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STUDY OF MEMORY EFFECT BETWEEN TWO MEMORY DEPENDENT INVENTORY MODELS

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Abstract. In this paper, a memory dependent inventory model has been developed utilizing the fractional order derivative and integration with cubic type demand rate. Then the minimized total average cost and the optimal ordering interval of of this model is compared numerically with another memory dependent inventory model which has quadratic type demand rate. One important observation can be summarize form the comparison of the results which is: for cubic type demand rate model, business policy falls down at long memory index but for quadratic type demand rare model business policy falls down at low memory index. It may be claimed from the numerical example that quadratic type demand rate inventory model is suitable for business profit compared to the cubic type demand rate inventory model.

1. Introduction

Fractional calculus is the most important mathematical tool which has been extensively used in the last three decades to study the system which has memory effect. Using the idea of fractional calculus the existence of memory effect in different real life problem such as in biology, financial processes etc. has been established by the authors[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], many new applications in science and engineering are governed by the fractional calculus. The generalization of derivative of any function to non-integer order using fractional calculus is a rather old problem, as demonstrated which is also tasted several months in 1695[3]. Fractional derivative has played an important role in various fields such as biology[2], economics[6, 7, 8], financial process[11, 12, 13, 28, 29] considering the fractional order derivatve as the index of memory [2].Recently, Tarasov et al[23] has written a nice introduction On History of Mathematical Economics with Application of Fractional Calculus.

Recently, Das and roy [1] applied fractional order derivative and integration in their article[1] which gives a direction to use applications of fractional order derivative and integration to the inventory model. In spite of inclusion of fractional order

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derivative and integration, there is no way to establish the existence of memory dependency of any system. Recently we have developed some papers considering the memory dependency in inventory model with different characteristic of demand as well as different characteristic cost with different situations[11, 12, 13, 28, 29].

Researchers	Application of memory effects in EOQ Model	Application of memory effects in Economics	Application of memory effects in Biological Model	Application of fractional calculus in another topics	Theoretically development fractional calculus
Miller.K.S, Ross.B[3]				√	
Podubly.I[4]				√	
M.Caputo [5]				√	
M.Saeedian, M. Khalighi, N. Azimi-Tafreshi,G. R. Jafari, M. Ausloos[2]			√		
V.E.Tarasov, V.V.Tarasova.[6, 7, 8, 20, 21]		√			
Tenreiro Machado J.,DurateF.B.,Duarte G.M.[9]		√			
T.Das, U.Ghosh, S.Sarkar and S.Das[10]			√		
Das.A.K, Roy.T.K[1]				√	
R.Pakhira,U.Ghosh, S.Sarkar[11, 12, 13]	√				
Das.S[14]				√	
Ghosh.U,Sengupta.S, Sarkar.S, Das.S.[15]					√
P. L. Butzer, U. Westphal, J. Douglas, W. R. Schneider, G. Zaslavsky,T. Nonnemacher[16]				√	
G. Rotundo [17]		√			
G.Rahman,D.Baleanu,M.Al-Qurashi, S.DPurohit,M.Arshad [18]					√
Gorenflo R.,Mainardi F.,Scalas E.,Raberto M[19]		√			

Table 1. Literature review and contribution of the development of fractional calculus.

Here, we want to develop a memory dependent inventory model using fractional calculus and a nice comparison of the results has been built using the paper [11].In the paper[11], quadratic type demand rate is assumed and but in this paper, cubic type demand rate is taken in model-I which is function of t^3 but in model-II ,demand rate is function of $t^{3\alpha}$ .Here, which type model is best for the real market situation with taking into account memory effect has been discussed. The quadratic type memory dependent inventory model is best for the real market situation because profit is more compared to the cubic type demand rate inventory model.

Inventory models are governed by the first order ordinary differential equations and ordinary integrations. It is known that the integer order derivative is not able to include memory effect or past experience effects. Actually, the classical inventory model suffers from amnesia of the past experiences because it is developed by integer order differential equation. Due to the above fact,to incorporate memory effect to the inventory system, we have used fractional order derivative and integrals. Here, it is established that the strength of memory is controlled by the order of fractional order derivative and integration.

In table-1, we have illustrated some developmental history of fractional calculus by the different branch of basic science.

The arrangement of the paper is as follows: Review of fractional calculus is given in the section-2, In section-3 contents assumption of the developed model, Fractional order inventory model with quadratic type demand rate is illustrated in the section-4, Formulation and analysis of memory less inventory model with cubic type demand rate is given in the section-5, Fractionalization of the Classical EOQ Model with cubic type demand rate is presented in the section-5.1, Analytic solution is given in the section-5.2, Numerical example is illustrated in the section 6,lastly some conclusion is presented in the section-7.

2. **Review of fractional calculus**

In this section we shall describe definitions and some basic properties of fractional derivative.

