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RELATIVE (p,q)-φ ORDER ORIENTED SOME GROWTH PROPERTIES OF P-ADIC ENTIRE FUNCTIONS

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Abstract. Let $\mathbb{K}$ be a complete ultrametric algebraically closed field and $\mathcal{A}(\mathbb{K})$ be the $\mathbb{K}$-algebra of entire function on $\mathbb{K}$. For any p adic entire functions $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. In this paper we introduce the concept of relative (p,q)-φ order where p, q are any two positive integers and $\varphi(r) : [0,+\infty) \to (0,+\infty)$ is a non-decreasing unbounded function of r. Then we study some growth properties of p-adic entire functions on the basis of their relative (p,q)-φ order.

1. Definitions

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0, complete with respect to a p-adic absolute value $|\cdot|$ (example $\mathbb{C}_p$). For any $\alpha \in \mathbb{K}$ and $R \in]0,+\infty[$, the closed disk $\{x \in \mathbb{K} : |x-\alpha| \leq R\}$ and the open disk $\{x \in \mathbb{K} : |x-\alpha| < R\}$ are denoted by $d(\alpha, R)$ and $d(\alpha, R^-)$ respectively. Also $C(\alpha, r)$ denotes the circle $\{x \in \mathbb{K} : |x-a| = r\}$. Moreover $\mathcal{A}(\mathbb{K})$ represents the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$ i.e. the set of power series with an infinite radius of convergence. For the most comprehensive study of analytic functions inside a disk or in the whole field $\mathbb{K}$, we refer the reader to the books [10, 11, 15, 17]. During the last several years the ideas of p-adic analysis have been studied from different aspects and many important results were gained (see [8] to [13], [12, 14]).

Let $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. Moreover, if f is not a constant, the $|f|(r)$ is strictly increasing function of r and tends to $+\infty$ with r therefore there exists its inverse function $\widehat{|f|} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim\limits_{s\to\infty} \widehat{|f|}(s) = \infty$.

Further $f \in \mathcal{A}(\mathbb{K})$ and $g \in \mathcal{A}(\mathbb{K})$ are said to be asymptotically equivalent if there exists l , $0 < l < \infty$ such that $\frac{|f|(r)}{|g|(r)} \to l$ as $r \to \infty$ and in this case we write $f \sim g$. If $f \sim g$ then clearly $g \sim f$.

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For $x \in [0,\infty)$ and $k \in \mathbb{N}$, we define $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ and $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. Throughout the paper, log denotes the Neperian logarithm. Further we assume that throughout the present paper p, q and m always denote positive integers. Taking this into account the (p,q)-th order and (p,q)-th lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ are defined as follows:

Definition 1. [3] *Let $f \in \mathcal{A}(\mathbb{K})$. Then the (p,q)-th order and (p,q)-th lower order of f are respectively defined as:*

$$\rho^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} |f|(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} |f|(r)}{\log^{[q]} r}.$$

Definition 1 avoids the restriction $p \geq q$ of the original definition of (p,q)-th order (respectively (p,q)-th lower order) of entire functions introduced by Juneja et al. [16] in complex context.

When $q = 1$, we get the definitions of generalized order and generalized lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ which symbolize as $\rho^{(p)}(f)$ and $\lambda^{(p)}(f)$ respectively. If $p = 2$ and $q = 1$ then we write $\rho^{(2,1)}(f) = \rho(f)$ and $\lambda^{(2,1)}(f) = \lambda(f)$ where $\rho(f)$ and $\lambda(f)$ are respectively known as order and lower order of $f \in \mathcal{A}(\mathbb{K})$ introduced by Boussaf et al. [8].

In this connection we just introduce the following definition:

Definition 2. *An entire function $f \in \mathcal{A}(\mathbb{K})$ is said to have index-pair (p,q) if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ otherwise. Moreover if $0 < \rho^{(p,q)}(f) < \infty$, then*

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$,

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

An entire function $f \in \mathcal{A}(\mathbb{K})$ of index-pair (p,q) is said to be of regular (p,q)-th growth if its (p,q)-th order coincides with its (p,q)-th lower order, otherwise f is said to be of irregular (p,q)-th growth.

