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SOLVING SOME $q-$DIFFERENCE EQUATIONS OF THE FOURTH ORDER

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Abstract. In this paper, we study the existence of the solutions of some $q-$difference equations. The solutions of some nonlinear of $q-$difference equations of order four are obtained. Also, we find the solutions of some non homogeneous linear $q-$difference equations with constant and variables coefficients.

1. Introduction and Preliminaries

The subject of q-calculus started appearing in the twentieth century in intensive works especially by F. H. Jackson [12], T. E. Mason [14], R. D. Carmichael [5],W. J. Trjitzinsky [16], C. R. Adams [1] and other authors such us Picard, Poincare, Ramanujan.

The q-difference has many applications in different mathematics such as statistical physics [15], fractal geometry [8],[9], number theory, quantum mechanics, orthogonal polynomials [11] and other sciences including quantum theory, mechanics and theory of relativity [4].

In this paper, we investigate the existence and the properties of the solution of the following:

I) The nonlinear boundary value problem (BVP) of the fourth-order of the form:

$$\begin{aligned} D_q^4 u(t) &= f(t,u(t)); \qquad 0 \le t \le 1, \quad 0 < q < 1, \\ u(0) &= 0, \quad D_q u(0) = 0, \quad D_q^2 u(0) = 0, \quad u(1) = 1, \end{aligned} \tag{1}$$

where f is a given continuous function.

II) The linear q-difference equations of order four

$$\Big[D^4_{q^{-1}} + a_1(x)D^3_{q^{-1}} + a_2(x)D^2_{q^{-1}} + a_3(x)D_{q^{-1}} + a_4(x)\Big]\, y(x) = f(x), \tag{2}$$

where $a_1(x), a_2(x), a_3(x)$ and $a_4(x)$ are continuous functions.

Let us recall some basic concepts of q-calculus ([7, 10, 13]).

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1- The q-analogue of the shifted factorial $(a)_n$ is defined by

$$\begin{aligned}(a;q) &= 1,\\ (a;q)_n &= (1)(1-qa)(1-q^2a)(1-q^3a)...(1-q^{n-1})\\ &= \prod_{m=0}^{n-1}(1-q^m a), \quad n \in N.\end{aligned}$$

2- The q-analogue of a complex number a and of the factorial function are defined by

$$\begin{aligned}[a]_q &= 1+q+q^2+...+q^{a-1}\\ &= \frac{1-q^a}{1-q}, \qquad q \in \mathbb{C}-\{1\}; \qquad a \in \mathbb{C},\end{aligned}$$

and

$$\begin{aligned}[n]_q! &= [1]_q[2]_q[3]_q...[n]_q\\ &= \prod_{m=1}^{n}[m]_q = \frac{(q;q)_n}{(1-q)^n}, \qquad q \neq 1; \quad n \in \mathbb{N}, \quad [0]! = 1; \quad 0 \leq q \leq 1.\end{aligned}$$

3- The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \qquad k = 0,1,2,...n.$$

4- The q-analogue of the function $(x+y)^n$ is defined as:

$$(x+y)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^{n-k} y^k, \qquad n \in \mathbb{N}_0.$$

5- The exponential functions are defined as:

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{((1-q)x;q)_\infty}, \qquad 0 < |q| < 1 \qquad ; |x| < |1-q|^{-1},$$

and

$$E_q(x) = \sum_{n=0}^{\infty} q^{k(k-1)/2} \frac{x^n}{[n]_q!} = (-(1-q)x;q)_\infty, \quad 0 < |q| < 1; \qquad x \in \mathbb{C}.$$

6- The functions $e_q(x)$ and $e_{q^{-1}}(-x)$ satisfy

$$e_q(x)\ e_{q^{-1}}(-x) = 1.$$

7- The functions $e_q(x)$ and $E_q(x)$ satisfy the following properties:

$$D_q e_q(x) = e_q(x) \quad \textit{and} \quad D_q E_q(x) = E_q(qx).$$

8- The q-derivative $D_q f$ of a function f also referred to as the Jackson derivative [12] defined as:

$$D_q f(x) = \frac{f(qx)-f(x)}{qx-x}, \qquad 0 < |q| < 1.$$

9- Derivative of a product

$$\begin{aligned} D_q(fg)(x) &= g(qx)D_qf(x)+f(x)D_qg(x) \\ &= f(qx)D_qg(x)+g(x)D_qf(x). \end{aligned}$$

10- Derivative of a ratio

$$D_q(\frac{f}{g})(x)=\frac{g(x)D_qf(x)-f(x)D_qg(x)}{g(qx)g(x)}.$$

11- Chain rule

$$\begin{aligned} D_q(f(g))(x) &= \frac{f(g(qx))-f(g(x))}{g(qx)-g(x)}.\frac{g(qx)-g(x)}{qx-x} \\ &= D_{q,g}f(g).D_{q,x}g(x). \end{aligned}$$