2.1. **Riemann-Liouville fractional derivative(R-L).**

The left Riemann-Liouville fractional derivative of any continuous function$f(x)$ of order α is defined as follows

$$ {}_aD_x^{\alpha}(f(x)) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dx}\right)^m \int_a^x (x-\tau)^{m-\alpha-1} f(\tau)d\tau \qquad (1)$$

where $x > a$

Right Riemann-Liouville fractional derivative of order α is defined as follows

$$ {}_xD_b^{\alpha}(f(x)) = \frac{1}{\Gamma(m-\alpha)}(-\left(\frac{d}{dx}\right)^m \int_x^b (\tau-x)^{m-\alpha-1} f(\tau)d\tau \qquad (2)$$

where $x > 0$

2.2. **Caputo fractional order derivative.**

Caputo [5] modifies the left and right Riemman-Liouville fractional derivative to overcome this difference. The Caputo modifications are as follows: The left Caputo fractional derivative[5] for any differentiable function $f(x)$ in $[a, b]$ is denoted by ${}_a^C D_x^\alpha(f(x))$ and is defined by

$$ {}_a^C D_x^\alpha(f(x)) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-\tau)^{m-\alpha-1} f^m(\tau) d\tau \tag{3} $$

where $0 \leq m-1 < \alpha < m$.
and the right Caputo fractional derivative is defined as follows

$$ {}_x^C D_b^\alpha(f(x)) = \frac{1}{\Gamma(m-\alpha)} \int_x^b (\tau-x)^{m-\alpha-1} f^m(\tau) d\tau \tag{4} $$

where $0 \leq m-1 < \alpha < m$. In terms of Caputo definition the derivative of any constant A is zero i.e.${}_a^C D_x^\alpha(A) = 0$.

2.3. **Fractional Laplace transforms Method.**

The Laplace transform of the function $f(t)$is defined as

$$ F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t) dt \tag{5} $$

where $s > 0$, is called the transform parameter. The Laplace transformation of n^{th} order derivative is defined as

$$ L(f^n(t)) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^k(0) \tag{6} $$

where $f^n(t)$ denotes n^{th}derivative of the function $f$with respect to t and for non integer m it is defined in generalized form[13] as,

$$ L(f^m(t)) = s^m F(s) - \sum_{k=0}^{n-1} s^k f^{m-k-1}(0) \tag{7} $$

where,$(n-1) < m \leq n$

3. **Assumptions**

In this paper, to develop the fractional order EOQ model the following assumptions are taken into consideration.
(i) Lead time is zero, (ii)Time horizon is infinite, (iii)There is no shortage, (iv)There is no deterioration.

4. **Fractional order inventory model with quadratic type demand rate**

The following two fractional EOQ models with quadratic type demand rate is already in [11]. Here we are stating those models only.

Model-I

In model-**I**, we have considered the demand rate is a polynomial of t in the following form $(a + bt + ct^2)$ then the memory dependent EOQ model can be written in the following form

$$ {}_0^C D_t^\alpha(I(t)) = -(a + bt + ct^2) \tag{8} $$

or equivalently

(i)$R(t)$:Demand rate	(ii)Q:Total order quantity
(iii)P:Per unit cost of the total order quantity	(iv)C_1:Inventory holding cost per unit
(v)C_3:ordering cost or set up cost per order	(vi)$I(t)$:Inventory level at time t
(vii)T:Ordering interval	(viii) $HOC_{\alpha,\beta}(T)$:Inventory holding cost
(ix)$T^*_{\alpha,\beta}$:Optimal ordering interval with fractional effect	(x)TOC^{av}:Total average cost
(xi)$TOC^*_{\alpha,\beta}(T)$:Minimized total average cost with fractional effect	(xii)$(B,.),\Gamma$:Beta and gamma function respectively

Table 2. Used symbols and Items.

$$\frac{d^{\alpha}(I(t)}{dt^{\alpha}} = -(a + bt + ct^2) \tag{9}$$

Model-II

In second model, we have modified the demand rate affected by the order of fractional derivative of the inventory level i.e. of the form $(a + bt^{\alpha} + ct^{2\alpha})$. Incorporating the modified demand rate the memory dependent EOQ model will be modified in the following form

$${}_0^C D_t^{\alpha}(I(t)) = -(a + bt^{\alpha} + ct^{2\alpha}) \tag{10}$$

or equivalently

$$\frac{d^{\alpha}\,(I(t))}{dt^{\alpha}} = -(a + bt^{\alpha} + ct^{2\alpha}) \tag{11}$$

where $0 < \alpha \leq 1$and $0 \leq t \leq T$

5. Classical order inventory model with cubic type demand rate

Here we consider the the inventory level reduces due to cubic type demand rate $R(t) = a + bt + ct^2 + dt^3$ where $a > 0, b, c, d \geq 0$and shortage is not allowed. The inventory reaches zero level at time $t = T$. Therefore, inventory level at any time during the time interval $[0, T]$ can be represented by the following first order ordinary differential equation as,

$$\frac{dI(t)}{dt} = -(a + bt + ct^2 + dt^3) \tag{12}$$

5.1. Fractionalization of the Classical EOQ Model with cubic type demand rate.

We will now develop the different fractional order inventory models considering fractional rate of change of the inventory level.

