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A NOTE ON ORDINARY HYPERGEOMETRIC SERIES AND BAILEY'Y TRANSFORM

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Abstract. In this paper, making use of Bailey's transform and certain known summation formula, we have established certain interesting transformation formula of ordinary hypergeometric series.

1. Introduction, Notations and Definition:

The generalized ordinary hyper geometric series ${}_rF_s$ is defined by

$$ {}_rF_s[a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; z] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n ... (a_r)_n}{n!(b_1)_n (b_2)_n ... (b_s)_n} z^n. \tag{1.1} $$

In the above series
(i) If $r \leq s$, then it converges for $|z| < \infty$.
(ii) If $r = s + 1$, then series converges for $|z| < 1$.
(iii) If $r > s + 1$, then series converges only at $z = 0$.
Gauss's hypergeometric series is defined as

$$ {}_2F_1[a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n, \tag{1.2} $$

where

$$ \begin{aligned} (a)_n &= a(a+1)(a+2)(a+3)...(a+n-1), n = 1, 2, .. \\ &= \frac{\Gamma(a+n)}{\Gamma(a)} \end{aligned} $$

$(a)_0 = 1.$
$(a)_{-r} = \frac{(-1)^r}{(1-a)_r}$
$(a)_{m+n} = (a)_m (a+m)_n$
Gauss- summation formula is

$$ {}_2F_1[a, b; c; 1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{1.3} $$

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provided that $R1(c-a-b)>0$. [6; (1.7.6)p.28]
Some interesting formula for multi-basic hypergeometric series appear in the work of $[1-7]$. Also, many useful summations and transformations for elliptic hypergeometric series have been established by $[8-19]$ In 1947, Bailey's established a remarkable, simple and useful transformation formula, which is given in the following form.
If

$$\beta_n \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.4}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n} \tag{1.5}$$

then subject to convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{1.6}$$

Where α_r, δ_r, u_r and v_r are functions of r alone.
In this paper, we shall also use the following summations.

$${}_3F_2[a,b,-n;(1+a-b),(1+a+n);1] = \frac{(1+a)_n(a+\frac{a}{2}-b)_n}{(1+\frac{a}{2})_n(1+a-b)_n} \tag{1.7}$$

[6; (2.3.3.6)p.52]

$${}_2F_1[a,b;(1+a+b);1]_n = \frac{(1+a)_n(1+b)_n}{n!(1+a+b)_n} \tag{1.8}$$

[6; (2.6.1.9)p.84]

$${}_5F_4[a,(1+\frac{a}{2}),b,c,d;\frac{a}{2},(1+a-b),(1+a-c),(1+a-d);1]_n = \frac{(1+a)_n(1+b)_n(1+c)_n(1+d)_n}{n!(1+a-b)_n(1+a-c)_n(1+a-d)_n} \tag{1.9}$$

[6; (2.3.4.6)p.56]

provided that, $a=b+c+d$

$${}_3F_2[a,b,c;d,(a+b+c-d);1]_n = \frac{(1+a)_n(1+b)_n(1+c)_n}{n!(d)_n(a+b+c-d)_n} \tag{1.10}$$

[6; (2.6.1.10)p.84]

$${}_{A+1}F_A[a_0,a_1,...,a_A;(1+b_1),(1+b_2),...(1+b_A);1]_N = \frac{(1+a_0)_N(1+a_1)_N(1+a_2)_N...(1+a_A)_N}{N!(1+b_1)_N(1+b_2)_N(1+b_3)_N...(1+b_A)_N}. \tag{1.11}$$

[6; (2.6.1.7)p.84]

Under the condition

$$a_0+a_1+a_2+...+a_A = b_1+b_2+b_3+...+b_A,$$

$$a_0a_1+a_1a_2+...+a_{A-1}a_A = b_1b_2+b_2b_3+...+b_{A-1}b_A,$$

$$..$$
$$a_0a_1a_2....a_A = b_1b_2b_3....b_A,$$

2. Main Results:

In this section, we shall establish our main results.
(i) Choosing $\alpha_r = \frac{(a)_r(b)_r(-1)^r}{r!(1+a-b)_r}$ and $u_r = \frac{1}{(1)_r}$, $v_r = \frac{1}{(1+a)_r}$ and $\delta_r = (\alpha)_r(\beta)_r$ in (1.7), then using (1.4) and (1.5), we get

$$\beta_n = \frac{(1+\frac{a}{2}-b)_n}{n!(1+\frac{a}{2})_n(1+a-b)_n} \quad and$$
$$\gamma_n = \frac{(\alpha)_n(\beta_n)}{(1+a-\alpha)_n(1+a-\beta)_n} \times \frac{\Gamma(1+a)\Gamma(1+a-\alpha-\beta)}{\Gamma(1+a-\alpha)\Gamma(1+a-\beta)}.$$

