SOME PROPERTIES OF VARIABLE-ORDER FRACTIONAL CALCULUS

SHUQIN ZHANG AND LEI HU

ABSTRACT. The topic of fractional calculus (derivatives and integrals of arbitrary orders) is enjoying growing interest not only in Mathematics, but also in Physics, Engineering and Mathematical Biology (see [?]-[?] for example). In this paper we are concerned with the fractional-order differential equations describe Games with non-uniform interaction rate [?] and Asymmetric games [?]. Existence, uniqueness and stability of the solutions of these systems are studied.

ABSTRACT. The operators of variable-order, which fall into a more complex category, are the derivatives and integrals whose orders are the functions of certain variables. In this work, we discuss some properties of variable-order fractional calculus.

1. INTRODUCTION

Recently, operators and differential equations of variable-order have been considered, see [2]-[29]. In these works, authors considered the applications of derivative of variable-order in various science such as anomalous diffusion modeling, mechanical applications, multi-fractional Gaussian noises. Among of these, there have many works dealing with numerical methods for some class of variable-order fractional differential equations, for instance, [8], [10], [16], [17], [18], [19], [22], [23], [26]. Moreover, a physical experimental study of variable-order operators has been considered in [15]. A comparative study of constant-order and variable-order models have been considered in [13]. The fractional differential equations are the generalization of differential equations of integer order. The fractional operators (fractional derivatives and integrals) are the generalization of integer-order differential and integral operators. It is well known that the motivation for those works rises from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, chemistry, aerodynamics, electro-dynamics of complex medium, polymer rheology, and so on.

In view of the Laplace transform, semigroup property of the constant order fractional integration operators and some other properties of fractional derivatives and integrals, the fractional differential equations can be transformed into their equivalent integral equations. However, we notice that there has few results for the variable-order fractional integrals and derivatives. Therefore, the main contribution of this paper is to show that the variable-order fractional integrals do not have semigroup properties, and that there is no explicit connection between the Laplace transforms of function $x(t)$ and variable-order integrals and derivatives to $x(t)$.

2. Preliminaries

We give several definitions of variable-order fractional integrals and derivatives, which can be found in [10]. We let $-\infty < a < b < \infty$.

Definition 2.1 ([10])**.** Let $p : [a,b] \times \mathbb{R} \to (0,+\infty)$, the left Riemann-Liouville fractional integral of order $p(t,x(t))$ for function $x(t)$ are defined as the following two types

$$I_{a+}^{p(t,x(t))}x(t) = \int_a^t \frac{(t-s)^{p(t,x(t))-1}}{\Gamma(p(t,x(t)))} x(s)ds, \ t > a, \tag{2.1}$$

$$I_{a+}^{p(t,x(t))}x(t) = \int_a^t \frac{(t-s)^{p(s,x(s))-1}}{\Gamma(p(s,x(s)))} x(s)ds, \ t > a. \tag{2.2}$$

Definition 2.2 ([10])**.** Let $p : [a,b] \times \mathbb{R} \to (0,+\infty)$, the left Riemann-Liouville fractional derivative of order $p(t,x(t))$ for function $x(t)$ are defined as the following two types

$$D_{a+}^{p(t,x(t))}x(t) = \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-p(t,x(t))-1}}{\Gamma(n-p(t,x(t)))} x(s)ds, \ t > a, \tag{2.3}$$

$$D_{a+}^{p(t,x(t))}x(t) = \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-p(s,x(s))-1}}{\Gamma(n-p(s,x(s)))} x(s)ds, \ t > a, \tag{2.4}$$

where $n \in N$ satisfies $n - 1 < p(t,x(t)) \leq n$ for all $(t,x) \in [a,b] \times \mathbb{R}$.

Obviously, the Riemann-Liouville fractional integral I_{a+}^{δ} and derivative D_{a+}^{δ} of function $x(t)$ are particular cases of $(2.1), (2.2)$ and $(2.3), (2.4)$ (see [1]), i.e.

$$I_{a+}^{\delta}x(t) = \frac{1}{\Gamma(\delta)} \int_a^t (t-s)^{\delta-1}x(s)ds, \ \ t > a,$$

$$D_{a+}^{\delta}x(t) = \left(\frac{d}{dt}\right)^n I_{a+}^{n-\delta}f(t) = \frac{1}{\Gamma(n-\delta)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\delta-1}x(s)ds, \ \ t > a,$$

where $n \in N$ satisfies $n - 1 < p(t,x(t)) \leq n$ for all $(t,x) \in (0,+\infty) \times \mathbb{R}$

The following are some properties for the Riemann-Liouville fractional integral and derivative, which can be founded in [1].