Model-I

To study the influence of memory effects, first the differential equation(12) is written using the memory kernel function in the following form [2].

$$\frac{dI(t)}{dt} = -\int k(t-t')\left(a + bt' + c(t')^2 + d(t')^3\right)dt' \tag{13}$$

In which the memory kernel $k(t-t')$ plays the important to reduce the integer order system to the fractional order system. For Markov process it is equal to the delta function $\delta(t-t')$ that generates the equation (12).In fact, any arbitrary function can be replaced by a sum of delta functions, thereby leading to a given type of time correlations. This type of kernel promises the existence of scaling features as it is often intrinsic in most natural phenomena. Thus, to generate the fractional order model we consider $k(t-t') = \frac{1}{\Gamma(\alpha-1)}(t-t')^{(\alpha-2)}$where$0 < \alpha \leq 1$ and $\Gamma(\alpha)$ denotes the gamma function. Using the definition of fractionalderivative [3], the equation(13) can be written to the form of fractional differential equations with the Caputo-type derivative in the following form as,

$$\frac{dI(t)}{dt} = -{}_0D_t^{-(\alpha-1)}\left(a + bt + ct^2 + dt^3\right) \tag{14}$$

Now, applying fractional Caputo derivative of order$(\alpha-1)$ on both sides of(14), and using the fact that Caputo fractional order derivative and fractional integral are inverse operators, the following fractional differential equations can be obtained for the model

$${}_0^CD_t^\alpha(I(t)) = -\left(a + bt + ct^2 + dt^3\right) \tag{15}$$

or equivalently

$$\frac{d^\alpha\left(I(t)\right)}{dt^\alpha} = -\left(a + bt + ct^2 + dt^3\right) \tag{16}$$

where $0 < \alpha \leq 1, 0 \leq t \leq T$ with boundary conditions $I(T) = 0, I(0) = Q$ and . α is considered as differential memory index.

Model-II

Now for model-**II**, we consider the demand rate as of $\left(a + bt^\alpha + ct^{2\alpha} + dt^{3\alpha}\right)$ then the memory dependent EOQ model will be of the following form (here we consider the exponent of t same as the order of fractional derivative)

$${}_0^CD_t^\alpha(I(t)) = -\left(a + bt^\alpha + ct^{2\alpha} + dt^{3\alpha}\right) \tag{17}$$

or equivalently

$$\frac{d^\alpha(I(t))}{dt^\alpha} = -\left(a + bt^\alpha + ct^{2\alpha} + dt^{3\alpha}\right) \tag{18}$$

where $0 < \alpha \leq 1, 0 \leq t \leq T$

5.2. **Analytical solution.** (i) **Model-I**

Here, we have considered the fractional order inventory model-**I**[11] which will be solved in this section. In operator form the fractional order differential equation in (16) can be represented as

$$D^\alpha(I(t)) = -(a + bt + ct^2 + dt^3) \tag{19}$$

where D^α stands for the Caputo fractional order derivative with the operator $(D^\alpha) = {}_0^CD_t^\alpha$

Using fractional Laplace transform and the corresponding inversion formula on the equation (19) we get the inventory level for this fractional order inventory model at time t which can be written as

$$I(t) = \left(Q - \frac{at^{\alpha}}{\Gamma(1+\alpha)} - \frac{bt^{\alpha+1}}{\Gamma(2+\alpha)} - \frac{2ct^{2+\alpha}}{\Gamma(3+\alpha)} - \frac{6dt^{3+\alpha}}{\Gamma(4+\alpha)}\right) \tag{20}$$

Using the boundary condition $I(0) = Q$ on the equation (20), The total order quantity is obtained as

$$Q = \left(\frac{aT^{\alpha}}{\Gamma(1+\alpha)} + \frac{bT^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{2cT^{\alpha+2}}{\Gamma(3+\alpha)} + \frac{6dT^{3+\alpha}}{\Gamma(4+\alpha)}\right) \tag{21}$$

and corresponding the inventory level at time t is as

$$I(t) = \frac{a\left(T^{\alpha} - t^{\alpha}\right)}{\Gamma\left(\alpha+1\right)} + \frac{b\left(T^{\alpha+1} - t^{\alpha+1}\right)}{\Gamma\left(\alpha+2\right)} + \frac{2c\left(T^{\alpha+2} - t^{\alpha+2}\right)}{\Gamma\left(\alpha+3\right)} + \frac{6d\left(T^{\alpha+3} - t^{\alpha+3}\right)}{\Gamma\left(\alpha+4\right)} \tag{22}$$

Purchasing cost is

$$PC = P\left(\frac{aT^{\alpha}}{\Gamma(1+\alpha)} + \frac{bT^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{2cT^{\alpha+2}}{\Gamma(3+\alpha)} + \frac{6dT^{3+\alpha}}{\Gamma(4+\alpha)}\right) \tag{23}$$

