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DYNAMICAL ANALYSIS OF A FRACTIONAL ORDER PREY-PREDATOR SYSTEM WITH A RESERVED AREA

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Abstract. This paper deals with a fractional order prey-predator system in a reserved area. The existence, uniqueness, non-negativity, boundedness and persistence of the solutions are proved for the considered model. Sufficient condition for local as well as global stability of the equilibrium points are derived. A numerical analysis for Hopf type bifurcation is presented. Finally numerical simulations are carried out to validate the results obtained.

1. Introduction

Fractional calculus is the branch of mathematics that generalizes derivatives and integration of arbitrary order. In particular, fractional differential equations are used to explain certain nonlinear phenomena [10]. The process of developing a differential system of integer order into fractional order becomes an important issue in dynamical system. Recently several studies are carried out in fractional order biological systems [2],[6],[9],[13]. The main reason for using fractional order systems is that they allow greater degree of freedom in the model. Moreover, fractional order systems are more realistic than integer order in biological modeling due to memory effects.

Lot of works have been done numerically in fractional systems still some problems are unsolved and a few analytical results have been developed. Results on stability are addressed in [11]. Recently, global stability analysis is discussed elaborately in [14].

In recent years, it is observed that many species in ecological system become extinct due to various reasons like, over exploitation, over predation, environmental pollution etc. To protect these species, appropriate measures such as restriction on harvesting, employing reserve zones/refuges should be taken so that species can grow in these regions without any external disturbances.

In view of above, Mukherjee [12] studied a prey-predator system with a reserved

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area with Holling type II predator functional response function as follows :

$$\begin{aligned}
\frac{dx}{dt} &= r_1x(1-\frac{x}{k_1}) - m_1x + m_2y - \frac{c_1xz}{h+x}, \\
\frac{dy}{dt} &= r_2y(1-\frac{y}{k_2}) + m_1x - m_2y, \\
\frac{dz}{dt} &= z(\frac{c_2x}{h+x} - d) \qquad (1)
\end{aligned}$$

Here $x(t)$ is the density of prey species inside the unreserved area. $y(t)$ is the density of prey species in the reserved area where predation is not allowed. $z(t)$ represents the predator density. r_1 and r_2 are the intrinsic growth rates of prey species inside the unserved and reserved area respectively. k_1 and k_2 are their respective carrying capacities. c_1, c_2 and d are the capturing rate, conversion rate and death rate of predator respectively. m_1 and m_2 are migration rates from the unreserved area to the reserved area and the reserved area to the unreserved area respectively. h is the half saturation constant. Stability, Hopf bifurcation and persistence in integer order system (1) with $h = 1$ are studied in [12]. In the present paper, we introduce the fractional order derivative by replacing the usual integer order derivative by fractional order Caputo-type derivative to obtain the following fractional order system :

$$\begin{aligned}
{}^cD^\alpha x(t) &= r_1x(1-\frac{x}{k_1}) - m_1x + m_2y - \frac{c_1xz}{h+x}, \\
{}^cD^\alpha y(t) &= r_2y(1-\frac{y}{k_2}) + m_1x - m_2y, \\
{}^cD^\alpha z(t) &= z(\frac{c_2x}{h+x} - d) \qquad (2)
\end{aligned}$$

with initial conditions
$x(0) = x_0 > 0, y(0) = y_0 > 0$ and $z(0) = z_0 > 0$,
where $\alpha \in (0, 1)$ and ${}^cD^\alpha$ is the standard Caputo differentiation. All the parameters of fractional order system (2) are non-negative for all time $t \geq 0$. The Caputo fractional derivative of order α is defined by

$$ {}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, n-1 < \alpha < n, n \in N.$$

From literature survey, the dynamics of a fractional order prey-predator system with reserved area has not studied before. Motivated by these observations, a fractional order prey-predator system with reserved area is considered. Contributions of these work can be summarized as follows : Firstly, we show existence and uniqueness of the solutions of system (2). Secondly, we prove non-negativity and boundedness of the solutions for system (2). Thirdly, sufficient conditions are derived to ensure local and global stability of the equilibrium points of system (2). Fourthly, persistence conditions are presented. Furthermore, the emergence of Hopf bifurcation in the fractional order system (2) is demonstrated. Finally, by numerical simulations, we discuss the effect of fractional order on the solutions of system (2).
The paper structured as follows. In Section 2, the existence, uniqueness, non-negativity, boundedness , stability analysis, persistence and Hopf bifurcation of fractional order system (2) are presented. In Section 3, numerical simulations are provided to validate the results obtained. Finally, a brief discussion concludes our

paper in Section 4.

2. Main results

In this section, we study the existence, uniqueness, non-negativity and boundedness of the solutions of a fractional order system (2). In addition, the stability of boundary equilibria, persistence, Hopf bifurcation and global stability of coexistence equilibrium point are analyzed.

2.1. **Existence and uniqueness.**

We now discuss the sufficient condition for existence and uniqueness of the fractional order system (2).

Theorem 1. For each non-negative initial condition, there exists a unique solution of fractional order system (2).

Proof. We study the existence and uniqueness of the fractional order system (2) in the region $B \times (0,T]$ where
$B = \{(x,y,z) \in \mathbb{R}^3 : \max(|x|,|y|,|z| \leq M)\}$. We follow the approach used in [7]. We denote $X = (x,y,z)$ and $\bar{X} = (\bar{x},\bar{y},\bar{z})$. Consider a mapping
$H(X) = (H_1(X), H_2(X), H_3(X))$ and

$$\begin{aligned} H_1(X) &= r_1 x(1-\frac{x}{k_1}) - m_1 x + m_2 y - \frac{c_1 xz}{h+x}, \\ H_2(X) &= r_2 y(1-\frac{y}{k_2}) + m_1 x - m_2 y, \\ H_3(X) &= z(\frac{c_2 x}{h+x} - d). \end{aligned} \tag{3}$$