Proposition 2.1 ([1])**.** The Riemann-Liouville fractional integral defined for function $x(t) \in L(a,b)$ exists almost everywhere.

Proposition 2.2 ([1])**.** The equality $I_{a+}^{\gamma}I_{a+}^{\delta}f(t) = I_{a+}^{\delta}I_{a+}^{\gamma}f(t) = I_{a+}^{\gamma+\delta}f(t), \gamma, \delta > 0$ holds for $f \in L(a,b)$.

Proposition 2.3 ([1])**.** The equality $D_{a+}^{\gamma}I_{a+}^{\gamma}f(t) = f(t), \ n - 1 < \gamma \leq n(n \in N^+)$ holds for $f \in L(a,b)$.

Proposition 2.4 ([1]). Let $n - 1 < \alpha \leq n(n \in N^+)$, then the differential equation

$$D_{a+}^{\alpha} u = 0$$

has a unique solution

$$u(t) = c_1(t - a)^{\alpha-1} + c_2(t - a)^{\alpha-2} + \cdots + c_n(t - a)^{\alpha-n}, c_i \in \mathbb{R}.$$

Proposition 2.5 ([1]). Let $n - 1 < \alpha \leq n(n \in N^+)$, $u \in L(a, b)$, $D_{a+}^{\alpha} u \in L(a, b)$. Then the following equality holds

$$I_{a+}^{\alpha} D_{a+}^{\alpha} u(t) = u(t) + c_1(t - a)^{\alpha-1} + c_2(t - a)^{\alpha-2} + \cdots + c_n(t - a)^{\alpha-n}, c_i \in \mathbb{R}.$$

Proposition 2.6 ([1]). Assume the Riemann-Liouville fractional integral I_{0+}^{δ} ($\delta > 0$) and Laplace transform of function $f(t)$ exist, then the Laplace transform of the Riemann-Liouville fractional integral $I_{0+}^{\delta} f(t)$ is

$$L\{I_{0+}^{\delta} f; s\} = s^{-\delta} F(s),$$

where $F(s)$ is Laplace transform of function $f(t)$.

Proposition 2.7 ([1]). Let the Riemann-Liouville fractional derivative D_{0+}^{δ} ($n - 1 < \alpha \leq n, n \in N^+$) and Laplace transform of function $f(t)$ exist, then the Laplace transform of the Riemann-Liouville fractional derivative $D_{0+}^{\delta} f(t)$ is

$$L\{D_{0+}^{\delta} f; s\} = s^{\delta} F(s) - \sum_{k=0}^{n-1} s^{n-k-1} D^k (I_{0+}^{n-\delta} f(0+)),$$

where $F(s)$ is Laplace transform of function $f(t)$.

3. Some Facts for variable-order fractional calculus

We first consider existence result for variable-order fractional integrals defined by (2.1) and (2.2).

Lemma 3.1. Let $p : [a, b] \times \mathbb{R} \to [p^*, q^*]$ $(0 < p^* < q^* < +\infty)$ be a real function. Then for $x \in L(a, b)$, variable-order fractional integrals (2.1) and (2.2) exist almost everywhere, provided that function $\frac{1}{\Gamma(p(t,x(t))}$ is bounded.

Proof. We only verify the result for variable-order fractional integral (2.2). The proof for variable-order fractional integral (2.1) is similar to it.

By the bounded assumption of $\frac{1}{\Gamma(p(t,x(t)))}$, letting $L_p = \sup_{a \leq t \leq b} |\frac{1}{\Gamma(p(t,x(t)))}|$. For $a \leq s \leq t \leq b$, when $p^* \leq p(s, x(s)) \leq q^*$, we have known that

$$\text{if } 0 \leq t - s \leq 1, \quad \text{then } (t - s)^{p(s,x(s))-1} \leq (t - s)^{p^*-1}; \tag{3.1}$$

$$\text{if } 1 < t - s < +\infty, \quad \text{then } (t - s)^{p(s,x(s))-1} \leq (t - s)^{q^*-1}. \tag{3.2}$$

Thus, from (3.1), (3.2), for $a \leq s \leq t \leq b$, we have

$$(t - s)^{p(s,x(s))-1} \leq \max\{(t - s)^{p^*-1}, (t - s)^{q^*-1}\} \doteq (t - s)^{\alpha-1}, \tag{3.3}$$

where α denotes p^* or q^*.