The β^{th} order total inventory holding cost is denoted as $HOC_{\alpha,\beta}(T)$ and defined as

$$\begin{aligned}
HOC_{\alpha,\beta}\left(T\right) &= C_1\left({}_0D_T^{-\beta}(I(t))\right) = \frac{C_1}{\Gamma\left(\beta\right)}\int_0^T \left(T-t\right)^{\beta-1} I\left(t\right)dt \\
&= \frac{C_1aT^{\alpha+\beta}}{\Gamma\left(\alpha+1\right)}\left(\frac{1}{\Gamma\left(\beta+1\right)} - \frac{B\left(\alpha+1,\beta\right)}{\Gamma\left(\beta\right)}\right) + \frac{C_1bT^{\alpha+\beta+1}}{\Gamma\left(\alpha+2\right)}\left(\frac{1}{\Gamma\left(\beta+1\right)} - \frac{B\left(\alpha+2,\beta\right)}{\Gamma\left(\beta\right)}\right) \\
&+ \frac{2C_1cT^{\alpha+\beta+2}}{\Gamma\left(\alpha+3\right)}\left(\frac{1}{\Gamma\left(\beta+1\right)} - \frac{B\left(\alpha+3,\beta\right)}{\Gamma\left(\beta\right)}\right) + \frac{6C_1dT^{\alpha+\beta+3}}{\Gamma\left(\alpha+4\right)}\left(\frac{1}{\Gamma\left(\beta+1\right)} - \frac{B\left(\alpha+4,\beta\right)}{\Gamma\left(\beta\right)}\right)
\end{aligned} \tag{24}$$

Here, β is considered another memory parameter (integral memory index) corresponding to the carrying cost which is the transportation related cost. Poor transportation service always has a bad impact on the business and good service has good impact on the business. To consider the past experience a memory parameter should be taken into account.

Therefore, the total average cost per unit time per cycle of this fractional model is,

$$\begin{aligned}
TOC_{\alpha},\beta(T) &= \frac{(PQ + HOC_{\alpha},\beta(T) + C_3)}{T} = AT^{\alpha+\beta+2} + B_1T^{\alpha+\beta+1} \\
&+ CT^{\alpha+\beta} + DT^{\alpha+\beta-1} + ET^{\alpha+2} + FT^{\alpha+1} + GT^{\alpha} + HT^{\alpha-1} + IT^{-1}
\end{aligned} \tag{25}$$

where $A = \frac{6dC_1}{\Gamma(\alpha+4)\Gamma(\beta)}\left(\frac{1}{\beta} - B(\alpha+4,\beta)\right), G = \frac{bP}{\Gamma(\alpha+2)}, I = C_3, B_1 = \frac{2cC_1}{\Gamma(\alpha+3)\Gamma(\beta)}\left(\frac{1}{\beta} - B(\alpha+3,\beta)\right), C = \frac{bC_1}{\Gamma(\alpha+2)\Gamma(\beta)}\left(\frac{1}{\beta} - B(\alpha+2,\beta)\right), D = \frac{aC_1}{\Gamma(\alpha+1)\Gamma(\beta)}\left(\frac{1}{\beta} - B(\alpha+1,\beta)\right), E = \frac{6dP}{\Gamma(\alpha+4)}, F = \frac{2cP}{\Gamma(\alpha+3)}, H = \frac{aP}{\Gamma(\alpha+1)}$.

(ii)Model-II

Here, we study the fractional order inventory **model-II** which will be solved by using Laplace transform method with the boundary conditions, given in the problem.In operator form the fractional differential equation in (18) can be represented as

$$D^{\alpha}(I(t)) = -\left(a + bt^{\alpha} + ct^{2\alpha} + dt^{3\alpha}\right) \tag{26}$$

where D^{α} stands for the Caputo fractional order derivative with the operator $(D^{\alpha}) ={}_0^C D_t^{\alpha}$.

We get the inventory level for this fractional order inventory model at time t which can be written as

$$I(t) = \left(Q - \frac{at^{\alpha}}{\Gamma(1+\alpha)} - \frac{b\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{c\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{d\Gamma(3\alpha+1)t^{4\alpha}}{\Gamma(4\alpha+1)}\right) \tag{27}$$

Using the boundary condition $I(0) = Q$ on the equation (27), the total order quantity is obtained as

$$Q = \left(\frac{at^{\alpha}}{\Gamma(1+\alpha)} + \frac{b\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{c\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{d\Gamma(3\alpha+1)t^{4\alpha}}{\Gamma(4\alpha+1)}\right) \tag{28}$$

and corresponding the inventory level at time t is as

$$I(t) = \frac{a(T^{\alpha} - t^{\alpha})}{\Gamma(1+\alpha)}) + \frac{b\Gamma(\alpha+1)(T^{2\alpha} - t^{2\alpha})}{\Gamma(2\alpha+1)} + \frac{c\Gamma(2\alpha+1)(T^{3\alpha} - t^{3\alpha})}{\Gamma(3\alpha+1)} + \frac{d\Gamma(3\alpha+1)(T^{4\alpha} - t^{4\alpha})}{\Gamma(4\alpha+1)} \tag{29}$$