For $X, \bar{X} \in B$, it follows from (3) that

$$\begin{aligned}
&\|H(X) - H(\bar{X})\| \\
&= |H_1(X) - H_1(\bar{X})| + |H_2(X) - H_2(\bar{X})| + |H_3(X) - H_3(\bar{X})| \\
&= |r_1 x(1-\tfrac{x}{k_1}) - m_1 x + m_2 y - \tfrac{c_1 xz}{h+x} - r_1\bar{x}(1-\tfrac{\bar{x}}{k_1}) + m_1\bar{x} - m_2\bar{y} + \tfrac{c_1\bar{x}\bar{z}}{h+\bar{x}}| \\
&+ |r_2 y(1-\tfrac{y}{k_2}) + m_1 x - m_2 y - r_2\bar{y}(1-\tfrac{\bar{y}}{k_2}) - m_1\bar{x} + m_2\bar{y}| + |z(\tfrac{c_2 xz}{h+x} - d) - \bar{z}(\tfrac{c_2\bar{x}\bar{z}}{h+\bar{x}} - d)| \\
&= |r_1(x-\bar{x}) - \tfrac{r_1}{k_1}(x^2-\bar{x}^2) - m_1(x-\bar{x}) + m_2(y-\bar{y}) + \tfrac{c_1 h(\bar{x}\bar{z}-xz) + c_1 x\bar{x}(\bar{z}-z)}{(h+x)(h+\bar{x})}| + |r_2(y - \\
&\bar{y}) - \tfrac{r_2}{k_2}(y^2-\bar{y}^2) + m_1(x-\bar{x}) - m_2(y-\bar{y})| + |\tfrac{c_2 h(xz-\bar{x}\bar{z}) + c_2 x\bar{x}(z-\bar{z})}{(h+x)(h+\bar{x})} - d(z-\bar{z})| \\
&\leq r_1|x-\bar{x}| + \tfrac{r_1}{k_1}|x+\bar{x}||x-\bar{x}| + m_1|x-\bar{x}| + m_2|y-\bar{y}| + \tfrac{c_1 h|\bar{x}\bar{z}-\bar{x}z+\bar{x}z-xz|}{(h+x)(h+\bar{x})} + \tfrac{c_1 x\bar{x}|z-\bar{z}|}{(h+x)(h+\bar{x})} + \\
&r_2|y-\bar{y}| + \tfrac{r_2}{k_2}|y+\bar{y}||y-\bar{y}| + m_1|x-\bar{x}| + m_2|y-\bar{y}| + \tfrac{c_2 h|xz-\bar{x}z+\bar{x}z-\bar{x}\bar{z}|}{(h+x)(h+\bar{x})} + \tfrac{c_2 x\bar{x}|z-\bar{z}|}{(h+x)(h+\bar{x})} + d|z-\bar{z}| \\
&\leq (r_1 + \tfrac{2Mr_1}{k_1} + m_1)|x-\bar{x}| + (r_2 + \tfrac{2Mr_2}{k_2} + m_2)|y-\bar{y}| + \tfrac{c_1 h|\bar{x}(\bar{z}-z) + z(\bar{x}-x)|}{(h+x)(h+\bar{x})} + \tfrac{c_2 h|z(x-\bar{x}) + \bar{x}(z-\bar{z})|}{(h+x)(h+\bar{x})} + \\
&\tfrac{(c_1+c_2)\bar{x}|z-\bar{z}|}{h+\bar{x}} + d|z-\bar{z}| \\
&\leq [r_1 + \tfrac{2r_1 M}{k_1} + m_1 + \{\tfrac{M(c_1+c_2)}{h+\bar{x}}\}]|x-\bar{x}| + [r_2 + \tfrac{2r_2 M}{k_2} + m_2]|y-\bar{y}| + \{\tfrac{2(c_1+c_2)\bar{x}}{h+\bar{x}} + d\}|z-\bar{z}| \\
&\leq L\|X - \bar{X}\|,
\end{aligned}$$

where
$L = \max\{r_1 + \frac{2r_1 M}{k_1} + \frac{M(c_1+c_2)}{h+\bar{x}} + m_1, r_2 + \frac{2r_2 M}{k_2} + m_2, \frac{2\bar{x}(c_1+c_2)}{h+\bar{x}} + d\}$.
Thus, $H(X)$ satisfies the Lipschitz condition. Consequently, the existence and uniqueness of fractional order system (2) follows.

2.2. **Non-negativity and boundedness.** For biological validity, we are only interested in solutions that are non-negative and bounded. The following result establishes the non-negativity and boundedness of the solutions of fractional order system (2).

Theorem 2 All the solutions of fractional order system (2) which start in $\mathbb{R}^3_+$ are uniformly bounded and non-negative.

Proof. Define the function
$V(t) = x(t) + y(t) + \frac{c_1}{c_2}z$. For each $d > 0$,
${}^cD^\alpha V(t) + dV(t)$
$= r_1x(1-\frac{x}{k_1})-m_1x+m_2y-\frac{c_1xz}{h+x}+r_2y(1-\frac{y}{k_2})+m_1x-m_2y+\frac{c_1xz}{h+x}-\frac{c_1dz}{c_2}+dx+dy+\frac{dc_1z}{c_2}$
$= -\frac{r_1}{k_1}x^2 + (r_1+d)x - \frac{r_2}{k_2}y^2 + (r_2+d)y$
$= -\frac{r_1}{k_1}\{x - \frac{k_1(r_1+d)}{2r_1}\}^2 + \frac{k_1(r_1+d)^2}{4r_1} - \frac{r_2}{k_2}\{y - \frac{k_2(r_2+d)}{2r_2}\}^2 + \frac{k_2(r_2+d)^2}{4r_2}$
$\leq \frac{k_1(r_1+d)^2}{4r_1} + \frac{k_2(r_2+d)^2}{4r_2}$.
By using the standard comparison theorem for fractional order [3],
$V(t) \leq V(0)E_\alpha(-d(t)^\alpha) + \{\frac{k_1(r_1+d)^2}{4r_1} + \frac{k_2(r_2+d)^2}{4r_2}\}(t)^\alpha E_{\alpha,\alpha+1}(-d(t)^\alpha)$
where E_α is the Mittag-Leffler function. According to Lemma 5 and Corollary 6 in [3],
$V(t) \leq \frac{k_1(r_1+d)^2}{4r_1} + \frac{k_2(r_2+d)^2}{4r_2}, t \to \infty$.
Therefore, all the solutions of fractional order system (2) starting in $\mathbb{R}^3_+$ are confined to the region Ω where

$$\Omega = \{(x,y,z) \in \mathbb{R}^3_+ : V \leq \frac{k_1(r_1+d)^2}{4r_1} + \frac{k_2(r_2+d)^2}{4r_2} + \epsilon, \epsilon > 0\}. \tag{4}$$