Putting the value of α_n, β_n, γ_n and δ_n in (1.6), we get

$$_4F_3[\alpha,\beta,a,b;(1+a-b),(1+a-\alpha),(1+a-\beta);-1] =$$

$$\frac{\Gamma(1+a)\Gamma(1+a-\alpha-\beta)}{\Gamma(1+a-\alpha)\Gamma(1+a-\beta)} \times_3 F_2[\alpha,\beta,(1+\frac{a}{2}-b);(1+a-b),(1+\frac{a}{2});1]. \quad (2.1)$$

(ii) Choosing

$$\alpha_r = \frac{(a)_r(b)_r}{r!(1+a+b)_r} \text{ and } u_r = v_r = 1, \text{ and } \delta_r = z^r$$

in (1.8), then using (1.4) and (1.5), we get

$$\beta_n = \frac{(1+a)_n(1+b)_n}{n!(1+a+b)_n} \text{ and } \gamma_n = \frac{z^n}{(1-z)}.$$

Putting the value of α_n, β_n, γ_n and δ_n in (1.6), we get

$$_2F_1[a,b;(1+a+b);z] = (1-z) \times_2 F_1[(1+a),(1+b);(1+a+b);z]. \quad (2.2)$$

(iii) Again by choosing

$$\alpha_r = \frac{(a)_r(b)_r}{r!(1+a+b)_r} \text{ and } u_r = v_r = 1, \text{ and } \delta_r = rz^r$$

in (1.8), then using (1.4) and (1.5), we get

$$\beta_n = \frac{(1+a)_n(1+b)_n}{n!(1+a+b)_n} \text{ and } \gamma_n = \{\frac{z^{n+1}}{(1-z)^2} + \frac{nz^n}{(1-z)}\}.$$

Putting the value of α_n, β_n, γ_n and δ_n in (1.6), we get

$$\frac{z}{(1-z)} \times_2 F_1[a,b;(1+a+b);z] + \frac{abz}{(1+a+b)(1-z)} \times_2 F_1[(1+a),(1+b);(2+a+b);z]$$
$$= \frac{(1+a)(1+b)z}{(1+a+b)} \times_2 F_1[(2+a),(2+b);(2+a+b);z] \quad (2.3)$$

(iv) Choosing

$$\alpha_r = \frac{(a)_r(1+\frac{a}{2})_r(b)_r(c)_r(d)_r}{r!(\frac{a}{2})_r(1+a-b)_r(1+a-c)_r(1+a-d)_r}$$

and $u_r = v_r = 1$, and $\delta_r = z^r$ in (1.9), then using (1.4) and (1.5), we get

$$\beta_n = \frac{(1+a)_n(1+b)_n(1+c)_n(1+d)_n}{n!(1+a-b)_n(1+a-c)_n(1+a-d)_n} \text{ and } \gamma_n = \frac{z^n}{(1-z)}.$$

Putting the value of α_n, β_n, γ_n and δ_n in (1.6), we get

$$ {}_5F_4[a,(1+\frac{a}{2}),b,c,(a-b-c);\frac{a}{2},(1+a-b),(1+a-c),(1+b+c);z]= $$
$$ z(1-z)\times{}_4F_3[(1+a),(1+b),(1+c),(1+a-b-c);(1+c-b),(1+a-c),(1+b+c);z] \tag{2.4} $$

(v) Again choosing

$$ \alpha_r=\frac{(a)_r(1+\frac{a}{2})_r(b)_r(c)_r(d)_r}{r!(\frac{a}{2})_r(1+a-b)_r(1+a-c)_r(1+a-d)_r} $$

and $u_r=v_r=1$, and $\delta_r=rz^r$ in (1.9), then using (1.4) and (1.5), we get

$$ \beta_n=\frac{(1+a)_n(1+b)_n(1+c)_n(1+d)_n}{n!(1+a-b)_n(1+a-c)_n(1+a-d)_n} \text{ and } \gamma_n=\{\frac{z^{n+1}}{(1-z)^2}+\frac{nz^n}{(1-z)}\}. $$