Purchasing cost is

$$PC = P\left(\frac{aT^{\alpha}}{\Gamma(\alpha+1)} + \frac{b\Gamma(\alpha+1)(T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{c\Gamma(2\alpha+1)(T^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{d\Gamma(3\alpha+1)(T^{4\alpha}}{\Gamma(4\alpha+1)}\right) \tag{30}$$

The β^{th} order total inventory holding cost is denoted as $HOC_{\alpha,\beta}(T)$ and defined as

$$HOC_{\alpha,\beta}(T) = C_1\left({}_0D_T^{-\beta}(I(t))\right) = \frac{C_1}{\Gamma(\beta)}\int_0^T (T-t)^{\beta-1}(I(t))dt$$

$$\frac{C_1 aT^{(\alpha+\beta)}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+1),\beta}{\Gamma(\beta)}\right) + \frac{C_1 bT^{(2\alpha+\beta)}\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(2\alpha+1,\beta)}{\Gamma(\beta)}\right) + \frac{C_1 cT^{(3\alpha+\beta)}\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(3\alpha+1),\beta}{\Gamma(\beta)}\right) + \frac{dC_1 T^{(4\alpha+\beta)}\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(4\alpha+1),\beta}{\Gamma(\beta)}\right) \tag{31}$$

Therefore, the total average cost per unit time per cycle of this fractional model is,

$$TOC_{\alpha,\beta}(T) = \frac{(PQ + HOC_{\alpha,\beta}(T) + C_3)}{T} = AT^{4\alpha+\beta-1} + B_1 T^{3\alpha+\beta-1} + CT^{2\alpha+\beta-1} + DT^{\alpha+\beta-1} + ET^{\alpha-1} + FT^{2\alpha-1} + GT^{3\alpha-1} + HT^{4\alpha-1} + IT^{-1} \tag{32}$$

α	β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.1	1.0	1.5348	$1.8663x10^4$
0.2	1.0	1.4811	$1.9087x10^4$
0.3	1.0	1.4111	$1.9376x10^4$
0.4	1.0	1.3224	$1.9497x10^4$
0.5	1.0	1.2113	$1.9413x10^4$
0.6	1.0	1.0731	$1.9077x10^4$
0.7	1.0	0.9041	$1.8428x10^4$
0.8	1.0	0.6902	$1.7379x10^4$
0.9	1.0	0.4469	$1.5812x10^4$
1.0	1.0	0.2392	$1.3647x10^4$

(a)

α	β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.1	1.0	362.6146	660.6534
0.2	1.0	60.7722	$3.1633x10^4$
0.3	1.0	17.5292	$8.0612x10^4$
0.4	1.0	6.3926	$1.3617x10^4$
0.5	1.0	3.0065	$1.7476x10^4$
0.6	1.0	1.7376	$1.9125x10^4$
0.7	1.0	1.1164	$1.9097x10^4$
0.8	1.0	0.7268	$1.7954x10^4$
0.9	1.0	0.4376	$1.6056x10^4$
1.0	1.0	0.2392	$1.3647x10^4$

(b)

Table 3. Optimal ordering interval and minimized total average cost for cubic type demand rate model **(a)(model-I)**,**(b)(model-II)** where $\beta = 1.0$ and $0 < \alpha \leq 1$.

where

$$A = \frac{C_1 d\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)\Gamma(\beta)}\left(\frac{1}{\beta} - B(4\alpha+1,\beta)\right), B_1 = \frac{cC_1\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\beta)}\left(\frac{1}{\beta} - B(3\alpha+1,\beta)\right),$$
$$C = \frac{bC_1\Gamma(\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\beta)}\left(\frac{1}{\beta} - B(2\alpha+1,\beta)\right), D = \frac{aC_1}{\Gamma(\alpha+1)\Gamma(\beta)}\left(\frac{1}{\beta} - B(\alpha+1,\beta)\right),$$
$$E = \frac{aP}{\Gamma(\alpha+1)}, F = \frac{bP\Gamma(\alpha+1)}{(\Gamma(2\alpha+1))}, G = \frac{cP\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}, H = \frac{dP\Gamma(3\alpha+1)}{(\Gamma(4\alpha+1))}, I = C_3. \tag{33}$$

6. Numerical Example

(a) To illustrate numerically the developed fractional order inventory model for cubic type demand rate we consider empirical values of the various parameters in proper units as $a = 40, b = 20, c = 2, d = 4, C_1 = 15, C_3 = 200, U = 300$ and required solution has been made using matlab minimization method.

In table-3 we have presented the values of Optimal ordering interval $(T*_\alpha, \beta)$ and minimized total average cost $(TOC*_\alpha, \beta)$ for different value of α and β for the model with cubic type demand rate((a)for model-I and (b) for model-II for $\alpha = 0.1, 0.2, 0.3, 0.4, ...1.0$).

It is clear from the table-**3(a)** that the minimized total average cost is maximum at $\alpha = 0.4$ but gradually decreases below and above. Hence,the profit falls down for a particular value of of the memory parameter then it gradually increases below and above with different optimal ordering interval. But for the model-II the minimized total average cost is maximum at $\alpha = 0.3$ and then similar behavior a previous. Thus introduction of the effect memory parameter in the demand rate maximize the minimized total average cost at lower value.