12- Derivative of the inverse function: Let $y=f(x)$ then $x=f^{-1}(y)$ where f^{-1} is the inverse function to f Applying the q-derivative on each side of the equality, implies

$$\begin{aligned} 1 &= D_q(x)=D_qf^{\ -1}(y)=\frac{f^{\ -1}(y(qx))-f^{\ -1}(y(x))}{y(qx)-y(x)}.\frac{y(qx)-y(x)}{qx-x} \\ &= D_{q,y}f^{\ -1}(y).D_{q,x}y(x), \end{aligned}$$

consequently

$$D_{q,y}f^{\ -1}(y)=\frac{1}{D_{q,x}y}.$$

13- The q-integral of a function f defined in the interval $[a,b]$ is given by

$$\int_a^x f(t)d_qt=\sum_{n=0}^{\infty}(1-q)q^n\left[x\ f(xq^n)-a\ f(q^na)\right],\quad x\in[a,b],$$

and for $a=0$, we denote

$$I_qf(x)=\int_0^x f(t)d_qt=\sum_{n=0}^{\infty}x(1-q)q^nf(xq^n),$$

provided the series converges. If $a\in[0,b]$ and f is defined on the interval $[0,b]$, then

$$\int_a^b f(t)d_qt=\int_0^b f(t)d_qt-\int_0^a f(t)d_qt.$$

Similarly we have

$$I_q^0f(t)=f(t),\qquad I_q^nf(t)=I_qI_q^{n-1}f(t),\quad n\in\mathbb{N}.$$

14- Fundamental principles of the q-analysis

i) $D_q\left[\int_a^x f(x)d_qx\right]=f(x).$

ii) $\int_a^x D_qf(x)d_qx=f(x)-f(a).$

15- Integration by parts: Consider the equality

$$D_q(fg)(x)=f(x)D_qg(x)+g(qx)D_qf(x),$$

then

$$f(x)D_qg(x) = D_q(fg)(x) - g(qx)D_qf(x).$$

Let $h(x) = f(x)g(x)$. We have

$$\int_a^b D_qh(x)d_qx = h(b) - h(a).$$

Hence

$$\int_a^b f(x)D_qg(x)d_qx = [fg]_a^b - \int_a^b g(qx)D_qf(x)d_qx.$$

2. Studying of Eq.(2)

In this section we consider the following two cases of Eq.(2).
The special case of Eq.(2) with constant coefficients of the form

$$\left[D^4_{q^{-1}} + a_1D^3_{q^{-1}} + a_2D^2_{q^{-1}} + a_3D_{q^{-1}} + a_4\right] y(x) = f(x), \tag{3}$$

where a_1, a_2, a_3 *and* a_4 *are* constants.

Recall that Eq.(3) is said to be a fourth-order constant coefficients linear nonhomogeneous q-difference equation. The corresponding homogenous equation reads

$$\left[D^4_{q^{-1}} + a_1D^3_{q^{-1}} + a_2D^2_{q^{-1}} + a_3D_{q^{-1}} + a_4\right] y(x) = 0, \tag{4}$$

In this case the equations can be solved explicitly.

Consider first the equation

$$D_{q^{-1}}y(x) = \lambda y(x),$$

which its solution reads

$$y(x) = e^{\lambda x}_{q^{-1}}. \tag{5}$$

Loading Eq.(5) in Eq.(4), one obtains the following algebraic equation in λ called the characteristic equation of Eq.(4):

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{6}$$

Lemma 1. *(I) If Eq.(6) has four distinct roots* λ_1, λ_2, λ_3 *and* λ_4 *then Eq.(4) admits as four linear independent solutions the functions*

$$y_i(x) = e^{\lambda_i x}_{q^{-1}}, \qquad i = 1, 2, 3, 4.$$

(II) If some of the roots of the characteristic equation are not distinct, then in that case also Eq.(4) admits four linear independent solutions.

(III) If for example a given root λ admits a multiplicity equal to m, so the corresponding independent solutions need to be searched among functions of the form

$$y(x) = \sum_{n=0}^{\infty} c_nx^n,$$

where the coefficients c_n satisfies

$$\sum_{i=0}^{m}\left[\binom{m}{i}(-\lambda)^{m-i}\prod_{k=0}^{i-1}\frac{1-q^{-(n+i)+k}}{1-q^{-1}}\right]c_{n+i}=0.$$

Proof. (I) Here the theorem is proved straightforwardly.

(II) and (III) To prove this part it suffices to load Eq.(5) in the following auxiliary equation

$$\left(D_{q^{-1}}-\lambda\right)^{m}y(x)=0, \qquad m=2,\ 3\ \textit{or}\ \ 4.$$

This proves the theorem. □

Theorem 2. *The solution of Eq.(3) is given by*

$$y(x)=C_1(x)y_1(x)+C_2(x)y_2(x)+C_3(x)y_3(x)+C_4(x)y_4(x),$$

where $C(x)=\begin{pmatrix} C_1(x) & C_2(x) & C_3(x) & C_4(x)\end{pmatrix}^t$ *is the solution of the system*

$$\Phi(qx)\ D_qC(x)=F(x),$$

and reads

$$C(x)=C+(1-q)x\sum_{i=0}^{\infty}q^i\Phi^{-1}(q^ix)F(q^ix),$$

$F(x)=\begin{pmatrix} 0 & 0 & 0 & f(x)\end{pmatrix}^t$ *and* $\Phi(x)$ *is a fundamental matrix of the solutions of the homogenous Eq.(4) and* $y_1(x)$, $y_2(x)$, $y_3(x)$ *and* $y_4(x)$ *a fundamental system of the solutions of Eq.(4).*