The concepts of (p,q)-φ order and (p,q)-φ lower order of entire functions in complex context were introduced by Shen et al. [18] where $p \geq q \geq 1$ and $\varphi : [0,+\infty) \to (0,+\infty)$ is a non-decreasing unbounded function. For details about (p,q)-φ order and (p,q)-φ lower order, one may see [18]. Considering the ideas developed by Shen et al. [18], one can define the (p,q)-φ order and (p,q)-φ lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ respectively in the following way:

Definition 3. *Let $f \in \mathcal{A}(\mathbb{K})$. Also let $\varphi(r) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function of r. The (p,q)-φ order $\rho^{(p,q)}(f,\varphi)$ and (p,q)-φ lower order $\lambda^{(p,q)}(f,\varphi)$ of f are respectively defined as:*

$$\rho^{(p,q)}(f,\varphi) = \limsup_{r\to\infty} \frac{\log^{[p]} |f|(r)}{\log^{[q]} \varphi(r)} \text{ and } \lambda^{(p,q)}(f,\varphi) = \liminf_{r\to\infty} \frac{\log^{[p]} |f|(r)}{\log^{[q]} \varphi(r)}.$$

If $\varphi(r) = r$, then Definition 1 is a special case of Definition 3.

Extending the notion of index-pair (p,q), one may also introduce the definition of index-pair (p,q)-φ in the following manner:

Definition 4. *An entire function $f \in \mathcal{A}(\mathbb{K})$ is said to have index-pair (p,q)-φ if $b < \rho^{(p,q)}(f,\varphi) < \infty$ and $\rho^{(p-1,q-1)}(f,\varphi)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ otherwise. Moreover if $0 < \rho^{(p,q)}(f,\varphi) < \infty$, then*

$$\begin{cases} \rho^{(p-n,q)}(f,\varphi) = \infty & \text{for} \quad n < p, \\ \rho^{(p,q-n)}(f,\varphi) = 0 & \text{for} \quad n < q, \\ \rho^{(p+n,q+n)}(f,\varphi) = 1 & \text{for} \quad n = 1,2,\cdots \end{cases}.$$

Similarly for $0 < \lambda^{(p,q)}(f,\varphi) < \infty$,

$$\begin{cases} \lambda^{(p-n,q)}(f,\varphi) = \infty & \text{for} \quad n < p, \\ \lambda^{(p,q-n)}(f,\varphi) = 0 & \text{for} \quad n < q, \\ \lambda^{(p+n,q+n)}(f,\varphi) = 1 & \text{for} \quad n = 1,2,\cdots \end{cases}.$$

An entire function $f \in \mathcal{A}(\mathbb{K})$ of index-pair (p,q)-φ is said to be of regular (p,q)-φ growth if its (p,q)-φ order coincides with its (p,q)-φ lower order, otherwise f is said to be of irregular (p,q)-φ growth.

However the notion of relative order was first introduced by Bernal [1]. In order to make some progress in the study of p-adic analysis, recently Biswas [2] introduced the definition of relative order and relative lower order of entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ in the following way:

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log \widehat{|g|}(|f|(r))}{\log r} \text{ and } \lambda_g(f) = \liminf_{r\to\infty} \frac{\log \widehat{|g|}(|f|(r))}{\log r}.$$

Further the function $f \in \mathcal{A}(\mathbb{K})$, for which relative order and relative lower order with respect to another function $g \in \mathcal{A}(\mathbb{K})$ are the same is called a function of regular relative growth with respect to g. Otherwise, f is said to be irregular relative growth with respect to g.

In the case of relative order, it therefore seems reasonable to define suitably the (p,q)-th relative order of entire function belonging to $\mathcal{A}(\mathbb{K})$. With this in view one may introduce the definition of (p,q)-th relative order $\rho_g^{(p,q)}(f)$ and (p,q)-th relative lower order $\lambda_g^{(p,q)}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$, in the light of index-pair which are as follows:

Definition 5. [3] *Let f, $g \in \mathcal{A}(\mathbb{K})$. Also let the index-pairs of f and g are (m,q) and (m,p), respectively. Then the (p,q)-th relative order $\rho_g^{(p,q)}(f)$ and (p,q)-th relative lower order $\lambda_g^{(p,q)}(f)$ of f with respect to g are defined as*

$$\rho_g^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[q]} r} = \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|g|}(r)}{\log^{[q]} \widehat{|f|}(r)}$$

and

$$\lambda_g^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[q]} r} = \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|g|}(r)}{\log^{[q]} \widehat{|f|}(r)}.$$