(b) Again to illustrate numerically the developed fractional order inventory model for quadratic type demand rate we consider empirical values of the various parameters in proper units as$a = 40, b = 20, c = 2, C_1 = 15, C_3 = 200, U = 300$ and required solution has been made using matlab minimization method.

α	β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.0951	1.0	3.3023	$1.3643x10^4$
0.1	1.0	3.2830	$1.3708x10^4$
0.2	1.0	2.8981	$1.5004x10^4$
0.3	1.0	2.5267	$1.6171x10^4$
0.4	1.0	2.1649	$1.7136x10^4$
0.5	1.0	1.8098	$1.7817x10^4$
0.6	1.0	1.4597	$1.8123x10^4$
0.7	1.0	1.1145	$1.7954x10^4$
0.8	1.0	0.7778	$1.7207x10^4$
0.9	1.0	0.4669	$1.5776x10^4$
1.0	1.0	0.2409	$1.3643x10^4$

(a)

α	β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.1	1.0	675.3851	427.1055
0.2	1.0	135.0484	$1.8042x10^3$
0.3	1.0	46.3908	$4.6288x10^3$
0.4	1.0	18.7019	$8.8477x10^3$
0.5	1.0	7.8573	$1.3414x10^4$
0.6	1.0	7.2745	$1.3643x10^4$
0.7	1.0	3.4517	$1.6771x10^4$
0.8	1.0	1.6676	$1.8113x10^4$
0.9	1.0	0.8742	$1.7670x10^4$
1.0	1.0	0.4621	$1.6008x10^4$

(b)

Table 4. Optimal ordering interval and minimized total average cost (a)(**model-I**),(b)(**model-II**)([11])for quadratic type demand rate model where $\beta = 1.0, 0 < \alpha \leq 1$ [11].

In table-4 we have presented the similar values as in table-3 for the models I and II with quadratic type demand rate. It is clear from the table-4 that minimized total average cost is maximum at$\alpha = 0.6$ for model-I but for for model-II the maximum value occurs at $\alpha = 0.8$.

In figure-1 and 2 we have presented the minimized total average cost for different value of memory index(α).It is clear from the figures that minimized total average cost for quadratic type demand rate model (I) is very low compared to the minimized total average cost for cubic type demand rate model(I) and the minimized total average cost for cubic type demand rate model-II is changing with respect to differential memory index as non -linear type but minimized total average cost for quadratic type demand rate model-II is linear type from $\alpha \in [0.1, 0.4]$ then it behaves non-linear type.

Again in figure-3 we have presented optimal ordering interval for cubic type demand rate model-I and quadratic type demand rate model-I. In figure-4 we have presented optimal ordering interval for cubic type demand rate model-II and quadratic type demand rate model-II. In figure-5,6 we have presented rate of change of inventory level for different values of α of model-I,II(Cubic type demand rate). In figure-7,we have presented rate of change of inventory level for different values of α of model-I(quadratic type demand rate). From fig-3, it is clear that differences of the optimal ordering interval gradually decreases with gradually decreasing memory effect.

From fig-4, it is clear that differences of the optimal ordering interval gradually decreases and almost tends to zero with gradually decreasing memory effect.

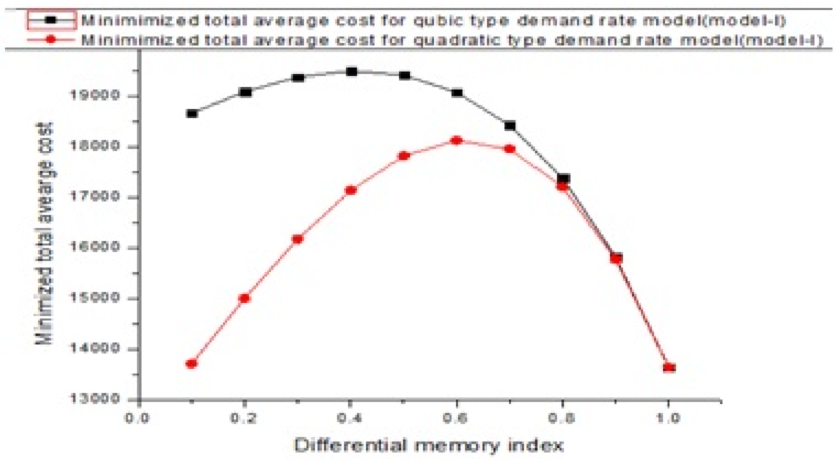

Figure 1. Differential memory index versus Minimized total average cost for model I.

From fig-5-7, it is clear that with memory affected inventory level changes differently compared to the without memory affected inventory level.