Proof. To find the general solution of Eq.(3) let

$$\begin{aligned} z_1(x) &= y(x),\\ z_2(x) &= D_{q^{-1}}y(x),\\ z_3(x) &= D^2_{q^{-1}}y(x),\end{aligned}$$

and

$$z_4(x)=D^3_{q^{-1}}y(x).$$

Thus we obtain the system

$$\begin{aligned} D_{q^{-1}}z_1(x) &= z_2(x),\\ D_{q^{-1}}z_2(x) &= z_3(x),\\ D_{q^{-1}}z_3(x) &= z_4(x),\end{aligned}$$

and

$$D_{q^{-1}}z_4(x)=-(a_1z_4(x)+a_2z_3(x)+a_3z_2(x)+a_4z_1(x))+f(x). \tag{7}$$

Now, replacing x by qx and then q^{-1} by q and by using the matrix form of Eq.(7) we obtain

$$D_qz(x)=Az(qx)+F(x). \tag{8}$$

where $z(x)=\begin{pmatrix} z_1(x) & z_2(x) & z_3(x) & z_4(x)\end{pmatrix}^t$,

$$A=\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -a_4 & -a_3 & -a_2 & -a_1\end{pmatrix},$$

and $F(x) = \begin{pmatrix} 0 & 0 & 0 & f(qx) \end{pmatrix}^t$.
Now consider the homogeneous part of Eq.(8)

$$D_q z(x) = Az(qx), \tag{9}$$

with the fundamental matrix

$$\Phi(x) = \begin{pmatrix} y_1(x) & y_2(x) & y_3(x) & y_4(x) \\ D_{q^{-1}} y_1(x) & D_{q^{-1}} y_2(x) & D_{q^{-1}} y_3(x) & D_{q^{-1}} y_4(x) \\ D^2_{q^{-1}} y_1(x) & D^2_{q^{-1}} y_2(x) & D^2_{q^{-1}} y_3(x) & D^2_{q^{-1}} y_4(x) \\ D^3_{q^{-1}} y_1(x) & D^3_{q^{-1}} y_2(x) & D^3_{q^{-1}} y_3(x) & D^3_{q^{-1}} y_4(x) \end{pmatrix}.$$

The general solution of Eq.(9) is found as

$$z(x) = \Phi(x)C(x),$$

where $C(x) = \begin{pmatrix} C_1(x) & C_2(x) & C_3(x) & C_4(x) \end{pmatrix}^t$ is the solution of the system

$$\Phi(x)D_q C(x) = F(x),$$

and reads

$$C(x) = C + (1-q)x \sum_{i=0}^{\infty} q^i \Phi^{-1}(q^i x)F(q^i x),$$

and the general solution of Eq.(3) reads

$$y(x) = z_1(x) = C_1(x)y_1(x) + C_2(x)y_2(x) + C_3(x)y_3(x) + C_4(x)y_4(x).$$

□

Remark 1. *Note that for the q−difference equation*

$$\left[D_q^4 + a_1 D_q + a_2 D_q + a_3 D_q + a_4\right] y(x) = f(x), \tag{10}$$

the solution of the corresponding system reads

$$z(x) = \Phi(x)C(x),$$

where

$$\Phi(x) = \begin{pmatrix} y_1(x) & y_2(x) & y_3(x) & y_4(x) \\ D_q y_1(x) & D_q y_2(x) & D_q y_3(x) & D_q y_4(x) \\ D_q^2 y_1(x) & D_q^2 y_2(x) & D_q^2 y_3(x) & D_q^2 y_4(x) \\ D_q^3 y_1(x) & D_q^3 y_2(x) & D_q^3 y_3(x) & D_q^3 y_4(x) \end{pmatrix},$$

and $C(x)$ is the solution of the system

$$\Phi(qx)D_q C(x) = F(x), \tag{11}$$

with $F(x) = \begin{pmatrix} 0 & 0 & 0 & f(x) \end{pmatrix}^t$, giving

$$C(x) = C + (1-q)x \sum_{i=0}^{\infty} q^i \Phi^{-1}(q^{i+1}x)F(q^i x).$$

Example 1. *Solve the q-difference equation*

$$D_q^4 y - 5D_q^3 y + 5D_q^2 y + 5D_q y - 6y = f(x), \tag{12}$$

with a) $f(x) = x^2$ *b)* $f(x) = xe^{\alpha x}$, $\alpha \in \mathbb{Z}$.

Solution The characteristic equation of Eq.(12) reads

$$\lambda^4 - 5\lambda^3 + 5\lambda^2 + 5\lambda - 6 = 0. \tag{13}$$

and have solutions $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 2$ and $\lambda_4 = 3$.

This leads to the fundamental system of the corresponding homogenous equation

$$y_1 = e_q^x, \qquad y_2 = e_q^{-x}, \qquad y_3 = e_q^{2x}, \qquad y_4 = e_q^{3x},$$

and the general solution of Eq.(11) is

$$y(x) = c_1(x)y_1 + c_2(x)y_2 + c_3(x)y_3 + c_4(x)y_4,$$

where

$$\Phi(qx)\begin{pmatrix} D_qc_1(x) \\ D_qc_2(x) \\ D_qc_3(x) \\ D_qc_4(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f(x) \end{pmatrix},$$

and

$$\Phi(x) = \begin{pmatrix} e_q^x & e_q^{-x} & e_q^{2x} & e_q^{3x} \\ e_q^x & -e_q^{-x} & 2e_q^{2x} & 3e_q^{3x} \\ e_q^x & e_q^{-x} & 4e_q^{2x} & 9e_q^{3x} \\ e_q^x & -e_q^{-x} & 8e_q^{2x} & 27e_q^{3x} \end{pmatrix}.$$