Next, we show that the solutions of the fractional order system (2) are non-negative. From Eq.(1) of system (2)

$${}^cD^\alpha x(t) = r_1x(1-\frac{x}{k_1}) - m_1x + m_2y - \frac{c_1xz}{h+x}. \tag{5}$$

From (4), one can see that

$$x + y + \frac{c_1}{c_2}z \leq \frac{k_1(r_1+d)^2}{4r_1} + \frac{k_2(r_2+d)^2}{4r_2} = k. \tag{6}$$

Based on (5) and (6), we get
${}^cD^\alpha x(t) \geq r_1x(1-\frac{x}{k_1}) - m_1x - \frac{c_2k}{h}x$
$= \{r_1(1-\frac{k}{k_1}) - m_1 - \frac{c_2k}{h}\}x = \beta x$
where
$\beta = r_1(1-\frac{k}{k_1}) - m_1 - \frac{c_2k}{h}\}$.
According to the standard comparison theorem for fractional order [3] and the positivity of Mittag-Leffler function $E_{\alpha,1}(t) > 0$ for any $\alpha \in (0,1)$ [15],
$x \geq x_0E_{\alpha,1}(\beta t^\alpha) \Rightarrow x \geq 0$.
From Eq.2 of system (2)
${}^cD^\alpha y(t) = r_2y(1-\frac{y}{k_2}) + m_1x - m_2y$
$\geq r_2y(1-\frac{k}{k_2}) - m_2y$
$= \beta_1 y$
where

$\beta_1 = r_2(1 - \frac{k}{k_2}) - m_2$.
Therefore, $y \geq y_0 E_{\alpha,1}(\beta_1 t^\alpha) \Rightarrow y \geq 0$.
Again, from Eq.3 of (2),
${}^c D^\alpha z(t) = z(\frac{c_2 xz}{h+x} - d) \geq -dz$.
Therefore,
$z \geq z_0 E_{\alpha,1}(-dt^\alpha) \Rightarrow z \geq 0$.
Thus, it has been proved that, the solutions of system (2) are non-negative.

2.3. **Equlibria and stability.** Equilibria of system (2) are solutions to the system

$$ {}^c D^\alpha x(t) = 0, {}^c D^\alpha y(t) = 0, {}^c D^\alpha z(t) = 0. $$

Then, the fractional order system (2) has three equilibrium points as follows :
1. The population free equilibrium point $E_0 = (0,0,0)$. The point E_0 always exist.
2. The predator free equilibrium point $E_1 = (\bar{x}, \bar{y}, 0)$. From system (2) we have

$$ r_1 \bar{x}(1 - \frac{\bar{x}}{k_1}) - m_1 \bar{x} + m_2 \bar{y} = 0, \tag{7} $$

$$ r_2 \bar{y}(1 - \frac{\bar{y}}{k_2}) + m_1 \bar{x} - m_2 \bar{y} = 0. \tag{8} $$

From (7) and (8) we find
$\bar{y} = \frac{1}{m_2}\{\frac{r_1 \bar{x}^2}{k_1} - (r_1 - m_1)\bar{x}\}$ and $\bar{x}$ is the positive root of the following equation

$$ b_0 x^3 + b_1 x^2 + b_2 x + b_3 = 0, \tag{9} $$

where
$b_0 = \frac{r_1^2 r_2}{m_2^2 k_1^2 k_2}, b_1 = -\frac{2r_1 r_2 (r_1 - m_1)}{k_1 k_2 m_2^2}, b_2 = \frac{r_2 (r_1 - m_1)}{k_2 m_2^2} - \frac{r_1 (r_2 - m_2)}{k_1 m_2}, b_3 = \frac{r_1 r_2 - r_2 m_1 - r_1 m_2}{m_2}$.
Eq. (9) has a positive root if $r_1 r_2 < r_1 m_2 + r_2 m_1$. For $\bar{y}$ to be positive, we must have $\bar{x} > \frac{k_1 (r_1 - m_1)}{r_1}$.
3. The coexistence equilibrium point $E_2(x^*, y^*, z^*)$. From system (2) we have
$x^* = \frac{dh}{c_2 - d}, y^* = \frac{k_2(r_2 - m_2) + \sqrt{k_2^2 (r_2 - m_2)^2 + 4k_2 m_1 x^* r_2}}{2r_2}, z^* = \frac{h + x^*}{c_1 x^*}\{r_1 x^*(1 - \frac{x^*}{k_1}) - m_1 x^* + m_2 y^*\}$.
The coexistence equilibrium point E_2 for the fractional order system (2) exists if $c_2 > d$ and $m_2 y^* + r_1 x^* > x^*(m_1 + \frac{r_1 x^*}{k_1})$.
The Jacobian matrix of system (2) at any point (x, y, z) is given by

$$ J(x,y,z) = \begin{pmatrix} r_1(1 - \frac{2x}{k_1}) - m_1 - \frac{c_1 zh}{(h+x)^2} & m_2 & -\frac{c_1 x}{h+x} \\ m_1 & r_2(1 - \frac{2y}{k_2}) - m_2 & 0 \\ \frac{c_2 hz}{(h+x)^2} & 0 & \frac{c_2 x}{h+x} - d \end{pmatrix} $$

Theorem 3. The equilibrium point E_0 of system (2) is an asymptotically stable equilibrium point if $r_i < m_i, i = 1, 2$ and a saddle point either $r_1 > m_1$ or $r_2 > m_2$.

Proof. The Jacobian matrix of system (2) at E_0 is given by

$$ J(E_0) = \begin{pmatrix} r_1 - m_1 & m_2 & 0 \\ m_1 & r_2 - m_2 & 0 \\ 0 & 0 & -d \end{pmatrix} $$

Eigenvalues of matrix $J(E_0)$ are obtained by solving the characteristic equation

$$P(\lambda) = det(J(E_0) - I\lambda) = \{\lambda^2 - \lambda(r_1 - m_1 + r_2 - m_2) + r_1 r_2 - r_1 m_2 - r_2 m_1\}(d + \lambda) = 0.$$