6.1. **Observations.**

(i) For quadratic type demand rate(**model-I**), profit is minimum i.e. minimized total average cost is maximum at $\alpha = 0.6$ but for cubic type demand rate, profit mostly loses at $\alpha = 0.8$.
(ii) In quadratic type demand rate(**model-II**), profit mostly loses at $\alpha = 0.8$ but in cubic type demand rate, profit mostly loses at$\alpha = 0.3$.
(iii) One important observation is observed that for cubic type demand rate model, business policy falls down at long memory index but for quadratic type demand rare model business policy falls down at low memory index.
(iv) Another observation from the paper is as quadratic type demand rate inventory model with memory effect is suitable for business profit compared to the cubic type demand rate inventory model.

7. Conclusion

In this paper, a comparative study has been constructed among four memory dependent EOQ model. The models are constructed considering the demand rate as quadratic and cubic type in two case and in another two case with the fractional order polynomials of order 2α and 3α. From the analysis we have established the profitable memory dependent model.It clear from the investigations of the models that the memory dependent inventory model with quadratic type demand rate is suitable compared to the cubic type demand rate for the business person. Secondly the models with fractional order polynomials demand rate is appropriate for short memory effect i.e. newly started business.
....

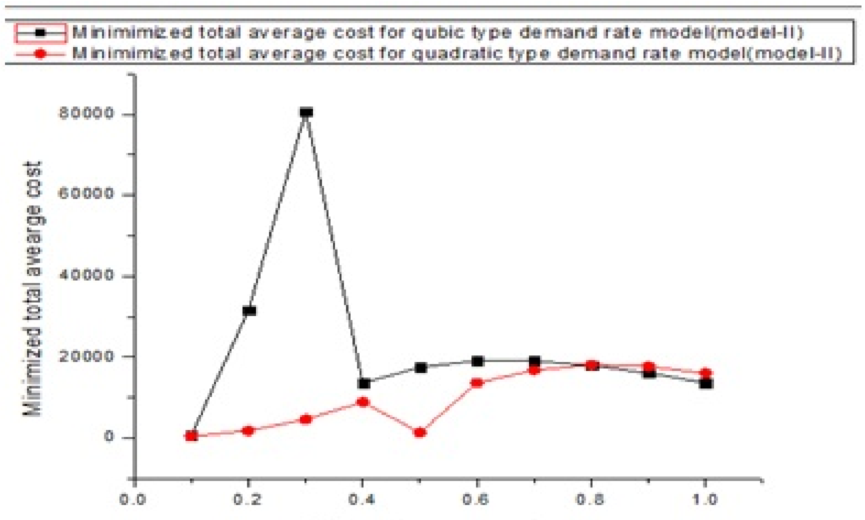

Figure 2. Differential memory index versus minimized total average cost for model II.