Then

$$\Phi^{-1}(x) = \begin{pmatrix} \frac{3}{2}e_{q^{-1}}^{-x} & \frac{1}{4}e_{q^{-1}}^{-x} & -e_{q^{-1}}^{-x} & \frac{1}{4}e_{q^{-1}}^{-x} \\ \frac{1}{4}e_{q^{-1}}^{x} & \frac{-11}{24}e_{q^{-1}}^{x} & \frac{1}{4}e_{q^{-1}}^{x} & \frac{-1}{24}e_{q^{-1}}^{x} \\ -e_{q^{-1}}^{-2x} & \frac{1}{3}e_{q^{-1}}^{-2x} & e_{q^{-1}}^{-2x} & \frac{-1}{3}e_{q^{-1}}^{-2x} \\ \frac{1}{4}e_{q^{-1}}^{-3x} & \frac{-1}{8}e_{q^{-1}}^{-3x} & \frac{-1}{4}e_{q^{-1}}^{-3x} & \frac{1}{8}e_{q^{-1}}^{-3x} \end{pmatrix},$$

and

$$\Phi^{-1}(qx) = \begin{pmatrix} \frac{3}{2}e_{q^{-1}}^{-qx} & \frac{1}{4}e_{q^{-1}}^{-qx} & -e_{q^{-1}}^{-qx} & \frac{1}{4}e_{q^{-1}}^{-qx} \\ \frac{1}{4}e_{q^{-1}}^{qx} & \frac{-11}{24}e_{q^{-1}}^{qx} & \frac{1}{4}e_{q^{-1}}^{qx} & \frac{-1}{24}e_{q^{-1}}^{qx} \\ -e_{q^{-1}}^{-2qx} & \frac{1}{3}e_{q^{-1}}^{-2qx} & e_{q^{-1}}^{-2qx} & \frac{-1}{3}e_{q^{-1}}^{-2qx} \\ \frac{1}{4}e_{q^{-1}}^{-3qx} & \frac{-1}{8}e_{q^{-1}}^{-3qx} & \frac{-1}{4}e_{q^{-1}}^{-3qx} & \frac{1}{8}e_{q^{-1}}^{-3qx} \end{pmatrix}.$$

Consequently

$$\begin{pmatrix} D_qc_1(x) \\ D_qc_2(x) \\ D_qc_3(x) \\ D_qc_4(x) \end{pmatrix} = \begin{pmatrix} \frac{3}{2}e_{q^{-1}}^{-qx} & \frac{1}{4}e_{q^{-1}}^{-qx} & -e_{q^{-1}}^{-qx} & \frac{1}{4}e_{q^{-1}}^{-qx} \\ \frac{1}{4}e_{q^{-1}}^{qx} & \frac{-11}{24}e_{q^{-1}}^{qx} & \frac{1}{4}e_{q^{-1}}^{qx} & \frac{-1}{24}e_{q^{-1}}^{qx} \\ -e_{q^{-1}}^{-2qx} & \frac{1}{3}e_{q^{-1}}^{-2qx} & e_{q^{-1}}^{-2qx} & \frac{-1}{3}e_{q^{-1}}^{-2qx} \\ \frac{1}{4}e_{q^{-1}}^{-3qx} & \frac{-1}{8}e_{q^{-1}}^{-3qx} & \frac{-1}{4}e_{q^{-1}}^{-3qx} & \frac{1}{8}e_{q^{-1}}^{-3qx} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ f(x) \end{pmatrix},$$

then

$$\begin{pmatrix} D_qc_1(x) \\ D_qc_2(x) \\ D_qc_3(x) \\ D_qc_4(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{4}f(x)e_{q^{-1}}^{-qx} \\ \frac{-1}{24}f(x)e_{q^{-1}}^{qx} \\ \frac{-1}{3}f(x)e_{q^{-1}}^{-2qx} \\ \frac{1}{8}f(x)e_{q^{-1}}^{-3qx} \end{pmatrix}.$$

This implies

$$\begin{aligned} D_q c_1(x) &= \frac{1}{4} f(x) e_{q^{-1}}^{-qx}, \\ D_q c_2(x) &= \frac{-1}{24} f(x) e_{q^{-1}}^{qx}, \\ D_q c_3(x) &= \frac{-1}{3} f(x) e_{q^{-1}}^{-2qx}, \end{aligned}$$

and

$$D_q c_4(x) = \frac{1}{8} f(x) e_{q^{-1}}^{-3qx}$$

a) If $f(x) = x^2$ we obtain

$$\begin{aligned} D_q c_1(x) &= \frac{1}{4} x^2 e_{q^{-1}}^{-qx}, \\ D_q c_2(x) &= \frac{-1}{24} x^2 e_{q^{-1}}^{qx}, \\ D_q c_3(x) &= \frac{-1}{3} x^2 e_{q^{-1}}^{-2qx}, \end{aligned}$$

and

$$D_q c_4(x) = \frac{1}{8} x^2 e_{q^{-1}}^{-3qx}.$$

Therefore q-Integrations by parts give

$$c_1(x) = c_1 - \frac{1}{4} e_{q^{-1}}^{-x} \left[x^2 + (q+1)x + (q+1) \right] + \frac{1}{4}(q+1),$$

$$c_2(x) = c_2 - \frac{1}{24} e_{q^{-1}}^{x} \left[x^2 - (q+1)x + (q+1) \right] + \frac{1}{24}(q+1),$$

$$c_3(x) = c_3 + e_{q^{-1}}^{-2x} \left[\frac{1}{6} x^2 + \frac{1}{12}(q+1)x + \frac{1}{24}(q+1) \right] - \frac{1}{24}(q+1),$$

and

$$c_4(x) = c_4 - e_{q^{-1}}^{-3x} \left[\frac{1}{24} x^2 + \frac{1}{72}(q+1)x + \frac{1}{216}(q+1) \right] + \frac{1}{216}(q+1).$$