Thus, for $x \in L(a, b)$, by the definition of variable-order fractional integral (2.2), using (3.3), we have

$$
\begin{aligned}
|I_{a+}^{p(t,x(t))} x(t)| &\leq \quad \left| \int_a^t \frac{(t-s)^{p(s,x(s))-1}}{\Gamma(p(s,x(s)))} x(s) ds \right| \\
&\leq \quad L_p \int_a^t (t-s)^{p(s,x(s))-1} |x(s)| ds \\
&\leq \quad L_p \int_a^t (t-s)^{\alpha-1} |x(s)| ds.
\end{aligned}
$$

It follows from Proposition 2.1 that

$$
\int_a^t (t-s)^{\alpha-1} |x(s)| ds = \Gamma(\alpha) I_{a+}^\alpha |x(t)|,
$$

which implies that variable-order fractional integral (2.2) exists almost everywhere for $x \in L(a, b)$. Thus we complete the proof. $\qquad \square$

Remark 3.1. If $p : [a, b] \to (0, 1)$ is a real continuous function. Then for $x \in C_r[a, b] \doteq \{x(t) \in C(a, b), t^r x(t) \in C[a, b]\}$ $(0 < r < \min_{a \leq t \leq b} |p(t)|)$, then variable-order fractional integrals (2.1) and (2.2) exist for any points on $[a, b]$.

In fact, it follows from the continuity of function $\Gamma(p(t))$ that $L_p = \max_{a \leq t \leq b} |\frac{1}{\Gamma(p(t))}|$ exists.

Let $\beta = \min_{a \leq t \leq b} p(t)$, then, for $a \leq s \leq t \leq b$, when $\beta \leq p(s, x(s)) < 1$, we have known that

$$
\text{if } 0 \leq t - s \leq 1, \quad \text{then } (t-s)^{p(s)-1} \leq (t-s)^{\beta-1}; \tag{3.4}
$$

$$
\text{if } 1 < t - s < +\infty, \quad \text{then } (t-s)^{p(s)-1} \leq 1. \tag{3.5}
$$

Thus, from (3.4), (3.5), for $a \leq s \leq t \leq b$, we have

$$
(t-s)^{p(s)-1} \leq \max\{(t-s)^{\beta-1}, 1\}. \tag{3.6}
$$

For $x \in C_r[a, b]$, by the definition of variable-order fractional integral (2.2), using (3.6), we have

$$
\begin{aligned}
\left| I_{a+}^{p(t)} x(t) \right| &\leq \quad L_p \int_a^t (t-s)^{p(s)-1} |x(s)| ds \\
&= \quad L_p \int_a^t (t-s)^{\beta-1} s^{-r} s^r |x(s)| ds \\
&\leq \quad L_p \max_{0 \leq t \leq b} t^r |x(t)| \int_a^t (t-s)^{\beta-1} s^{-r} ds \\
&= \quad \frac{L_p \max_{0 \leq t \leq b} t^r |x(t)| (t-a)^{\beta-r} \Gamma(\beta) \Gamma(1-r)}{\Gamma(1-r+\beta)} \\
&\leq \quad \frac{L_p \max_{0 \leq t \leq b} t^r |x(t)| (b-a)^{\beta-r} \Gamma(\beta) \Gamma(1-r)}{\Gamma(1-r+\beta)},
\end{aligned}
$$

or

$$\left| I_{a+}^{p(t)} x(t) \right| \leq \quad L_p \int_a^t (t-s)^{p(s)-1} |x(s)| ds$$

$$= \quad L_p \int_a^t (t-s)^{\beta-1} s^{-r} s^r |x(s)| ds$$

$$\leq \quad L_p \max_{0 \leq t \leq b} t^r |x(t)| \int_a^t s^{-r} ds$$

$$= \quad \frac{L_p \max_{0 \leq t \leq b} t^r |x(t)| (t-a)^{1-r}}{1-r}$$

$$\leq \quad \frac{L_p \max_{0 \leq t \leq b} t^r |x(t)| (b-a)^{1-r}}{1-r},$$

both the two estimations imply that variable-order fractional integral (2.2) exists everywhere for $x \in C_r[a, b]$. The proof for variable-order fractional integral (2.1) is similar to that.

Example 3.1 Let $x(t) = t^{\frac{2}{3}}$, $p(t, x(t)) = \frac{1}{3} + (x(t))^{\frac{3}{2}} + 0.5\sin(\pi t) = \frac{1}{3} + t + 0.5\sin(\pi t)$, $0 \leq t \leq 1$, we will calculate the variable order integral $I_{0+}^{p(t,x(t))} x(t)$ defined in (2.1).

By (2.1), for $0 \leq t \leq 1$, we obtain that

$$I_{0+}^{p(t,x(t))} x(t) = \frac{1}{\Gamma(\frac{1}{3} + t + 0.5\sin(\pi t))} \int_0^t (t-s)^{t+0.5\sin(\pi t) - \frac{2}{3}} s^{\frac{2}{3}} ds = \frac{\Gamma(\frac{5}{3}) t^{1+t+0.5\sin(\pi t)}}{\Gamma(2 + t + 0.5\sin(\pi t))},$$

obviously, $\frac{\Gamma(\frac{5}{3}) t^{1+t+0.5\sin(\pi t)}}{\Gamma(2+t+0.5\sin(\pi t))}$ is bounded for $t \in [0, 1]$.