Now in order to make some progress in the study of relative order, one may introduce the definitions of relative (p,q)-φ order and relative (p,q)-φ lower order of entire functions belonging to $\mathcal{A}(\mathbb{K})$ and to investigate some of its properties, which we attempt in this paper. With this in view one may introduce the definition of relative (p,q)-φ order $\rho_g^{(p,q)}(f,\varphi)$ and relative (p,q)-φ lower order $\lambda_g^{(p,q)}(f,\varphi)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ which are as follows:

Definition 6. *Let f, $g \in \mathcal{A}(\mathbb{K})$ and $\varphi(r) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function of r. Also let the index-pairs of f and g are (m,q)-φ and (m,p), respectively. The relative (p,q)-φ order denoted as $\rho_g^{(p,q)}(f,\varphi)$ and relative (p,q)-φ lower order denoted by $\lambda_g^{(p,q)}(f,\varphi)$ of f with respect to g are defined as*

$$\rho_g^{(p,q)}(f,\varphi) = \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[q]} \varphi(r)} = \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|g|}(r)}{\log^{[q]} \varphi\left(\widehat{|f|}(r)\right)}$$

and

$$\lambda_g^{(p,q)}(f,\varphi) = \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[q]} \varphi(r)} = \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|g|}(r)}{\log^{[q]} \varphi\left(\widehat{|f|}(r)\right)}.$$

If $\varphi(r) = r$, then Definition 5 is a special case of Definition 6. Further if relative (p,q)-φ order and the relative (p,q)-φ lower order of f with respect to g are the same, then f is called a function of regular relative (p,q)-φ growth with respect to g. Otherwise, f is said to be irregular relative (p,q)-φ growth with respect to g.

The main aim of this paper is to establish some results related to the growth rates of p-adic entire functions on the basis of relative (p,q)-φ order and relative (p,q)-φ lower order.

2. Results

First of all, we recall one related known property which will be needed in order to prove our results, as we see in the following lemma.

Lemma 1. [6] *Let $f \in \mathcal{A}(\mathbb{K})$ and $\alpha > 1$, $0 < \beta < \alpha$, then for all large r,*

$$\beta|f|(r) \le |f|(\alpha r).$$

Theorem 1. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(m,q)}(f,\varphi) \le \rho_h^{(m,q)}(f,\varphi) < \infty$ and $0 < \lambda_h^{(m,p)}(g) \le \rho_h^{(m,p)}(g) < \infty$. Then*

$$\frac{\lambda_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)} \le \lambda_g^{(p,q)}(f,\varphi) \le \min\left\{\frac{\lambda_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}\right\}$$
$$\le \max\left\{\frac{\lambda_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}\right\} \le \rho_g^{(p,q)}(f,\varphi) \le \frac{\rho_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}.$$

Proof. From the definitions of $\rho_g^{(p,q)}(f,\varphi)$ and $\lambda_g^{(p,q)}(f,\varphi)$ we get that

$$\log \rho_g^{(p,q)}(f) = \limsup_{r\to\infty} \left(\log^{[p+1]} \widehat{|g|}(r) - \log^{[q+1]} \varphi\left(\widehat{|f|}(r)\right)\right), \tag{1}$$

$$\log \lambda_g^{(p,q)}(f) = \liminf_{r\to\infty} \left(\log^{[p+1]} \widehat{|g|}(r) - \log^{[q+1]} \varphi\left(\widehat{|f|}(r)\right)\right). \tag{2}$$

Now from the definitions of $\rho_h^{(m,q)}(f,\varphi)$ and $\lambda_h^{(m,q)}(f,\varphi)$, it follows that

$$\log\rho_h^{(m,q)}(f) = \limsup_{r\to\infty}\left(\log^{[m+1]}\widehat{|h|}(r)-\log^{[q+1]}\varphi\left(\widehat{|f|}(r)\right)\right), \tag{3}$$

$$\log\lambda_h^{(m,q)}(f) = \liminf_{r\to\infty}\left(\log^{[m+1]}\widehat{|h|}(r)-\log^{[q+1]}\varphi\left(\widehat{|f|}(r)\right)\right). \tag{4}$$