Consequently the general solution of Eq.(12) where $f(x) = x^2$ is

$$\begin{aligned} y(x) = \; & c_1 e_q^x - \frac{1}{4} \left[x^2 + (q+1)x + (q+1 \right] + \frac{1}{4}(q+1) e_q^x \\ & + c_2 e_q^{-x} - \frac{1}{24} \left[x^2 - (q+1)x + (q+1) \right] + \frac{1}{24}(q+1) e_q^{-x} \\ & + c_3 e_q^{2x} + \left[\frac{1}{6} x^2 + \frac{1}{12}(q+1)x + \frac{1}{24}(q+1) \right] - \frac{1}{24}(q+1) e_q^{2x} \\ & + c_4 e_q^{3x} - \left[\frac{1}{24} x^2 + \frac{1}{72}(q+1)x + \frac{1}{216}(q+1) \right] + \frac{1}{216}(q+1) e_q^{3x}. \end{aligned}$$

b) If $f(x) = xe_q^{\alpha x}$ we obtain

$$\begin{aligned} D_q c_1(x) &= \frac{1}{4} x e_q^{\alpha x} e_{q^{-1}}^{-qx}, \\ D_q c_2(x) &= \frac{-1}{24} x e_q^{\alpha x} e_{q^{-1}}^{qx}, \\ D_q c_3(x) &= \frac{-1}{3} x e_q^{\alpha x} e_{q^{-1}}^{-2qx}, \end{aligned}$$

and

$$D_q c_4(x) = \frac{1}{8} x e_q^{\alpha x} e_{q^{-1}}^{-3qx}.$$

Therefore q-Integrations by parts give

$$c_1(x) = c_1 + \frac{1}{4(\alpha-1)(\alpha q-1)} \left[((\alpha-1)x-1) e_q^{\alpha x} e_{q^{-1}}^{-x} + 1 \right], \quad \alpha \neq 1,$$

$$c_2(x) = c_2 - \frac{1}{24(\alpha+1)(\alpha q+1)} \left[((\alpha+1)x-1) e_q^{\alpha x} e_{q^{-1}}^{x} + 1 \right], \quad \alpha \neq -1,$$

$$c_3(x) = c_3 - \frac{1}{3(\alpha-2)(\alpha q-2)} \left[((\alpha-2)x-1) e_q^{\alpha x} e_{q^{-1}}^{-2x} + 1 \right], \quad \alpha \neq 2,$$

and

$$c_4(x) = c_4 + \frac{1}{8(\alpha-3)(\alpha q-3)} \left[((\alpha-3)x-1) e_q^{\alpha x} e_{q^{-1}}^{-3x} + 1 \right], \quad \alpha \neq 3.$$

Consequently the general solution of Eq.(12)where $f(x) = xe_q^{\alpha x}$ is

$$\begin{aligned} y(x) &= c_1 e_q^x + \frac{1}{4(\alpha-1)(\alpha q-1)} \left[((\alpha-1)x-1) e_q^{\alpha x} + e_q^x \right] \\ &+ c_2 e_q^{-x} - \frac{1}{24(\alpha+1)(\alpha q+1)} \left[((\alpha+1)x-1) e_q^{\alpha x} + e_q^{-x} \right] \\ &+ c_3 e_q^{2x} - \frac{1}{3(\alpha-2)(\alpha q-2)} \left[((\alpha-2)x-1) e_q^{\alpha x} + e_q^{2x} \right] \\ &+ c_4 e_q^{3x} + \frac{1}{8(\alpha-3)(\alpha q-3)} \left[((\alpha-3)x-1) e_q^{\alpha x} + e_q^{3x} \right]. \end{aligned}$$

The general case of Eq.(2) of the form

$$\left[D_{q^{-1}}^4 + a_1(x) D_{q^{-1}}^3 + a_2(x) D_{q^{-1}}^2 + a_3(x) D_{q^{-1}} + a_4(x) \right] y(x) = f(x), \tag{14}$$

where $a_1(x)$, $a_2(x)$, $a_3(x)$ and $a_4(x)$ are continuous functions.
The corresponding homogenous equation of Eq.(14) is given by

$$\left[D_{q^{-1}}^4 + a_1(x) D_{q^{-1}}^3 + a_2(x) D_{q^{-1}}^2 + a_3(x) D_{q^{-1}} + a_4(x) \right] y(x) = 0. \tag{15}$$

Theorem 3. *The solution of equation Eq.(14) is given by*

$$y(x) = C_1(x) y_1(x) + C_2(x) y_2(x) + C_3(x) y_3(x) + C_4(x) y_4(x),$$

where $C(x) = \begin{pmatrix} C_1(x) & C_2(x) & C_3(x) & C_4(x) \end{pmatrix}^t$ *is the solution of the system*

$$\Phi(qx) D_q C(x) = F(x),$$

and reads

$$C(x) = C + (1-q)x \sum_{i=0}^{\infty} q^i \Phi^{-1}(q^i x) F(q^i x),$$

$F(x) = \begin{pmatrix} 0 & 0 & 0 & f(x) \end{pmatrix}^t$ *and* $\Phi(x)$ *is a fundamental matrix of solution of the homogenous equation Eq.(15) and* $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$ *a fundamental system of solution of Eq.(15).*