Example 3.2 Let $x(t) = t^{\frac{2}{3}}$, $p(t, x(t)) = \frac{1}{3} + (x(t))^{\frac{3}{2}} + 0.5\sin(\pi t) = \frac{1}{3} + t + 0.5\sin(\pi t)$, $0 \leq t \leq 1$, we will calculate the variable order integral $I_{0+}^{p(t,x(t))} x(t)$ defined in (2.2).

By (2.2), we obtain that

$$I_{0+}^{p(t,x(t))} x(t) = \int_0^t \frac{(t-s)^{s+0.5\sin(\pi s) - \frac{2}{3}}}{\Gamma(\frac{1}{3} + s + 0.5\sin(\pi s))} s^{\frac{2}{3}} ds, 0 \leq t \leq 1.$$

It follows from the continuity of Gamma function, we know that $\frac{1}{\Gamma(\frac{1}{3}+s+0.5\sin(\pi s))}$ is bounded for $s \in [0, 1]$, denoted by M_Γ. Moreover, for $0 \leq s \leq t \leq 1$, $(t-s)^{s+0.5\sin(\pi s) - \frac{2}{3}} \leq (t-s)^{-\frac{2}{3}}$, as a result, we have that

$$\left| I_{0+}^{p(t,x(t))} x(t) \right| \leq \quad M_\Gamma \int_0^t (t-s)^{s+0.5\sin(\pi s) - \frac{2}{3}} s^{\frac{2}{3}} ds$$

$$\leq \quad M_\Gamma \int_0^t (t-s)^{-\frac{2}{3}} s^{\frac{2}{3}} ds = M_\Gamma \Gamma(\frac{1}{3}) \Gamma(\frac{5}{3}) t \leq M_\Gamma \Gamma(\frac{1}{3}) \Gamma(\frac{5}{3}),$$

which implies that $I_{0+}^{p(t,x(t))} x(t)$ is bounded for $t \in [0, 1]$.

The following are curves for variable-order integral $I_{0+}^{p(t,x(t))} x(t) = \frac{\Gamma(\frac{5}{3}) t^{1+t+0.5\sin(\pi t)}}{\Gamma(2+t+0.5\sin(\pi t))}$,

$0 \le t \le 1$ defined in 2.1, and variable-order integral $I_{0+}^{p(t,x(t))} x(t) = \int_0^t \frac{(t-s)^{s+0.5\sin(\pi s)-\frac{2}{3}}}{\Gamma(\frac{1}{3}+s+0.5\sin(\pi s))} s^{\frac{2}{3}} ds, 0 \le t \le 1$ defined in 2.2.
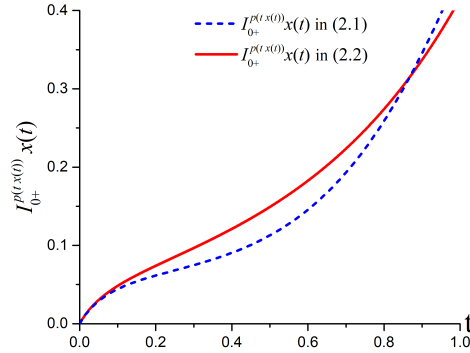

FIGURE 1. $I_{0+}^{p(t,x(t))} x(t)$, where $x(t) = t^{\frac{2}{3}}$, $p(t, x(t)) = \frac{1}{3} + t + 0.5\sin(\pi t)$, $0 \le t \le 1$.

Now, let's take an example to consider results of Proposition 2.2 for variable-order fractional integrals 2.1 and 2.2.

Let $p(t, x(t)) \doteq p(t), q(t, x(t)) \doteq q(t)$ be not constant functions, for $p, q \in C[0, b], 0 < b < +\infty, f(t) \equiv 1, t \in [0, T]$, we calculate $I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)$ and $I_{0+}^{p(t)+q(t)} f(t)$. First, according to (2.1), we have

$$
\begin{aligned}
I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) = \quad & \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} f(\tau) d\tau ds \\
= \quad & \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds \\
= \quad & \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \frac{s^{q(s)}}{\Gamma(1+q(s))} ds \\
= \quad & \int_0^1 \frac{t^{p(t)+q(ts)} s^{p(ts)} (1-s)^{p(t)-1}}{\Gamma(p(t))\Gamma(1+q(ts))} ds,
\end{aligned}
$$

$$
I_{0+}^{p(t)+q(t)} f(t) = \int_0^t \frac{(t-s)^{p(t)+q(t)-1}}{\Gamma(p(t)+q(t))} ds = \frac{t^{p(t)+q(t)}}{\Gamma(1+p(t)+q(t))}.
$$