Similarly, from the definitions of $\rho_h^{(m,p)}(g)$ and $\lambda_h^{(m,p)}(g)$, we obtain that

$$\log\rho_h^{(m,p)}(g) = \limsup_{r\to\infty}\left(\log^{[m+1]}\widehat{|h|}(r)-\log^{[p+1]}\widehat{|g|}(r)\right), \tag{5}$$

$$\log\lambda_h^{(m,p)}(g) = \liminf_{r\to\infty}\left(\log^{[m+1]}\widehat{|h|}(r)-\log^{[p+1]}\widehat{|g|}(r)\right). \tag{6}$$

Therefore from (2), (4) and (5), we get that

$$\log\lambda_g^{(p,q)}(f,\varphi)=\liminf_{r\to\infty}\Big[\log^{[m+1]}\widehat{|h|}(r)-\log^{[q+1]}\varphi\left(\widehat{|f|}(r)\right) - \left(\log^{[m+1]}\widehat{|h|}(r)-\log^{[p+1]}\widehat{|g|}(r)\right)\Big]$$

$$i.e.,\ \log\lambda_g^{(p,q)}(f,\varphi)\geq\Big[\liminf_{r\to\infty}\left(\log^{[m+1]}\widehat{|h|}(r)-\log^{[q+1]}\varphi\left(\widehat{|f|}(r)\right)\right) - \limsup_{r\to\infty}\left(\log^{[m+1]}\widehat{|h|}(r)-\log^{[p+1]}\widehat{|g|}(r)\right)\Big]$$

$$i.e.,\ \log\lambda_g^{(p,q)}(f,\varphi)\geq\left(\log\lambda_h^{(m,q)}(f,\varphi)-\log\rho_h^{(m,p)}(g)\right). \tag{7}$$

Similarly, from (1), (3) and (6), it follows that

$$\log\rho_g^{(p,q)}(f,\varphi)=\limsup_{r\to\infty}\Big[\log^{[m+1]}\widehat{|h|}(r)-\log^{[q+1]}\varphi\left(\widehat{|f|}(r)\right) - \left(\log^{[m+1]}\widehat{|h|}(r)-\log^{[p+1]}\widehat{|g|}(r)\right)\Big]$$

$$i.e.,\ \log\rho_g^{(p,q)}(f,\varphi)\leq\Big[\limsup_{r\to\infty}\left(\log^{[m+1]}\widehat{|h|}(r)-\log^{[q+1]}\varphi\left(\widehat{|f|}(r)\right)\right) - \underline{\lim}_{r\to\infty}\left(\log^{[m+1]}\widehat{|h|}(r)-\log^{[p+1]}\widehat{|g|}(r)\right)\Big]$$

$$i.e.,\ \log\rho_g^{(p,q)}(f,\varphi)\leq\left(\log\rho_h^{(m,q)}(f,\varphi)-\log\lambda_h^{(m,p)}(g)\right). \tag{8}$$

Again, in view of (2) we obtain that

$$\log\lambda_g^{(p,q)}(f,\varphi)=\liminf_{r\to\infty}\Big[\log^{[m+1]}\widehat{|h|}(r)-\log^{[q+1]}\varphi\left(\widehat{|f|}(r)\right) - \left(\log^{[m+1]}\widehat{|h|}(r)-\log^{[p+1]}\widehat{|g|}(r)\right)\Big].$$

By taking $A=\left(\log^{[m+1]}\widehat{|h|}(r)-\log^{[q+1]}\varphi\left(\widehat{|f|}(r)\right)\right)$ and $B=\left(\log^{[m+1]}\widehat{|h|}(r)-\log^{[p+1]}\widehat{|g|}(r)\right)$, we get from above that

$$\log\lambda_g^{(p,q)}(f,\varphi)\leq\min\left(\liminf_{r\to\infty}A+\limsup_{r\to\infty}-B,\limsup_{r\to\infty}A+\liminf_{r\to\infty}-B\right)$$

$$i.e., \ \log \lambda_g^{(p,q)}(f,\varphi) \leq \min\left(\liminf_{r\to\infty} A - \liminf_{r\to\infty} B, \limsup_{r\to\infty} A - \limsup_{r\to\infty} B\right).$$