Proof. To find the general solution of (14) supposing

$$\begin{aligned} z_1(x) &= y(x), \\ z_2(x) &= D_{q^{-1}} y(x), \\ z_3(x) &= D^2_{q^{-1}} y(x), \end{aligned}$$

and

$$z_4(x) = D^3_{q^{-1}} y(x).$$

We obtain the system

$$\begin{aligned} D_{q^{-1}} z_1(x) &= z_2(x), \\ D_{q^{-1}} z_2(x) &= z_3(x), \\ D_{q^{-1}} z_3(x) &= z_4(x), \end{aligned}$$

and

$$D_{q^{-1}} z_4(x) = -(a_1(x)z_4(x) + a_2(x)z_3(x) + a_3(x)z_2(x) + a_4(x)z_1(x)) + f(x). \quad (16)$$

In matrices terms we have

$$D_q z(x) = A(x) z(qx) + F(x), \quad (17)$$

where $z(x) = \begin{pmatrix} z_1(x) & z_2(x) & z_3(x) & z_4(x) \end{pmatrix}^t$,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4(qx) & -a_3(qx) & -a_2(qx) & -a_1(qx) \end{pmatrix},$$

and $F(x) = \begin{pmatrix} 0 & 0 & 0 & f(qx) \end{pmatrix}^t$. So it follows from Eq.(16) that the existence of a unique solution of Eq.(14) under the initial constraints

$$y(x_0) = y_0, \quad D_{q^{-1}} y(x_0) = y_1, \quad D^2_{q^{-1}} y(x_0) = y_2, \quad D^3_{q^{-1}} y(x_0) = y_3,$$

is equivalent to the existence of a unique solution of Eq.(17) under the constraints

$$\begin{pmatrix} z_1(x_0) & z_2(x_0) & z_3(x_0) & z_4(x_0) \end{pmatrix}^t = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \end{pmatrix}^t.$$

As a consequence the existence of a fundamental system of solution $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$ of Eq.(14) is equivalent to the existence of a fundamental system

$$\begin{pmatrix} y_1(x) & D_{q^{-1}} y_1(x) & D^2_{q^{-1}} y_1(x) & D^3_{q^{-1}} y_1(x) \end{pmatrix}^t, ..., \begin{pmatrix} y_4(x) & D_{q^{-1}} y_4(x) & D^2_{q^{-1}} y_4(x) & D^3_{q^{-1}} y_4(x) \end{pmatrix}^t,$$

of the homogeneous part of Eq.(17)

$$D_q z(x) = A(x) z(qx),$$

with the fundamental matrix

$$\Phi(x)=\begin{pmatrix} y_1(x) & y_2(x) & y_3(x) & y_4(x) \\ D_{q^{-1}}y_1(x) & D_{q^{-1}}y_2(x) & D_{q^{-1}}y_3(x) & D_{q^{-1}}y_4(x) \\ D^2_{q^{-1}}y_1(x) & D^2_{q^{-1}}y_2(x) & D^2_{q^{-1}}y_3(x) & D^2_{q^{-1}}y_4(x) \\ D^3_{q^{-1}}y_1(x) & D^3_{q^{-1}}y_2(x) & D^3_{q^{-1}}y_3(x) & D^3_{q^{-1}}y_4(x) \end{pmatrix}. \tag{18}$$

Indeed if

$$\alpha_1 y_1(x)+\alpha_2 y_2(x)+\alpha_3 y_3(x)+\alpha_4 y_4(x)=0,$$

then

$$\begin{aligned} \alpha_1 D_{q^{-1}}y_1(x)+\alpha_2 D_{q^{-1}}y_2(x)+\alpha_3 D_{q^{-1}}y_3(x)+\alpha_4 D_{q^{-1}}y_4(x) &= 0, \\ D^2_{q^{-1}}\alpha_1 y_1(x)+\alpha_2 D^2_{q^{-1}}y_2(x)+\alpha_3 D^2_{q^{-1}}y_3(x)+\alpha_{4D^2_{q^{-1}}}y_4(x) &= 0, \end{aligned}$$

and

$$\alpha_1 D^3_{q^{-1}}y_1(x)+\alpha_2 D^3_{q^{-1}}y_2(x)+\alpha_3 D^3_{q^{-1}}y_3(x)+\alpha_4 D^3_{q^{-1}}y_4(x)=0,$$

or

$$\Phi(x)\alpha=0.$$

where $\Phi(x)$ is in Eq.(18) and $\alpha=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{pmatrix}^t$. Hence the system y_i; $i=1,\,2,\,3,\,4$ is linear independent if the matrix $\Phi(x)$ in Eq.(18) is nonsingular. The matrix $\Phi(x)$ can naturally be called the $q-$Wronskian or $q-$Casoratian of the equation Eq.(15) corresponding to the continuous or discrete cases. Consider now the equation of deriving the solution of the non homogenous Eq.(4). If $y_1(x), y_2(x), y_3(x), y_4(x)$ is a fundamental system of solution of the homogenous equation Eq.(15) corresponding to the fundamental matrix $\Phi(x)$, then according to the general theory of $q-$difference systems, the general solution of Eq.(17) is found as