The eigenvalues corresponding to the equilibrium point E_0 are $\lambda_{1,2} = \frac{r_1 - m_1 + r_2 - m_2 \pm \sqrt{(r_1 - m_1 + r_2 - m_2)^2 - 4(r_1 r_2 - r_1 m_2 - r_2 m_1)}}{2}$ and $\lambda_3 = -d$. If $r_1 + r_2 > m_1 + m_2$ and $(r_1 - m_1 + r_2 - m_2)^2 > 4(r_1 r_2 - r_1 m_2 - r_2 m_1)$ then $|\arg \lambda_1| = 0 < \frac{\alpha\pi}{2}, |\arg \lambda_2| = 0 < \frac{\alpha\pi}{2}$ and $|\arg \lambda_3| = \pi > \frac{\alpha\pi}{2}$.
According to the Matignon's condition [11], the equilibrium point E_0 is a saddle point if $r_1 + r_2 > m_1 + m_2$ and $(r_1 - m_1 + r_2 - m_2)^2 > 4(r_1 r_2 - r_1 m_2 - r_2 m_1)$. If $r_1 + r_2 < m_1 + m_2$ and $(r_1 - m_1 + r_2 - m_2)^2 > 4(r_1 r_2 - r_1 m_2 - r_2 m_1)$ then $|\arg \lambda_i| = \pi > \frac{\alpha\pi}{2}, i = 1, 2$. Then E_0 is locally asymptotically stable.
Theorem 4. The equilibrium point E_1 of system (2) is locally asymptotically stable if $\frac{c_2 \bar{x}}{h + \bar{x}} < d$.

Proof. The Jacobian matrix of system (2) at E_1 is given by

$$J(E_1) = \begin{pmatrix} r_1(1 - \frac{2\bar{x}}{k_1}) - m_1 & m_2 & -\frac{c_1 \bar{x}}{h + \bar{x}} \\ m_1 & r_2(1 - \frac{2\bar{y}}{k_2}) - m_2 & 0 \\ 0 & 0 & \frac{c_2 \bar{x}}{h + \bar{x}} - d \end{pmatrix}$$

Eigenvalues of matrix $J(E_1)$ are $\lambda_1 = \frac{c_2 \bar{x}}{h + \bar{x}} - d$ and the other λ_2, λ_3 are obtained by solving the equation

$$\lambda^2 - \{r_1(1 - \frac{2\bar{x}}{k_1}) - m_1 + r_2(1 - \frac{2\bar{y}}{k_2}) - m_2\}\lambda + \{r_1(1 - \frac{2\bar{x}}{k_1}) - m_1\}\{r_2(1 - \frac{2\bar{y}}{k_2}) - m_2\} - m_1 m_2 = 0 \quad (10)$$

Now, $r_1(1 - \frac{\bar{x}}{k_1}) - m_1 + \frac{m_2 \bar{y}}{\bar{x}} = 0$.
Therefore, $r_1(1 - \frac{2\bar{x}}{k_1}) - m_1 = -\frac{r_1 \bar{x}}{k_1} - \frac{m_2 \bar{y}}{\bar{x}}$.
Similarly,
$r_2(1 - \frac{2\bar{y}}{k_2}) - m_2 = -\frac{r_2 \bar{y}}{k_2} - \frac{m_1 \bar{x}}{\bar{y}}$. Thus (10) becomes

$$\lambda^2 + (\frac{r_1 \bar{x}}{k_1} + \frac{m_2 \bar{y}}{\bar{x}} + \frac{r_2 \bar{y}}{k_2} + \frac{m_1 \bar{x}}{\bar{y}})\lambda + (\frac{r_1 \bar{x}}{k_1} + \frac{m_2 \bar{y}}{\bar{x}})(\frac{r_2 \bar{y}}{k_2} + \frac{m_1 \bar{x}}{\bar{y}}) - m_1 m_2 = 0.$$

Clearly, $(\frac{r_1 \bar{x}}{k_1} + \frac{m_2 \bar{y}}{\bar{x}} + \frac{r_2 \bar{y}}{k_2} + \frac{m_1 \bar{x}}{\bar{y}})^2 > 4\{(\frac{r_1 \bar{x}}{k_1} + \frac{m_2 \bar{y}}{\bar{x}})(\frac{r_2 \bar{y}}{k_2} + \frac{m_1 \bar{x}}{\bar{y}}) - m_1 m_2\}$. So $\lambda_2, \lambda_3 < 0$
Thus, E_1 is locally asymptotically stable if $|\arg \lambda_i| > \frac{\alpha\pi}{2}, i = 1$ .Thus, E_1 is locally asymptotically stable if $|\arg \lambda_i| > \frac{\alpha\pi}{2}, i = 1, 2, 3$. Now $|\arg \lambda_1| = \pi$ if $\frac{c_2 \bar{x}}{h + \bar{x}} < d$. Furthermore,$|\arg \lambda_{2,3}| = \pi > \frac{\alpha\pi}{2}$. This completes the proof.
To analyze the stability of equilibrium point E_2, we compute $J(E_2)$. Now

$$J(E_2) = \begin{pmatrix} r_1(1 - \frac{2x^*}{k_1}) - m_1 - \frac{c_1 z^* h}{(h + x^*)^2} & m_2 & -\frac{c_1 d}{c_2} \\ m_1 & r_2(1 - \frac{2y^*}{k_2}) - m_2 & 0 \\ \frac{c_2 h z^*}{(h + x^*)^2} & 0 & 0 \end{pmatrix}$$

Eigenvalues of matrix $J(E_2)$ are obtained solving the by solving the characteristic equation

$$\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0,$$

where

$a_1 = -\{r_1(1-\frac{2x^*}{k_1}) - m_1 - \frac{c_1 z^* h}{(h+x^*)^2} + r_2(1-\frac{2y^*}{k_2}) - m_2\}, a_2 = \{r_1(1-\frac{2x^*}{k_1}) - m_1 - \frac{c_1 z^* h}{(h+x^*)^2}\}\{r_2(1-\frac{2y^*}{k_2}) - m_2\} - m_1 m_2 + \frac{c_1 dhz^*}{(h+x^*)^2}, a_3 = \frac{c_1 dhz^*}{(h+x^*)^2}\{\frac{m_1 x^*}{y^*} + \frac{r_2 y^*}{k_2}\}$, and its discriminant is given by :

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3a_1^3 - 4a_2^3 - 27a_3^2.$$

We note that $a_3 > 0$.

Theorem 5. The equilibrium point E_2 of system (2) is locally asymptotically stable if one of the following conditions are satisfied.

1. $D(P) > 0, a_1 > 0$, and $a_1a_2 > a_3$.
2. $D(P) < 0, a_1 \geq 0, a_2 \geq 0$, and $\alpha < \frac{2}{3}$.
3. $D(P) < 0, a_1 > 0, a_2 > 0, a_1a_2 = a_3$ and for all $\alpha \in (0,1)$.