Therefore in view of (3), (4), (5) and (6) we get from above that

$$\log \lambda_g^{(p,q)}(f,\varphi) \leq$$

$$\min\left\{\log \lambda_h^{(m,q)}(f,\varphi) - \log \lambda_h^{(m,p)}(g), \log \rho_h^{(m,q)}(f,\varphi) - \log \rho_h^{(m,p)}(g)\right\}. \quad (9)$$

Further from (1) it follows that

$$\log \rho_g^{(p,q)}(f,\varphi) = \limsup_{r\to\infty}\left[\log^{[m+1]}\widehat{|h|}(r) - \log^{[q+1]}\varphi\left(\widehat{|f|}(r)\right) - \left(\log^{[m+1]}\widehat{|h|}(r) - \log^{[p+1]}\widehat{|g|}(r)\right)\right]$$

By taking $A = \left(\log^{[m+1]}\widehat{|h|}(r) - \log^{[q+1]}\varphi\left(\widehat{|f|}(r)\right)\right)$ and $B = \left(\log^{[m+1]}\widehat{|h|}(r) - \log^{[p+1]}\widehat{|g|}(r)\right)$, we obtain from above that

$$\log \rho_g^{(p,q)}(f,\varphi) \geq \max\left(\liminf_{r\to\infty} A + \limsup_{r\to\infty} - B, \limsup_{r\to\infty} A + \liminf_{r\to\infty} - B\right)$$

$$i.e., \ \log \rho_g^{(p,q)}(f,\varphi) \geq \max\left(\liminf_{r\to\infty} A - \liminf_{r\to\infty} B, \limsup_{r\to\infty} A - \limsup_{r\to\infty} B\right).$$

Therefore in view of (3), (4), (5) and (6), it follows from above that

$$\log \rho_g^{(p,q)}(f,\varphi) \geq$$

$$\max\left\{\log \lambda_h^{(m,q)}(f,\varphi) - \log \lambda_h^{(m,p)}(g), \log \rho_h^{(m,q)}(f,\varphi) - \log \rho_h^{(m,p)}(g)\right\}. \quad (10)$$

Thus the theorem follows from (7), (8), (9) and (10). □

The conclusion of the following remark can be carried out after applying the same technique of Theorem 1 and therefore its proof is omitted.

Remark 1. *Let f, $g \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda^{(m,q)}(f,\varphi) \leq \rho^{(m,q)}(f,\varphi) < \infty$ and $0 < \lambda^{(m,p)}(g) \leq \rho^{(m,p)}(g) < \infty$. Then*

$$\frac{\lambda^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)} \leq \lambda_g^{(p,q)}(f,\varphi) \leq \min\left\{\frac{\lambda^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)}, \frac{\rho^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)}\right\}$$
$$\leq \max\left\{\frac{\lambda^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)}, \frac{\rho^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)}\right\} \leq \rho_g^{(p,q)}(f,\varphi) \leq \frac{\rho^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)}.$$

Remark 2. *From the conclusion of Theorem 1, one may write $\rho_g^{(p,q)}(f,\varphi) = \frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}$ and $\lambda_g^{(p,q)}(f,\varphi) = \frac{\lambda_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}$ when $\lambda_h^{(m,p)}(g) = \rho_h^{(m,p)}(g)$. Similarly $\rho_g^{(p,q)}(f,\varphi) = \frac{\lambda_h^{(m,q)}(f,\varphi)}{\lambda_h^{(m,p)}(g)}$ and $\lambda_g^{(p,q)}(f,\varphi) = \frac{\rho_h^{(m,q)}(f,\varphi)}{\rho_h^{(m,p)}(g)}$ when $\lambda_h^{(m,q)}(f,\varphi) = \rho_h^{(m,q)}(f,\varphi)$.*

Theorem 2. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. If $g \sim h$ then $\rho_g^{(p,q)}(f,\varphi) = \rho_h^{(p,q)}(f,\varphi)$ and $\lambda_g^{(p,q)}(f,\varphi) = \lambda_h^{(p,q)}(f,\varphi)$.*