Thus, for $f(t) \equiv 1$, we see that, if

$$
\int_0^1 \frac{t^{p(t)+q(ts)} s^{p(ts)} (1-s)^{p(t)-1}}{\Gamma(p(t))\Gamma(1+q(ts))} ds = \frac{t^{p(t)+q(t)}}{\Gamma(1+p(t)+q(t))}, \tag{3.7}
$$

then we could obtain

$$
I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) = I_{0+}^{p(t)+q(t)} f(t).
$$

However, we can't assert (3.7) holds. We only get

$$
\int_0^1 \frac{t^{p(t)+q(t)} s^{p(t)} (1-s)^{p(t)-1}}{\Gamma(p(t))\Gamma(1+q(t))} ds = \frac{t^{p(t)+q(t)}}{\Gamma(1+p(t)+q(t))}.
$$

Second, according to (2.2), we have

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) = \int_0^t \frac{(t-s)^{p(s)-1}}{\Gamma(p(s))} \int_0^s \frac{(s-\tau)^{q(\tau)-1}}{\Gamma(q(\tau))} f(\tau) d\tau ds$$

$$= \int_0^t \frac{(t-s)^{p(s)-1}}{\Gamma(p(s))} \int_0^s \frac{(s-\tau)^{q(\tau)-1}}{\Gamma(q(\tau))} d\tau ds$$

$$= \int_0^t \frac{(t-s)^{p(s)-1}}{\Gamma(p(s))} \int_0^1 \frac{s^{q(s\mu)}(1-\mu)^{q(s\mu)-1}}{\Gamma(q(s\mu))} d\mu ds$$

$$= \int_0^1 \int_0^1 \frac{t^{p(tr)}(1-r)^{p(tr)-1}}{\Gamma(p(tr))} \frac{(tr)^{q(tr\mu)}(1-\mu)^{q(t\mu r)-1}}{\Gamma(q(t\mu r))} d\mu dr$$

$$= \int_0^1 \int_0^1 \frac{t^{p(tr)+q(t\mu r)}(1-r)^{p(tr)-1}}{\Gamma(p(tr))} \frac{r^{q(tr\mu)}(1-\mu)^{q(t\mu r)-1}}{\Gamma(q(t\mu r))} d\mu dr,$$

$$I_{0+}^{p(t)+q(t)} f(t) = \int_0^t \frac{(t-s)^{p(s)+q(s)-1}}{\Gamma(p(s)+q(s))} f(s) ds = \int_0^1 \frac{t^{p(tr)+q(tr)}(1-r)^{p(tr)+q(tr)-1}}{\Gamma(p(tr)+q(tr))} dr.$$

Thus, for $f(t) \equiv 1$, we see that, if

$$\int_0^1 \frac{t^{p(tr)+q(t\mu r)}(1-r)^{p(tr)-1}}{\Gamma(p(tr))} \frac{r^{q(tr\mu)}(1-\mu)^{q(t\mu r)-1}}{\Gamma(q(t\mu r))} d\mu = \frac{t^{p(tr)+q(tr)}(1-r)^{p(tr)+q(tr)-1}}{\Gamma(p(tr)+q(tr))}, \tag{3.8}$$

then we could obtain

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) = I_{0+}^{p(t)+q(t)} f(t).$$

However, we can't assert (3.8) holds. We only get

$$\int_0^1 \frac{t^{p(tr)+q(tr)}(1-r)^{p(tr)-q(tr)-1}}{\Gamma(p(tr))} \frac{\mu^{p(tr)-1}(1-\mu)^{q(tr)-1}}{\Gamma(q(tr))} d\mu = \frac{t^{p(tr)+q(tr)}(1-r)^{p(tr)+q(tr)-1}}{\Gamma(p(tr)+q(tr))}.$$

Therefore, from the above arguments, we could obtain the following result.

Lemma 3.2. Let $x(t), p(t, x(t)), q(t, x(t))$ be real functions on finite interval $[0, T]$, assume that variable-order fractional integrals $I_{0+}^{p(t,x(t))} x(t)$, $I_{0+}^{q(t,x(t))} x(t)$ and $I_{0+}^{p(t,x(t))+q(t,x(t))} x(t)$ defined by $(2.1), (2.2)$ exist. In general case, we could claim that

$$I_{0+}^{p(t),x(t)} I_{0+}^{q(t,x(t))} x(t) \neq I_{0+}^{p(t,x(t))+q(t,x(t))} x(t), \text{for some points in } [0, T].$$

In particular, for general functions $0 < p(t, x(t)) < 1$ and $x(t)$, we have

$$I_{0+}^{p(t,x(t))} I_{0+}^{1-p(t,x(t))} x(t) \neq I_{0+}^{p(t,x(t))+1-p(t,x(t))} x(t) = I_{0+}^1 x(t), \text{for some points in } [0, T].$$

In order to demonstrate the claim of Lemma 3.2, we give the following two examples about variable order fractional integrals (2.1) and (2.2).