$$z(x)=\Phi(x)C(x),$$

where $C(x)=\begin{pmatrix} C_1(x) & C_2(x) & C_3(x) & C_4(x)\end{pmatrix}^t$, is the solution of the system

$$\Phi(qx)D_qC(x)=F(x),$$

and reads

$$C(x)=C+(1-q)x\sum_{i=0}^{\infty}q^i\Phi^{-1}(q^ix)F(q^ix),$$

and the general solution of Eq.(14) reads

$$y(x)=z_1(x)=C_1(x)y_1(x)+C_2(x)y_2(x)+C_3(x)y_3(x)+C_4(x)y_4(x).$$

□

Remark 2. *Note that for the q-difference equation*

$$\left[D_q^4+a_1(x)D_q+a_2(x)D_q+a_3(x)D_q+a_4(x)\right]y(x)=f(x).$$

the solution of the corresponding system reads

$$z(x)=\Phi(x)C(x),$$

where

$$\Phi(x)=\begin{pmatrix} y_1(x) & y_2(x) & y_3(x) & y_4(x)\\ D_qy_1(x) & D_qy_2(x) & D_qy_3(x) & D_qy_4(x)\\ D_q^2y_1(x) & D_q^2y_2(x) & D_q^2y_3(x) & D_q^2y_4(x)\\ D_q^3y_1(x) & D_q^3y_2(x) & D_q^3y_3(x) & D_q^3y_4(x)\end{pmatrix},$$

and $C(x)$ is the solution of the system

$$\Phi(qx)D_qC(x)=F(x),$$

with $F(x)=\begin{pmatrix} 0 & 0 & 0 & f(x)\end{pmatrix}^t$, giving

$$C(x)=C+(1-q)x\sum_{i=0}^{\infty}q^i\Phi^{-1}(q^{i+1}x)F(q^ix).$$

3. Studying of Eq.(1)

We will need to the following lemma to prove Theorem (5) below.

Lemma 4. *([2],[3])*

$$\int_0^t\int_0^v f(s,u(s))d_qsd_qv=\int_0^t\int_{qs}^t f(s,u(s))d_qvd_qs.$$

Theorem 5. *Let* $u\in\mathbb{C}\,[0,1]$ *then the boundary value problems Eq.(1) has a unique solution is given by:*

$$u(t)=\int_0^1 G(t,s;q)\ f(s,u(s))d_qs, \tag{19}$$

where

$$G(t,s;q)=\frac{1}{1+q}\begin{cases} qs(t-1)\left[\frac{q^5s^2(1+t+t^2)}{1+q+q^2}+t^2-q^2st(t+1)\right], & 0\le s\le t\le 1,\\ t^3\left[\frac{-1+q^6s^3}{1+q+q^2}-q^3s^2+qs\right], & 0\le t\le s\le 1.\end{cases}$$

Proof. Integrating Eq.(1) gives

$$D_q^3u(t)=\int_0^t f(s,u(s))d_qs+a_1$$

$$\begin{aligned} D_q^2u(t) &= \int_0^t\int_0^v f(s,u(s))d_qsd_qv=\int_0^t\int_{qs}^t f(s,u(s))d_qvd_qs\\ &= \int_0^t (t-qs)f(s,u(s))d_qs+a_1t+a_2.\end{aligned}$$

Similarly we have

$$\begin{aligned}
D_q u(t) &= \int_0^t \left(\int_0^v (v-qs) f(s,u(s)) d_q s \right) d_q v + a_1 t^2 + a_2 t + a_3 \\
&= \int_0^t \left(\int_{qs}^t (v-qs) f(s,u(s)) d_q v \right) d_q s + a_1 t^2 + a_2 t + a_3 \\
&= \int_0^t \left[\left(\frac{v^2}{1+q} - qvs \right) f(s,u(s)) \right]_{qs}^t d_q s + a_1 t^2 + a_2 t + a_3 \\
&= \int_0^t \left(\frac{t^2}{1+q} - qts - \frac{q^2 s^2}{1+q} + q^2 s^2 \right) f(s,u(s)) d_q s + a_1 t^2 + a_2 t + a_3 \\
&= \int_0^t \left(\frac{t^2 - q^2 s^2 + q^2 s^2 + q^3 s^2}{1+q} - qts \right) f(s,u(s)) d_q s + a_1 t^2 + a_2 t + a_3 \\
&= \int_0^t \left(\frac{t^2 + q^3 s^2}{1+q} - qts \right) f(s,u(s)) d_q s + a_1 t^2 + a_2 t + a_3,
\end{aligned}$$

then

$$D_q u(t) = \int_0^t \left(\frac{t^2 + q^3 s^2}{1+q} - qts \right) f(s,u(s)) d_q s + a_1 t^2 + a_2 t + a_3.$$