Proof. Proceeding along the lines in [2], we can prove Theorem 5.

2.4. **Persistence.** Persistence in biological systems implies long term survival of the population. So study of persistence is important as it is concerned with the stability of some equilibrium solutions of the dynamical system.

We have already proved in Theorem 2 that

$$x + y + \frac{c_1}{c_2}z \leq \frac{k_1(r_1+d)^2}{4r_1} + \frac{k_2(r_2+d)^2}{4r_2} = k.$$

This implies that $x(t) \leq k, y(t) \leq k, z(t) \leq \frac{kc_2}{c_1}$.

The solution of Eq.1 of system (2) is

$x(t)$

$= x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{r_1x(s)(1-\frac{x(s)}{k_1}) - m_1x(s) + m_2y(s) - \frac{c_1x(s)z(s)}{h+x(s)}\}ds$

$> \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)\{r_1(1-\frac{k}{k_1}) - m_1 - \frac{c_2}{h}\}ds$

$= \frac{\beta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds$

$= E_\alpha(\beta t^\alpha) = p_1 > 0$ where $\beta = r_1(1-\frac{k}{k_1}) - m_1 - \frac{c_2}{h}$.

Similarly one can show that

$y(t) > E_\alpha(\gamma t^\alpha) > 0$ if $\gamma > 0$ where $\gamma = r_2(1-\frac{k}{k_2}) - m_2$. Now using the estimate of $x(t)$, we get

$z(t) > \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}z(s)\{\frac{c_2p_1}{h+p_1} - d\}ds = E_\alpha(\delta t^\alpha) > 0$ if $\delta > 0$ where $\delta = \frac{c_2p_1}{h+p_1} - d$.

Thus we have the following theorem.

Theorem 6. The solution of system (2) is persistence if
$r_1(1-\frac{k}{k_1}) > m_1 + \frac{c_2k}{h}, r_2(1-\frac{k}{k_2}) > m_2$ and $\frac{c_2p_1}{h+p_1} > d$ holds.

2.5. **Global stability.** In this section, we present global stability of coexistence equilibrium point E_2.

Theorem 7. The coexistence equilibrium point E_2 of system (2) is globally asymptotically stable if $\frac{r_1h}{k_1} \geq \frac{c_1z^*}{h+x^*}$.

Proof. Consider the following positive definite function about E_2

$$V(x,y,z) = (x - x^* - x^* ln\frac{x}{x^*}) + q_1(y - y^* - y^* ln\frac{y}{y^*}) + q_2(z - z^* - z^* ln\frac{z}{z^*})$$

where q_1 and q_2 are positive constants to be chosen later.
We compute the α order derivative of $V(x,y,z)$ along the solution of system (2) by applying Lemma 3.1 in [14]. Thus we have
${}^cD^\alpha V(x,y,z)$
$\leq (1-\frac{x^*}{x}){}^cD^\alpha x(t) + q_1(1-\frac{y^*}{y}){}^cD^\alpha y(t) + q_2(1-\frac{z^*}{z}){}^cD^\alpha z(t)$
$= (x-x^*)\{r_1(1-\frac{x}{k_1}) - m_1 + \frac{m_2 y}{x} - \frac{c_1 z}{h+x}\} + q_1(y-y^*)\{r_2(1-\frac{y}{k_2}) + \frac{m_1 x}{y} - m_2\} + q_2(z-z^*)(\frac{c_2 x}{h+x} - d)$
$= (x-x^*)\{-\frac{r_1}{k_1}(x-x^*) + \frac{m_2(yx^*-xy^*)}{xx^*} + \frac{c_1(h+x^*)(z^*-z)+c_1 z^*(x-x^*)}{(h+x)(h+x^*)}\} + q_1(y-y^*)\{-\frac{r_2}{k_2}(y-y^*) + \frac{m_1(xy^*-yx^*)}{yy^*}\} + \frac{q_2 c_2(z-z^*)(x-x^*)}{(h+x)(h+x^*)}$.
$\leq -(\frac{r_1}{k_1} - \frac{c_1 z^*}{h(h+x^*)})(x-x^*)^2 - \frac{y^* m_2 r_2 (y-y^*)^2}{k_2 m_1 x^*} - \frac{m_2(x^*y-xy^*)^2}{xyx^*}$
where $q_1 = \frac{m_2 y^*}{m_1 x^*}$ and $q_2 = \frac{c_1(h+x^*)}{c_2}$
Consequently, ${}^cD^\alpha V(x,y,z) \leq 0$, when $\frac{r_1 h}{k_1} \geq \frac{c_1 z^*}{h+x^*}$.
The result follows by the application of Lemma 4.6 in Huo et al.[8].

2.6. **Hopf bifurcation.** In this section, we identify the parameters which gives Hopf bifurcation in the fractional order system (2). Due to mathematical complexity, we carry out numerical simulations for finding the outcomes of the dynamical behavior. To determine this, we follow the approach developed in [1]. In the fractional order system, the stability of E_2 is determined by the sign of $f_i(\alpha, k_1) = \frac{\alpha\pi}{2} - |\arg(\lambda_i(k_1))|, i = 1,2,3$. Hopf bifurcation result in fractional order system can be described as
$D(P_{E_2}(k_1^*)) < 0, f_{1,2}(\alpha, k_1^*) = 0$, and $\lambda_3(k_1^*) \neq 0, \frac{\partial f_i}{\partial k_1}|_{k_1=k_1^*} \neq 0$.

3. Numerical simulations

In this section, we numerically simulate the theoretical results to justify and develop this paper for different fractional orders $0 < \alpha \leq 1$. We have applied Adamas-type predictor corrector method for the fractional order differential equation (FODE)[4, 5]. We first have replaced the FODE system (2) by the equivalent fractional order integral system and then used the approach of PECE (predict, evaluate, correct, evaluate) method.
In Fig. 1, we have showed the global stability of positive equilibrium point of system (2) with the choice of parameters set $r_1 = 1.2$, $k_1 = 1.5$, $c_1 = 4.2$, $r_2 = 1.25$, $k_2 = 5$, $c_2 = 2$, $d = 1$, $m_1 = 1$, $m_2 = 2$, $h = 2$. Here we have taken different values of α (including integer value also). Such choice of parameters, Theorem 7 is satisfied. Fig. 2 and Fig. 3 represent the phase portrait of system (2) which show that all solution of system (2) with different initial points converge to the coexistence equilibrium point.
There is a role of fractional order α to stabilize the system. We have considered a parameter set $r_1 = 85/22$, $k_1 = 7.5$, $c_1 = 90/11$, $r_2 = 2$, $k_2 = 2$, $c_2 = 2$, $d = 1$, $m_1 = 1$, $m_2 = 2$, $h = 1$. At $\alpha = 0.98$, a limit cycle occurs around coexistence equilibrium point (see Fig. 4). When we decrease the value α, limit