Putting the value of α_n, β_n, γ_n and δ_n in (1.6), we get

$$ \frac{z}{(1-z)^2}\times{}_5F_4[a,(1+\frac{a}{2}),b,c,(a-b-c);\frac{a}{2},(1+a-b),(1+a-c),(1+b+c);z]+ $$
$$ \frac{za(1+\frac{a}{2})bc(a-b-c)}{(1-z)\frac{a}{2}(1+a-b)(1+a-c)(1+b+c)}\times $$
$$ {}_5F_4[(1+a),(2+\frac{a}{2}),(1+b),(1+c),(a-b-c+1);(1+\frac{a}{2}),(2+a-b),(2+a-c),(2+b+c);z] $$
$$ =\frac{z(1+a)(1+b)(1+c)(1+a-b-c)}{(1+a-c)(1+c-b)(1+b+c)}\times $$
$$ {}_4F_3[(2+a),(2+b),(2+c),(2+a-b-c);(2+c-b),(2+a-c),(2+b+c);z] \tag{2.5} $$

(vi) Choosing

$$ \alpha_r=\frac{(a)_r(b)_r(c)_r}{r!(d)_r(a+b+c-d)_r} \text{ and } u_r=v_r=1, \text{ and } \delta_r=z^r $$

in (1.10), then using (1.4) and (1.5), we get

$$ \beta_n=\frac{(1+a)_n(1+b)_n(1+c)_n}{n!(d)_n(a+b+c-d)_n} \text{ and } \gamma_n=\frac{z^n}{(1-z)} $$

Putting the value of α_n, β_n, γ_n and δ_n in (1.6), we get

$$ {}_3F_2[a,b,c;d,(a+b+c-d);z]=(1-z){}_3F_2[(1+a),(1+b),(1+c);d,(a+b+c-d);z] \tag{2.6} $$

(vii) Again choosing

$$ \alpha_r=\frac{(a)_r(b)_r(c)_r}{r!(d)_r(a+b+c-d)_r} \text{ and } u_r=v_r=1, \text{ and } \delta_r=rz^r $$

in (1.10), then using (1.4) and (1.5), we get

$$ \beta_n=\frac{(1+a)_n(1+b)_n(1+c)_n}{n!(d)_n(a+b+c-d)_n} \text{ and } \gamma_n=\frac{z^{n+1}}{(1-z)^2}+\frac{nz^n}{(1-z)}. $$

Putting the value of α_n, β_n, γ_n and δ_n in (1.6), we get

$$ \frac{z}{(1-z)}\times{}_3F_2[a,b,c;d,(a+b+c-d);z]+\frac{abcz}{(1-z)d(a+b+c-d)}\times $$
$$ {}_3F_2[(1+a),(1+b),(1+c);(1+d),(1+a+b+c-d);z]= $$
$$ \frac{(1+a)(1+b)(1+c)z}{d(a+b+c-d)}\times{}_3F_2[(2+a),(2+b),(2+c);(1+d),(1+a+b+c-d);z] \tag{2.7} $$

($viii$) Choosing

$$\alpha_r = \frac{(a_0)_r(a_1)_r(a_2)_r....(a_A)_r}{N!(1+b_1)_r(1+b_2)_r(1+b_3)_r...(1+b_A)_r}$$

and $u_r = v_r = 1$, and $\delta_r = z^r$ in (1.11), then using (1.4) and (1.5), we get

$$\beta_n = \frac{(1+a_0)_N(1+a_1)_N(1+a_2)_N....(1+a_A)_N}{N!(1+b_1)_N(1+b_2)_N(1+b_N)_r...(1+b_A)_N} \text{ and } \gamma_n = \frac{z^n}{(1-z)}.$$

Putting the value of α_n, β_n, γ_n and δ_n in (1.6), we get

$$_{A+1}F_A[a_0, a_1, ..., a_A; (1+b_1), (1+b_2), ...(1+b_A); z] =$$
$$(1-z)_{A+1}F_A[(1+a_0), (1+a_1), ..., (1+a_A); (1+b_1), (1+b_2), ...(1+b_A); z]. \quad (2.8)$$

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