \{n_0 + \max\{n_1, n_2\} \|x\|_X + \max\{n_3, n_4\} \|y\|_Y\} \\ &\quad + \frac{v_4}{\Gamma(2-\gamma)} \{n_0 + \max\{n_1, n_2\} \|x\|_X + \max\{n_3, n_4\} \|y\|_Y\}. \end{aligned}$$

Then, we can find that

$$\|(x, y)\|_{X \times Y} = \|x\|_X + \|y\|_Y \leq \phi + \max\{\psi, \chi\} \|(x, y)\|_{X \times Y},$$

which implies that

$$\|(x, y)\|_{X \times Y} \leq \frac{\phi}{1 - \max\{\Psi, \chi\}}.$$

This shows that the set \mathcal{E} is bounded thus by Theorem 3.2, the operator \mathcal{H} has at least one fixed point. In consequence, the problem (1.1)-(1.2) has at least one mild solution on J . The proof is completed. \square

4. EXAMPLES

Example 4.1. Consider the following coupled system

$$\begin{cases} {}^C D^{\frac{5}{3}} x(t) = \frac{1}{\sqrt{25+t}} + \frac{1}{4\sqrt{3600+t^2}} \left(\frac{|x(t)|}{1+|x(t)|} + \sin\left({}^C D^{\frac{1}{2}} x(t)\right) + \sin(y(t)) + {}^C D^{\frac{3}{4}} y(t) \right), \\ {}^C D^{\frac{4}{3}} y(t) = \frac{e^{-t}}{4(1+t^4)} + \frac{1}{100\sqrt{t^3+1}} \left(x(t) + \arctan\left({}^C D^{\frac{1}{2}} x(t)\right) + \frac{|y(t)|}{1+|y(t)|} + \sin {}^C D^{\frac{3}{4}} y(t) \right), \end{cases}$$

for $t \in J = [0, 1]$, supplemented with non-periodic conditions:

$$\begin{cases} x(1) = 2x(0), \quad x'(1) = \frac{1}{4}x'(0), \\ y(1) = 3y(0), \quad y'(1) = \frac{1}{2}y'(0). \end{cases}$$

Simple calculations lead to the following values: $L_1 = \frac{1}{60}$, $L_2 = \frac{1}{25}$, $N_1 = \frac{1}{5}$, $N_2 = \frac{1}{4}$, $\nu_1 \approx 4.28$, $\nu_2 \approx 2.59$, $\nu_3 \approx 4.6$, $\nu_4 \approx 3.36$, $\omega_1 \approx 0.071$, $\omega_2 \approx 0.043$, $\omega_3 \approx 0.184$, $\omega_4 \approx 0.134$, $\sigma_1 \approx 0.856$, $\sigma_2 \approx 0.518$, $\sigma_3 \approx 1.15$, $\sigma_4 \approx 0.84$. Then $\omega_1 + \frac{\omega_2}{\Gamma(\frac{3}{2})} \approx 0.12 < \frac{1}{2}$, and $\omega_3 + \frac{\omega_4}{\Gamma(2-\gamma)} \approx 0.33 < \frac{1}{2}$. Obviously, all conditions of Theorem 3.1 are satisfied, then the coupled system of Example 4.1 has a unique mild solution on J .

Example 4.2. Consider the coupled system of fractional differential equations:

$$\begin{cases} {}^C D^{\frac{5}{3}} x(t) = \frac{1}{\sqrt{9+t^2}} + \frac{1}{100} e^{-t} \sin x(t) + \frac{{}^C D^{\frac{1}{4}} x(t)}{100(3+t)} + \frac{1}{120} \sin(y(t)) + \frac{{}^C D^{\frac{1}{2}} y(t)}{10(20+t)}, \\ {}^C D^{\frac{3}{2}} y(t) = \frac{\cos t}{\sqrt{121+t^2}} + \frac{x(t)}{400(1+y^3(t))} + \frac{1}{600} {}^C D^{\frac{1}{4}} x(t) + \frac{y(t)}{3(50+t^{\frac{1}{3}})} + \frac{{}^C D^{\frac{1}{2}} y(t)}{2\sqrt{25+t^2}}, \end{cases}$$

equipped with non-periodic conditions:

$$\begin{cases} x(1) = 2x(0), & x'(1) = \frac{2}{3}x'(0), \\ y(1) = \frac{3}{2}y(0), & y'(1) = 3y'(0), \end{cases}$$

Clearly

$$\begin{aligned} & \left| f\left(t, x(t), {}^C D^{\frac{1}{4}} x(t), y(t), {}^C D^{\frac{1}{2}} y(t)\right) \right| \\ & \leq \frac{1}{3} + \frac{1}{100} \|x\| + \frac{1}{300} \left\| {}^C D^{\frac{1}{4}} x \right\| + \frac{1}{120} \|y\| + \frac{1}{200} \left\| {}^C D^{\frac{1}{2}} y \right\|, \end{aligned}$$

and

$$\begin{aligned} & \left| g\left(t, x(t), {}^C D^{\frac{1}{4}} x(t), y(t), {}^C D^{\frac{1}{2}} y(t)\right) \right| \\ & \leq \frac{1}{11} + \frac{1}{400} \|x\| + \frac{1}{600} \left\| {}^C D^{\frac{1}{4}} x \right\| + \frac{1}{150} \|y\| + \frac{1}{10} {}^C D^{\frac{1}{2}} y. \end{aligned}$$

Thus $m_0 = \frac{1}{3}$, $m_1 = \frac{1}{100}$, $m_2 = \frac{1}{300}$, $m_3 = \frac{1}{120}$, $m_4 = \frac{1}{200}$, $n_0 = \frac{1}{11}$, $n_1 = \frac{1}{400}$, $n_2 = \frac{1}{600}$, $n_3 = \frac{1}{150}$, $n_4 = \frac{1}{10}$, $v_1 \approx 8.42$, $v_2 \approx 4.44$, $v_3 \approx 3.91$, $v_4 \approx 1.68$, $\psi \approx 0.16$, $\chi \approx 0.69$, $\max\{\psi, \chi\} < 1$. The hypotheses of Theorem 3.3 are satisfied, therefore, there exist at least one mild solution of the system in Example 4.2 on J .

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