Therefore

$$\begin{aligned}
u(t) &= \int_0^t \left[\int_0^v \left(\frac{v^2 + q^3 s^2}{1+q} - qvs \right) f(s,u(s)) d_q s \right] d_q v + a_1 t^3 + a_2 t^2 + a_3 t + a_4 \\
&= \int_0^t \left[\int_{qs}^t \left(\frac{v^2 + q^3 s^2}{1+q} - qvs \right) f(s,u(s)) d_q v \right] d_q s + a_1 t^3 + a_2 t^2 + a_3 t + a_4 \\
&= \int_0^t \left[\left(\frac{v^3}{(1+q)(1+q+q^2)} + \frac{q^3 s^2 v}{1+q} - \frac{qsv^2}{1+q} \right) f(s,u(s)) \right]_{qs}^t d_q s + a_1 t^3 + a_2 t^2 + a_3 t + a_4 \\
&= \frac{1}{1+q} \int_0^t \left[\frac{t^3}{1+q+q^2} + q^3 s^2 t - qst^2 - \frac{q^3 s^3}{1+q+q^2} - q^4 s^3 + q^3 s^3 \right] f(s,u(s)) d_q s \\
&\quad + a_1 t^3 + a_2 t^2 + a_3 t + a_4 \\
&= \frac{1}{1+q} \int_0^t \left[\frac{t^3 - q^3 s^3 - q^4 s^3 - q^5 s^3 - q^6 s^3 + q^3 s^3 + q^4 s^3 + q^5 s^3}{1+q+q^2} - qt^2 s + q^3 ts^2 \right] f(s,u(s)) d_q s \\
&\quad + a_1 t^3 + a_2 t^2 + a_3 t + a_4.
\end{aligned}$$

Thus

$$u(t)=\frac{1}{1+q}\int_0^t\left[\frac{t^3-q^6s^3}{1+q+q^2}-qt^2s+q^3ts^2\right]f(s,u(s))d_qs+a_1t^3+a_2t^2+a_3t+a_4, \tag{20}$$

where a_1, a_2, a_3 *and* a_4 are arbitrary constants. Now using the boundary conditions of Eq.(1) we have $a_2=a_3=a_4=0$ and

$$a_1=\frac{-1}{1+q}\int_0^1\left[\frac{1-q^6s^3}{1+q+q^2}-qs+q^3s^2\right]f(s,u(s))d_qs.$$

then

$$\begin{aligned} u(t) &= \frac{1}{1+q}\int_0^t\left[\frac{t^3-q^6s^3}{1+q+q^2}-qt^2s+q^3ts^2\right]f(s,u(s))d_qs \\ &\quad -\frac{t^3}{1+q}\int_0^1\left[\frac{1-q^6s^3}{1+q+q^2}-qs+q^3s^2\right]f(s,u(s))d_qs, \end{aligned}$$

or

$$\begin{aligned} u(t) &= \frac{1}{[2]_q}\int_0^t\left[\frac{t^3-q^6s^3}{[3]_q}-qt^2s+q^3ts^2\right]f(s,u(s))d_qs \\ &\quad -\frac{t^3}{[2]_q}\int_0^1\left[\frac{1-q^6s^3}{[3]_q}-qs+q^3s^2\right]f(s,u(s))d_qs \\ &= \int_0^1 G(t,s;q)\ f(s,u(s))d_qs.;[2]_q=1+q,[3]_q=1+q+q^2. \end{aligned} \tag{21}$$

□

Example 2. *Find the general solution of the following Boundry Value Problem*

$$\begin{aligned} D_q^4u(t) &= t^2, \quad 0\le t\le 1, \\ u(0) &= 0, \quad D_qu(0)=0, \quad D_q^2u(0)=0, \quad u(1)=1. \end{aligned} \tag{22}$$

Solution Let $f(t,u(t))=t^2\to f(s,u(s))=s^2$, it follows from Eq.(21) that

$$\begin{aligned} u(t) &= \frac{1}{[2]_q}\int_0^t\left[\frac{t^3-q^6s^3}{[3]_q}-qt^2s+q^3ts^2\right]s^2d_qs-\frac{t^3}{[2]_q}\int_0^1\left[\frac{1-q^6s^3}{[3]_q}-qs+q^3s^2\right]s^2d_qs \\ &= \frac{1}{[2]_q}\int_0^t\left[\frac{t^3s^2}{[3]_q}-\frac{q^6s^5}{[3]_q}-qt^2s^3+q^3ts^4\right]d_qs-\frac{t^3}{[2]_q}\int_0^1\left[\frac{s^2}{[3]_q}-\frac{q^6s^5}{[3]_q}-qs^3+q^3s^4\right]d_qs \\ &= \frac{1}{[2]_q}\left[\frac{t^6}{[3]_q[3]_q}-\frac{q^6t^6}{[3]_q[6]_q}-\frac{q}{[4]_q}t^6+\frac{q^3}{[5]_q}t^6\right]-\frac{t^3}{[2]_q}\left[\frac{1}{[3]_q[3]_q}-\frac{q^6}{[3]_q[6]_q}-\frac{q}{[4]_q}+\frac{q^3}{[5]_q}\right] \\ &= \frac{1}{[2]_q}\left[\frac{1}{[3]_q[3]_q}-\frac{q^6}{[3]_q[6]_q}-\frac{q}{[4]_q}+\frac{q^3}{[5]_q}\right]\left(t^6-t^3\right), \end{aligned}$$

then

$$u(t)=\frac{1}{[2]_q}\left[\frac{1}{[3]_q[3]_q}-\frac{q^6}{[3]_q[6]_q}-\frac{q}{[4]_q}+\frac{q^3}{[5]_q}\right]\left(t^6-t^3\right). \quad (23)$$

Clearly when $q\longrightarrow 1$, the solution of Eq.(23) will be as follows

$$u(t)=\frac{1}{360}\left(t^6-t^3\right),$$

which represents the solution of the equation

$$\begin{aligned} D^4u(t) &= t^2; \qquad 0\le t\le 1,\\ u(0) &= 0, \quad Du(0)=0, \quad D^2u(0)=0, \quad u(1)=1. \end{aligned}$$

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