cycle disappears and coexistence equilibrium point becomes stable (see Fig. 5). We draw the bifurcation diagram (see Fig. 6) of system (2) around the coexistence equilibrium point $E_2(0.959, 0.9672, 1.0314)$ when α is taken as parameter with $r_1 = 85/22, k_1 = 8, c_1 = 90/11, r_2 = 2, k_2 = 2, c_2 = 2, d = 1, m_1 = 1, m_2 = 1, h = 1$. Fig. 7 is developed by Theorem 5 which denote the stability regions of the fractional order with the carrying capacity of the prey. In integer order system (1) it was shown k_1 as a bifurcation parameter [12]. So two parameters bifurcation diagram in $\alpha - k_1$ plane are generated. The stability regions are divided into three parts : red colored region corresponds to locally asymptotically stable region, yellow region represents oscillatory zone and black colored line denotes Hopf line and unstable region. This indicates the fact that a system with fraction order is more stable than a system with integer order.

4. CONCLUSION

In this paper, we have investigated a fractional order prey-predator model in a reserved area. The existence, uniqueness, non-negativity and boundedness of the model system have been shown. Local stability of the boundary equilibrium points is discussed. Persistence result is discussed. Sufficient condition for global stability of the coexistence equilibrium point has been derived by constructing a suitable Lyapunov function. Moreover, we address the emergence of Hopf bifurcation in the the fractional order system (2). The analytical results obtained in this paper are verified through numerical simulations. The simulations shows vital dynamics of the fractional order system (2). The numerical studies also show the impact of fractional order α on each population densities. The fractional order model (2) is more stable than the integer model (2).

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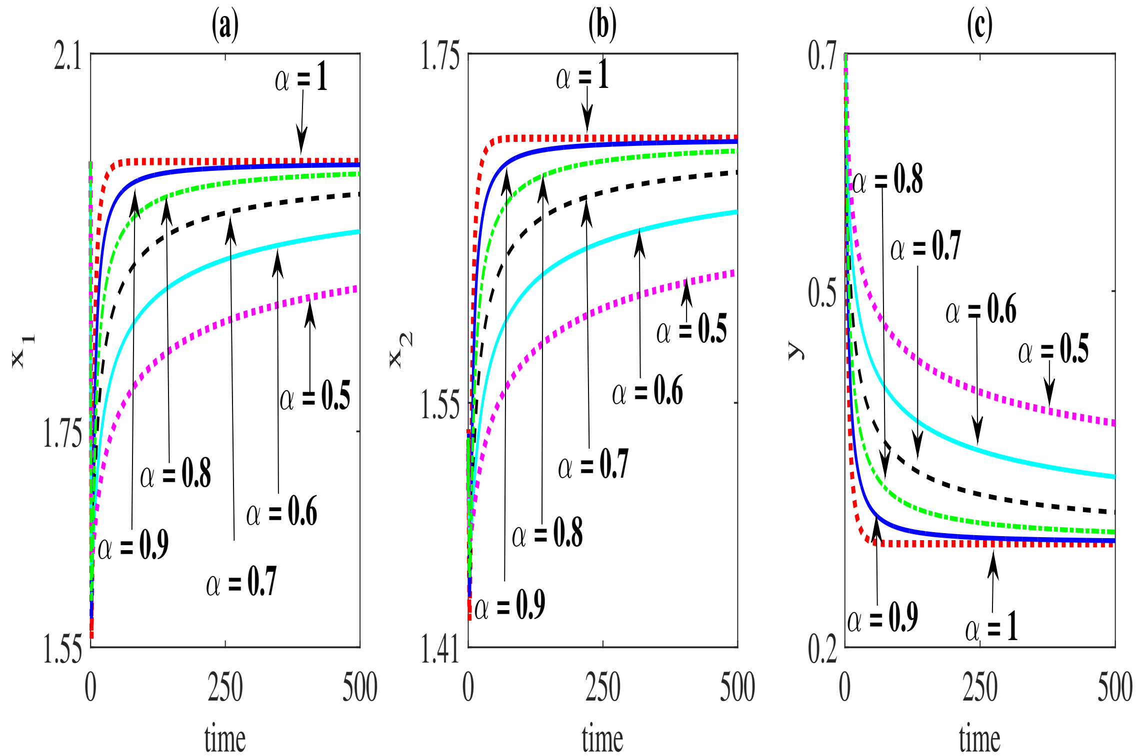

FIGURE 1. Time series of x_1, x_2 and y population for $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9, 1$ and $r_1 = 1.2$; $k_1 = 1.5$; $c_1 = 4.2$; $r_2 = 1.25$; $k_2 = 5$; $c_2 = 2$; $d = 1$; $m_1 = 1$; $m_2 = 2$; $h = 2$.