Example 3.3 Let $p(t) = t, 0 \leq t \leq 3$, $q(t) = \frac{t}{3} + \frac{1}{3}$, $f(t) = 1, 0 \leq t \leq 3$. Now, we calculate $I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=2}$ and $I_{0+}^{p(t)+q(t)}|_{t=2}$ defined in (2.1).

We see that

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) = \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} f(\tau) d\tau ds$$

$$= \int_0^t \frac{(t-s)^{t-1}}{\Gamma(t)} \int_0^s \frac{(s-\tau)^{\frac{s}{3}+\frac{1}{3}-1}}{\Gamma(\frac{s}{3}+\frac{1}{3})} d\tau ds$$

$$= \int_0^t \frac{(t-s)^{t-1} s^{\frac{1}{3}+\frac{s}{3}}}{\Gamma(t)\Gamma(\frac{4}{3}+\frac{s}{3})} ds$$

$$= \int_0^2 \frac{(t-s)^{t-1} s^{\frac{1}{3}+\frac{s}{3}}}{\Gamma(t)\Gamma(\frac{4}{3}+\frac{s}{3})} ds + \int_2^t \frac{(t-s)^{t-1} s^{\frac{1}{3}+\frac{s}{3}}}{\Gamma(t)\Gamma(\frac{4}{3}+\frac{s}{3})} ds.$$

We set $M_1 = \max_{2 \le t \le 3} |\frac{1}{\Gamma(p(t))}|$ and $M_2 = \max_{2 \le s \le 3} |\frac{1}{\Gamma(\frac{4}{3}+\frac{s}{3})}|$. For $2 \le t \le 3$, it follows from

$$\left| \int_2^t \frac{(t-s)^{t-1} s^{\frac{1}{3}+\frac{s}{3}}}{\Gamma(t)\Gamma(\frac{4}{3}+\frac{s}{3})} ds \right| \le \int_2^t |3^{t-1}| \cdot |(\frac{t-s}{3})^{t-1}| \cdot \frac{s^{\frac{1}{3}+\frac{s}{3}}}{\Gamma(t)\Gamma(\frac{4}{3}+\frac{s}{3})} ds$$

$$\le M_1 M_2 \int_2^t 3^{3-1} (\frac{t-s}{3})^{2-1} s^2 ds$$

$$\le M_1 M_2 \int_2^t 3^2 \frac{t-s}{3} t^2 ds$$

$$= \frac{3 M_1 M_2 t^2 (t-2)^2}{2}$$

that

$$\left[\int_2^t \frac{(t-s)^{t-1} s^{\frac{1}{3}+\frac{s}{3}}}{\Gamma(t)\Gamma(\frac{4}{3}+\frac{s}{3})} ds \right]_{t=2} = 0.$$

So, we get

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=2} = \int_0^2 \frac{(2-s)^{2-1} s^{\frac{1}{3}+\frac{s}{3}}}{\Gamma(2)\Gamma(\frac{4}{3}+\frac{s}{3})} ds \approx 1.79$$

and

$$I_{0+}^{p(t)+q(t)} f(t)|_{t=2} = \int_0^2 \frac{(2-s)^{p(2)+q(2)-1}}{\Gamma(p(2)+q(2))} ds = \frac{2^{p(2)+q(2)}}{\Gamma(1+p(2)+q(2))} = \frac{4}{3}.$$

Therefore,

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=2} \ne I_{0+}^{p(t)+q(t)} f(t)|_{t=2}.$$

Example 3.4 Let $p(t) = t$, $q(t) = 1$, $f(t) = 1, 0 \le t \le 3$. Now, we calculate $I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1}$ and $I_{0+}^{p(t)+q(t)} f(t)|_{t=1}$ defined in (2.2).