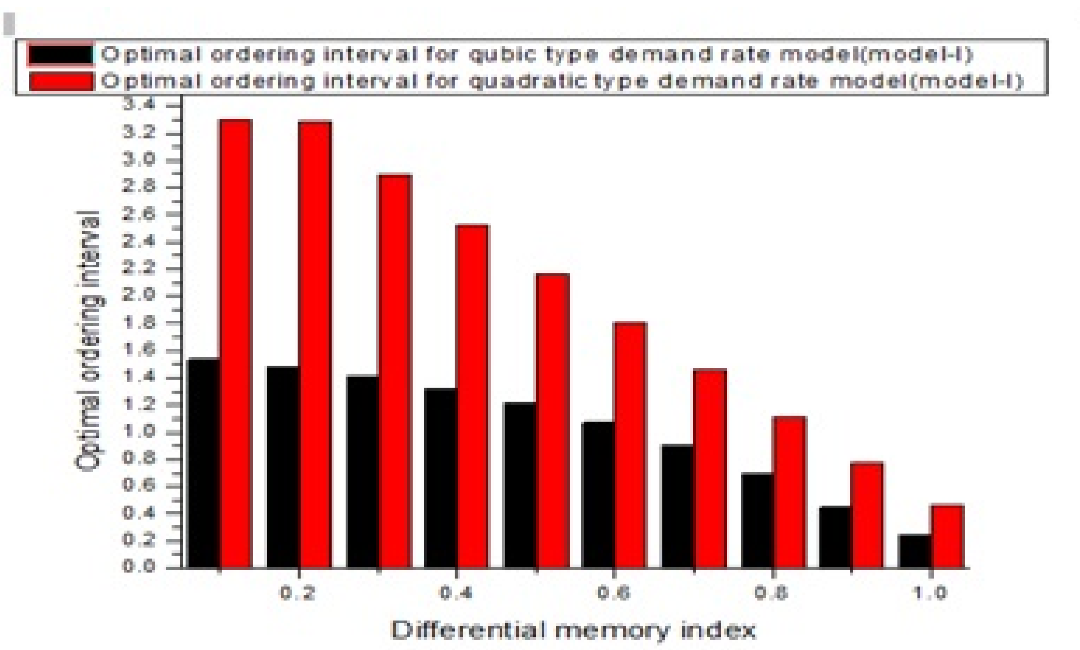

Figure 3. Differential memory index versus Optimal Ordering intervalfor model I

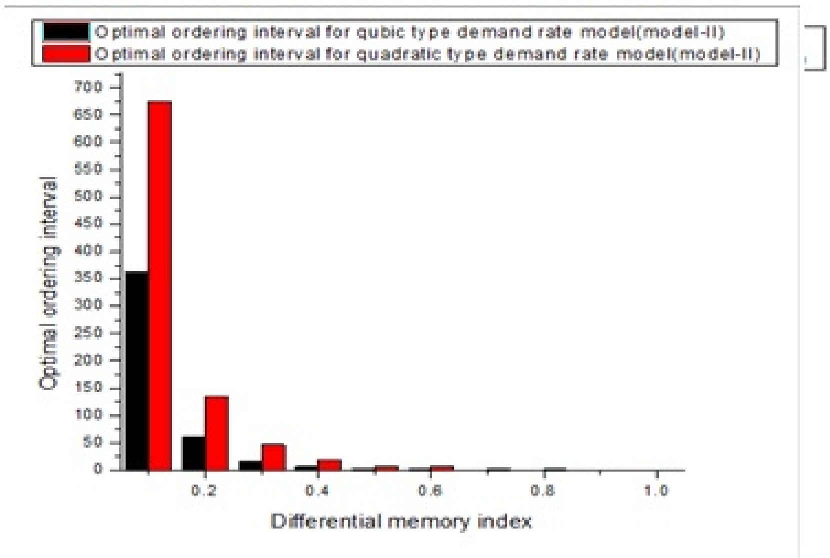

Figure 4. Differential memory index versus Optimal Ordering interval for model II.

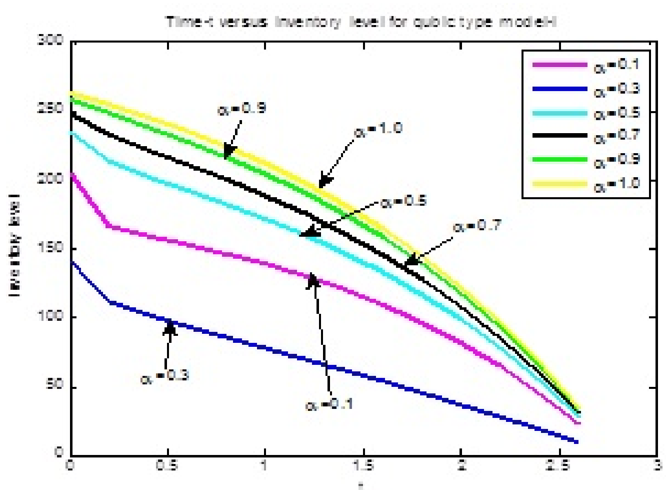

Figure 5. Time-t versus Inventory level for model-I for the cubic type demand rate model

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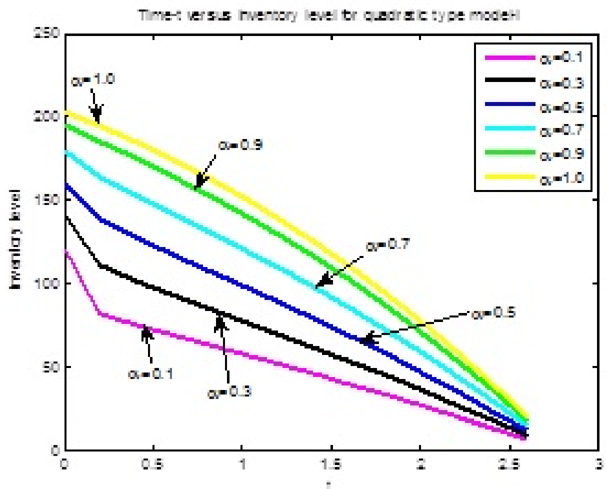

Figure 6. Time-t versus Inventory level for model-II of the cubic type demand rate model

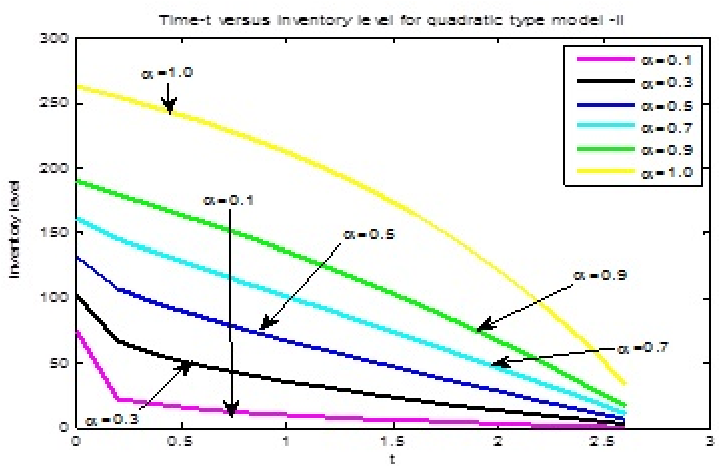

Figure 7. Time-t versus Inventory level for model-I of the quadratic type demand rate model

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