Proof. Let $\varepsilon > 0$. Since $g \sim h$, for any l $(0 < l < \infty)$ it follows for all sufficiently large positive numbers of r that

$$|g|(r) < (l+\varepsilon)|h|(r).$$

Now for $\alpha > \max\{1, (l+\varepsilon)\}$, we get by Lemma 1 and above for all sufficiently large positive numbers of r that

$$|g|(r) < |h|(\alpha r)$$

$$i.e., \ \widehat{|h|}(r) < \alpha\widehat{|g|}(r). \tag{11}$$

Therefore we get from (11) that

$$\rho_h^{(p,q)}(f,\varphi) = \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f|(r))}{\log^{[q]} \varphi(r)} \le \limsup_{r\to\infty} \frac{\log^{[p]} \alpha\widehat{|g|}(|f|(r))}{\log^{[q]} \varphi(r)}$$

$$i.e., \ \rho_h^{(p,q)}(f,\varphi) \le \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r)) + O(1)}{\log^{[q]} \varphi(r)}.$$

Therefore from above we get that $\rho_h^{(p,q)}(f,\varphi) \le \rho_g^{(p,q)}(f,\varphi)$. The reverse inequality is clear because $h \sim g$ and so $\rho_g^{(p,q)}(f,\varphi) = \rho_h^{(p,q)}(f,\varphi)$.

In a similar manner, $\lambda_h^{(p,q)}(f,\varphi) = \lambda_g^{(p,q)}(f,\varphi)$.

This proves the theorem. □

Theorem 3. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. If $f \sim h$ then $\rho_g^{(p,q)}(h,\varphi) = \rho_g^{(p,q)}(f,\varphi)$ and $\lambda_g^{(p,q)}(h,\varphi) = \lambda_g^{(p,q)}(f,\varphi)$ where $\varphi(r) : [0,+\infty) \to (0,+\infty)$ is a non-decreasing unbounded function with $\lim_{r\to\infty} \frac{\log^{[q]} \varphi(\alpha r)}{\log^{[q]} \varphi(r)} = 1$ for any $\alpha > 0$.*

Proof. Since $f \sim h$, for any $\varepsilon > 0$ we obtain that

$$|f|(r) < (l+\varepsilon)|h|(r),$$

where $0 < l < \infty$.

Therefore for $\alpha > \max\{1, (l+\varepsilon)\}$ and in view of Lemma 1, we get from above for all sufficiently large positive numbers of r that

$$|f|(r) < |h|(\alpha r). \tag{12}$$

Now we obtain from (12) that

$$\begin{aligned}
\rho_g^{(p,q)}(f,\varphi) &= \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[q]} \varphi(r)} \\
&\le \limsup_{r\to\infty} \left(\frac{\log^{[p]} \widehat{|g|}(|h|(\alpha r))}{\log^{[q]} \varphi(\alpha r)} \cdot \frac{\log^{[q]} \varphi(\alpha r)}{\log^{[q]} \varphi(r)} \right) \\
&\le \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|g|}(|h|(\alpha r))}{\log^{[q]} \varphi(\alpha r)} \cdot \lim_{\sigma\to+\infty} \frac{\log^{[q]} \varphi(\alpha r)}{\log^{[q]} \varphi(r)}.
\end{aligned}$$

Now from above we get that $\rho_g^{(p,q)}(f,\varphi) \le \rho_g^{(p,q)}(h,\varphi)$. Further $f \sim h \Rightarrow h \sim f$, so we also obtain that $\rho_g^{(p,q)}(h,\varphi) \le \rho_g^{(p,q)}(f,\varphi)$ and therefore $\rho_g^{(p,q)}(h,\varphi) = \rho_g^{(p,q)}(f,\varphi)$.

In a similar manner, $\lambda_g^{(p,q)}(h,\varphi) = \lambda_g^{(p,q)}(f,\varphi)$.

This proves the theorem. □

Theorem 4. *Let* $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$*. If* $g \sim h$ *and* $f \sim k$ *then* $\rho_f^{(p,q)}(g,\varphi) = \rho_k^{(p,q)}(h,\varphi) = \rho_f^{(p,q)}(h,\varphi) = \rho_k^{(p,q)}(g,\varphi)$ *and* $\lambda_f^{(p,q)}(g,\varphi) = \lambda_k^{(p,q)}(h,\varphi) = \lambda_f^{(p,q)}(h,\varphi) = \lambda_k^{(p,q)}(g,\varphi)$ *where* $\varphi(r) : [0,+\infty) \to (0,+\infty)$ *is a non-decreasing unbounded function with* $\lim_{r\to\infty} \frac{\log^{[q]} \varphi(ar)}{\log^{[q]} \varphi(r)} = 1$ *for any* $\alpha > 0$*.*

Theorem 4 follows from Theorem 2 and Theorem 3.