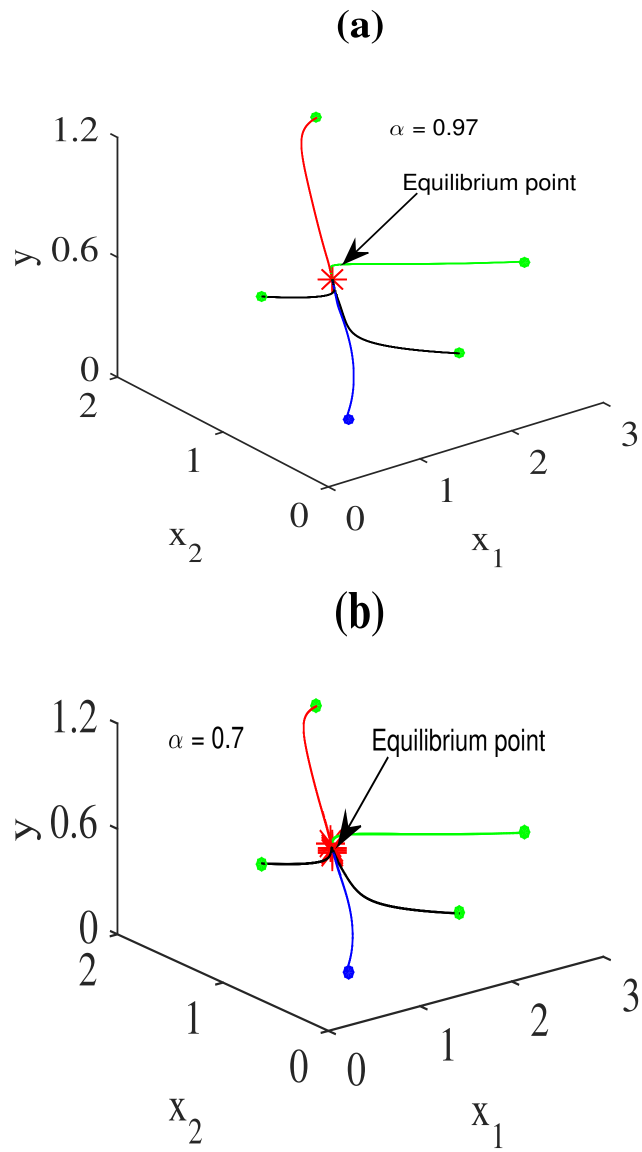

FIGURE 2. Phase portrait of system (2) with different values of α and many initial points. The parameters set as in Fig.1

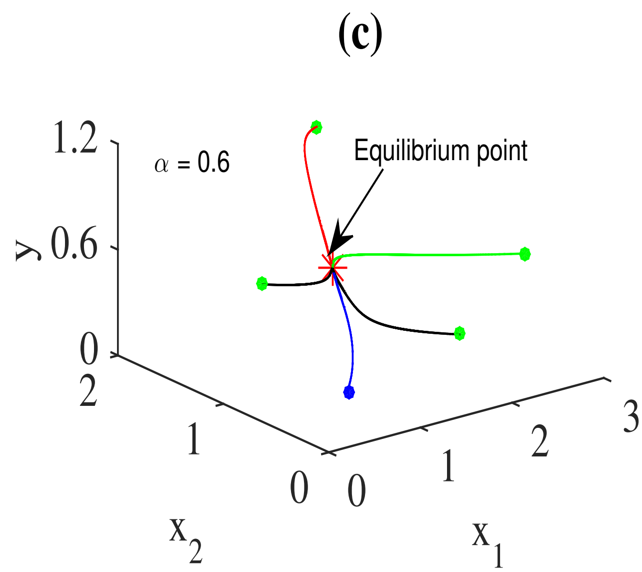

FIGURE 3. Phase portrait of system (2) with different values of α and many initial points. The parameters set as in Fig.1

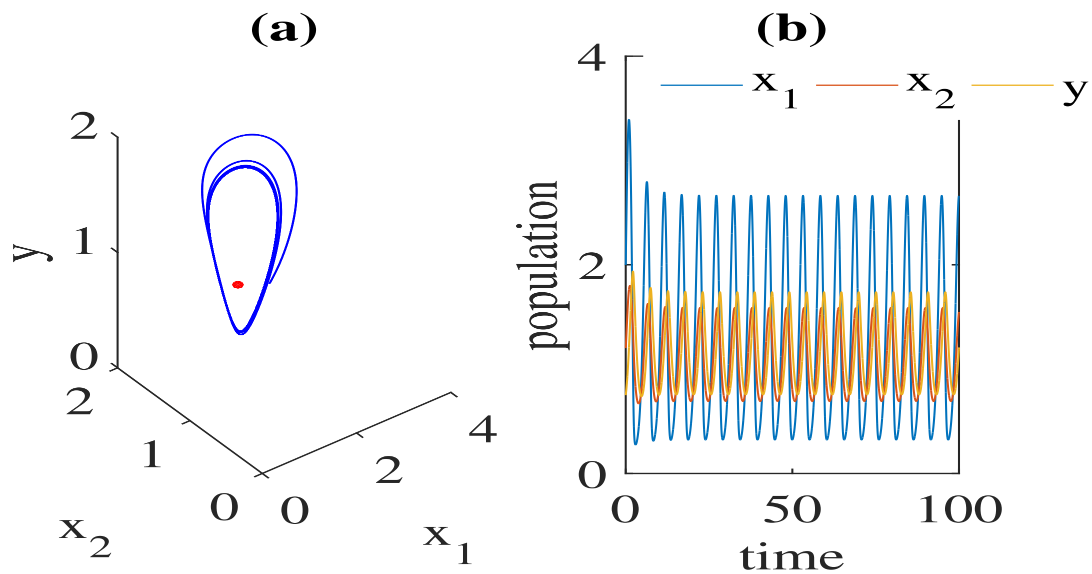

FIGURE 4. (a) Phase portrait of system (2) and (b) Numerical values of x_1, x_2 and y respect to time. Parameters values are $\alpha = 0.98$, $r_1 = 85/22$, $k_1 = 7.5$, $c_1 = 90/11$, $r_2 = 2$, $k_2 = 2$, $c_2 = 2$, $d = 1$, $m_1 = 1$, $m_2 = 2$, $h = 1$.

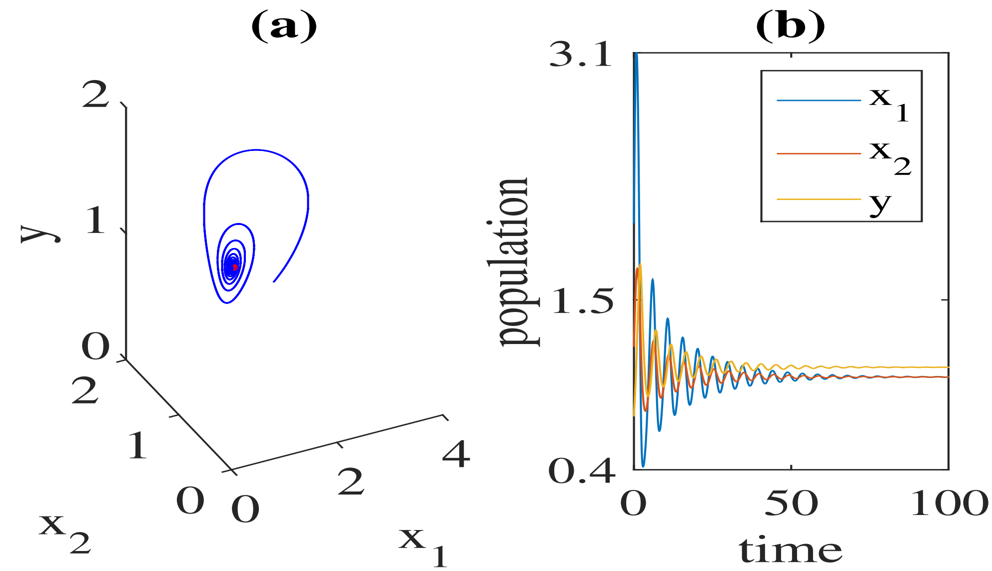

FIGURE 5. (a) Phase portrait of system (2) and (b) Numerical values of x_1, x_2 and y respect to time. Parameters values are $\alpha = 0.9$, $r_1 = 85/22$, $k_1 = 7.5$, $c_1 = 90/11$, $r_2 = 2$, $k_2 = 2$, $c_2 = 2$, $d = 1$, $m_1 = 1$, $m_2 = 2$, $h = 1$.