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1} = \int_0^1 \frac{(1-s)^{s-1}}{\Gamma(s)} \int_0^s \frac{(s-\tau)^{1-1}}{\Gamma(1)} d\tau ds = \int_0^1 \frac{(1-s)^{s-1} s}{\Gamma(s)} ds \approx 0.472.$$

and

$$I_{0+}^{p(t)+q(t)} f(t)|_{t=1} = \int_0^1 \frac{(1-s)^s}{\Gamma(s+1)} ds = \int_0^1 \frac{(1-s)^s}{s\Gamma(s)} ds \approx 0.686.$$

Therefore,

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1} \ne I_{0+}^{p(t)+q(t)} f(t)|_{t=1}.$$

Remark 3.2. According to Lemma 3.2, we could claim that variable-order calculus of non-constant functions $p(t, x(t))$ for $x(t)$ defined by $(2.1), (2.2)$ don't have the properties like Propositions 2.3-2.5.

Integral and derivative of variable-order $(2.1), (2.2)$ and $(2.3), (2.4)$, defined on a finite interval $[a, b]$ of the real line $\mathbb{R}$, are naturally extended to the half-axis on $\mathbb{R}^+$. The variable-order integration and differentiation constructions, corresponding to the ones in $(2.1), (2.2)$ and $(2.3), (2.4)$, having the following forms

$$I_{0+}^{p(t,x(t))}x(t) = \int_0^t \frac{(t-s)^{p(t,x(t))-1}}{\Gamma(p(t,x(t)))}x(s)ds, \ t > 0, \tag{3.9}$$

$$I_{0+}^{p(t,x(t))}x(t) = \int_0^t \frac{(t-s)^{p(s,x(s))-1}}{\Gamma(p(s,x(s)))}x(s)ds, \ t > 0. \tag{3.10}$$

$$D_{0+}^{p(t,x(t))}x(t) = \left(\frac{d}{dt}\right)^n \int_0^t \frac{(t-s)^{n-p(t,x(t))-1}}{\Gamma(n-p(t,x(t)))}x(s)ds, \ t > 0, \tag{3.11}$$

$$D_{0+}^{p(t,x(t))}x(t) = \left(\frac{d}{dt}\right)^n \int_0^t \frac{(t-s)^{n-p(s,x(s))-1}}{\Gamma(n-p(s,x(s)))}x(s)ds, \ t > 0. \tag{3.12}$$

Let $L[x(t); s]$, $L[I_{0+}^{p(t)}x(t); s]$, $L[D_{0+}^{p(t)}x(t); s]$ denote the Laplace transforms of functions $x(t)$, $I_{0+}^{p(t)}x(t)$ and $D_{0+}^{p(t)}x(t)$. We find that there there is no explicit connection between $L[x(t); s]$ and $L[I_{0+}^{p(t)}x(t); s]$, as a result, there is also no explicit connection between $L[x(t); s]$ and $L[D_{0+}^{p(t)}x(t); s]$.

Lemma 3.3. Let $x(t), p(t, x(t))$ be real functions, assume that variable-order fractional integrals $I_{0+}^{p(t,x(t))}x(t)$ defined by (3.3) exists, and the Laplace transforms $L[x(t); s]$ and $L[I_{0+}^{p(t,x(t))}x(t); s]$ exist. There is no explicit connection between $L[x(t); s]$ and $L[I_{0+}^{p(t,x(t))}x(t); s]$.

Proof. By definitions of variable-order fractional integral (3.9) and Laplace transform, we get

$$L[I_{0+}^{p(t,x(t))}x(t); s] = \int_0^\infty e^{-st} \int_0^t \frac{(t-\tau)^{p(t,x(t))-1}}{\Gamma(p(t,x(t)))}x(\tau)d\tau dt$$

$$= \int_0^\infty e^{-st} \int_\tau^\infty \frac{(t-\tau)^{p(t,x(t))-1}}{\Gamma(p(t,x(t)))}x(\tau)dt d\tau$$

$$= \int_0^\infty e^{-s(\tau+r)} \int_0^\infty \frac{r^{p(\tau+r,x(\tau+r))-1}}{\Gamma(p(\tau+r,x(\tau+r)))}x(\tau)dr d\tau$$

$$= \int_0^\infty e^{-s\tau}x(\tau) \int_0^\infty e^{-sr} \frac{r^{p(\tau+r,x(\tau+r))-1}}{\Gamma(p(\tau+r,x(\tau+r)))}dr d\tau$$

$$= \int_0^\infty e^{-s\tau}x(\tau)L\left[\frac{r^{p(r+\tau,x(t+\tau))-1}}{\Gamma(p(r+\tau),x(r+\tau))}; s\right]d\tau.$$

Since we couldn't know what $L\left[\frac{r^{p(r+\tau,x(t+\tau))-1}}{\Gamma(p(r+\tau),x(r+\tau))}; s\right]$ equals to, this implies that we couldn't know the explicit connection between $L[x(t); s]$ and $L[I_{0+}^{p(t,x(t))}x(t); s]$. Thus, we complete the proof. □