Now we state the following four theorems which can easily be carried out from the definitions of relative (p,q)-φ order and relative (p,q)-φ lower order and with the help of Theorem 2, Theorem 3 and Theorem 4 and therefore their proofs are omitted.

Theorem 5. *Let* $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$*. Also let* $g \sim h$*,* $0 < \lambda_g^{(p,q)}(f,\varphi) \leq \rho_g^{(p,q)}(f,\varphi) < \infty$ *and* $0 < \lambda_h^{(p,q)}(f,\varphi) \leq \rho_h^{(p,q)}(f,\varphi) < \infty$*. Then*

$$\liminf_{\sigma\to+\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[p]} \widehat{|h|}(|f|(r))} \leq 1 \leq \limsup_{\sigma\to+\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[p]} \widehat{|h|}(|f|(r))}.$$

Theorem 6. *Let* $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$*. Also let* $f \sim h$*,* $0 < \lambda_g^{(p,q)}(f,\varphi) \leq \rho_g^{(p,q)}(f,\varphi) < \infty$ *and* $0 < \lambda_g^{(p,q)}(h,\varphi) \leq \rho_g^{(p,q)}(h,\varphi) < \infty$ *where* $\varphi(r) : [0,+\infty) \to (0,+\infty)$ *is a non-decreasing unbounded function with* $\lim_{r\to\infty} \frac{\log^{[q]} \varphi(ar)}{\log^{[q]} \varphi(r)} = 1$ *for any* $\alpha > 0$*.*

$$\liminf_{\sigma\to+\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[p]} \widehat{|g|}(|h|(r))} \leq 1 \leq \limsup_{\sigma\to+\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[p]} \widehat{|g|}(|h|(r))}.$$

Theorem 7. *Let* $f,\ g,\ h,\ k \in \mathcal{A}(\mathbb{K})$*. Also let* $f \sim h$ *and* $g \sim k$*,* $0 < \lambda_g^{(p,q)}(f) \leq \rho_g^{(p,q)}(f) < \infty$ *and* $0 < \lambda_k^{(p,q)}(h) \leq \rho_k^{(p,q)}(h) < \infty$ *where* $\varphi(r) : [0,+\infty) \to (0,+\infty)$ *is a non-decreasing unbounded function with* $\lim_{r\to\infty} \frac{\log^{[q]} \varphi(ar)}{\log^{[q]} \varphi(r)} = 1$ *for any* $\alpha > 0$*.*

$$\liminf_{\sigma\to+\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[p]} \widehat{|k|}(|h|(r))} \leq 1 \leq \limsup_{\sigma\to+\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[p]} \widehat{|k|}(|h|(r))}.$$

Theorem 8. *Let* $f,\ g,\ h,\ k \in \mathcal{A}(\mathbb{K})$*. Also let* $f \sim h$ *and* $g \sim k$*,* $0 < \lambda_g^{(p,q)}(h) \leq \rho_g^{(p,q)}(h) < \infty$ *and* $0 < \lambda_k^{(p,q)}(f) \leq \rho_k^{(p,q)}(f) < \infty$ *where* $\varphi(r) : [0,+\infty) \to (0,+\infty)$ *is a non-decreasing unbounded function with* $\lim_{r\to\infty} \frac{\log^{[q]} \varphi(ar)}{\log^{[q]} \varphi(r)} = 1$ *for any* $\alpha > 0$*.*

$$\liminf_{\sigma\to+\infty} \frac{\log^{[p]} \widehat{|g|}(|h|(r))}{\log^{[p]} \widehat{|k|}(|f|(r))} \leq 1 \leq \limsup_{\sigma\to+\infty} \frac{\log^{[p]} \widehat{|g|}(|h|(r))}{\log^{[p]} \widehat{|k|}(|f|(r))}.$$

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