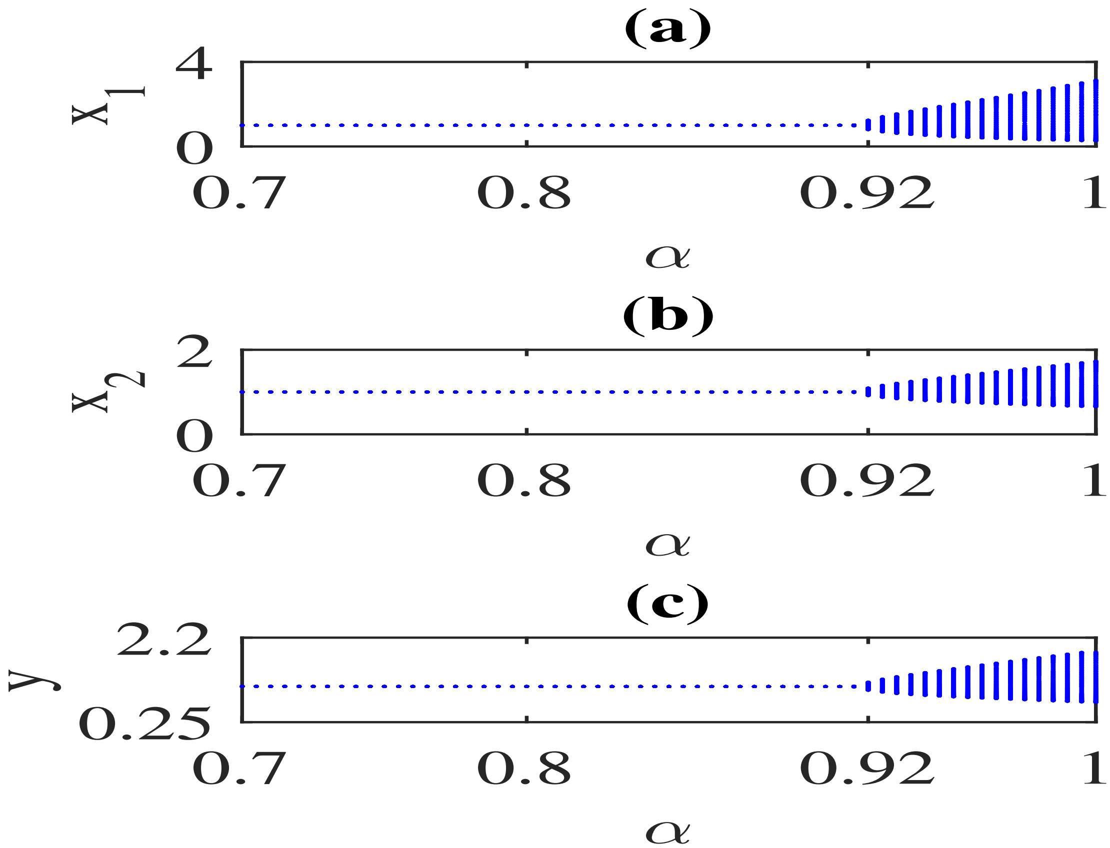

FIGURE 6. One parameter bifurcation diagram with respect to α. The values of the parameters are $r_1 = 85/22$, $k_1 = 8$, $c_1 = 90/11$, $r_2 = 2$, $k_2 = 2$, $c_2 = 2$, $d = 1$, $m_1 = 1$, $m_2 = 2$, $h = 1$

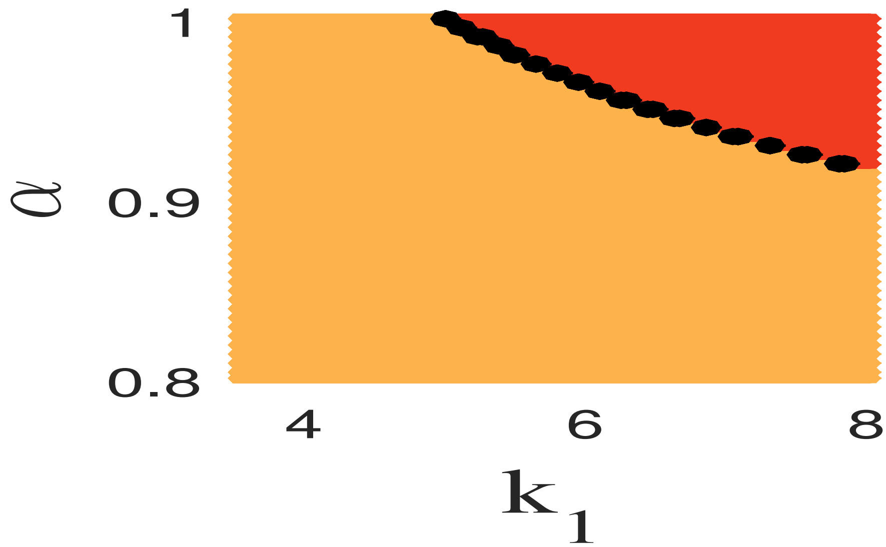

FIGURE 7. Bifurcation diagram in (α, k_1) space. Red colored region stands for stability of positive equilibrium point of system (2), yellow colored region represents limit cycle oscillations and black colored line denotes the Hopf line between stable and unstable region when $r_1 = 85/22$, $c_1 = 90/11$, $k_1 = 6$, $r_2 = 2$, $k_2 = 3$, $c_2 = 2$, $d = 1$, $m_1 = 1$, $m_2 = 2$, $h = 1$.