Example 3.5 We consider the Laplace transforms of functions $t^\lambda, \lambda > -1$ and

$$I_{0+}^{p(t)} t^\lambda = \int_0^t \frac{(t-s)^{t^2-1}}{\Gamma(t^2)} s^\lambda ds = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+t^2)} t^{t^2+\lambda}.$$

We could know

$$L[t^\lambda; s] = \int_0^\infty e^{-st} t^\lambda dt = \frac{\Gamma(1+\lambda)}{s^{1+\lambda}},$$

$$L[I_{0+}^{p(t)} t^\lambda; s] = \int_0^\infty e^{-st} \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+t^2)} t^{t^2+\lambda} dt, \qquad (3.13)$$

$$
\begin{aligned}
L[I_{0+}^{p(t)} t^\lambda; s] = & \int_0^\infty e^{-st} \int_0^t \frac{(t-\tau)^{t^2-1}}{\Gamma(t^2)} \tau^\lambda d\tau dt \\
= & \int_0^\infty e^{-st} \int_\tau^\infty \frac{(t-\tau)^{t^2-1}}{\Gamma(t^2)} \tau^\lambda dt d\tau \\
= & \int_0^\infty e^{-s(\tau+r)} \int_0^\infty \frac{r^{(\tau+r)^2-1}}{\Gamma((\tau+r)^2)} \tau^\lambda dr d\tau \\
= & \int_0^\infty e^{-s\tau} \tau^\lambda \int_0^\infty e^{-sr} \frac{r^{(\tau+r)^2-1}}{\Gamma((\tau+r)^2)} dr d\tau \\
= & \int_0^\infty e^{-s\tau} \tau^\lambda L\left[\frac{r^{(r+\tau)^2-1}}{\Gamma((r+\tau)^2)}; s \right] d\tau. \qquad (3.14)
\end{aligned}
$$

By (3.13) and (3.14), we could assure that there is no explicit connection between $L[x(t); s]$ and $L[I_{0+}^{p(t,x(t))} x(t); s]$.

By the similar arguments, we could get the same result as Lemma 3.2 for variable-order fractional integral defined by (3.10).

Lemma 3.4. Let $x(t), p(t, x(t))$ be real functions, assume that variable-order fractional integrals $I_{0+}^{p(t,x(t))} x(t)$ defined by (3.4) exists, and the Laplace transform $L[x(t); s]$ and $L[I_{0+}^{p(t,x(t))} x(t); s]$ exist. There is no explicit connection between $L[x(t); s]$ and $L[I_{0+}^{p(t,x(t))} x(t); s]$.

Proof. By definition of variable-order fractional integral (3.10) and Laplace transform, we get that

$$L[I_{0+}^{p(t,x(t))}x(t);s] = \int_0^\infty e^{-st} \int_0^t \frac{(t-\tau)^{p(\tau,x(\tau))-1}}{\Gamma(p(\tau,x(\tau)))} x(\tau)d\tau dt$$

$$= \int_0^\infty e^{-st} \int_t^\infty \frac{(t-\tau)^{p(\tau,x(\tau))-1}}{\Gamma(p(\tau,x(\tau)))} x(\tau)dt d\tau$$

$$= \int_0^\infty e^{-s(\tau+r)} \int_0^\infty \frac{r^{p(\tau,x(\tau))-1}}{\Gamma(p(\tau,x(\tau)))} x(\tau)dr d\tau$$

$$= \int_0^\infty \frac{e^{-s\tau}}{\Gamma(p(\tau,x(\tau)))} x(\tau) \int_0^\infty e^{-sr} r^{p(\tau,x(\tau))-1}dr d\tau$$

$$= \int_0^\infty \frac{e^{-s\tau}}{\Gamma(p(\tau,x(\tau)))} x(\tau) L[r^{p(\tau,x(\tau))-1};s]d\tau$$

$$= \int_0^\infty \frac{e^{-s\tau}}{\Gamma(p(\tau,x(\tau)))} x(\tau) \frac{\Gamma(p(\tau,x(\tau)))}{s^{p(\tau,x(\tau))}}d\tau$$

$$= \int_0^\infty e^{-s\tau} s^{-p(\tau,x(\tau))} x(\tau)d\tau.$$

These imply that we couldn't know explicit connection between $L[x(t);s]$ and $L[I_{0+}^{p(t,x(t))}x(t);s]$. Thus, we complete the proof.   □

Example 3.6 We consider the Laplace transforms of functions $t^\lambda, \lambda > -1$ and $I_{0+}^{p(t)}t^\lambda = \int_0^t \frac{(t-s)^{s^2-1}}{\Gamma(s^2)} s^\lambda ds$. We could know

$$L[I_{0+}^{p(t)}t^\lambda;s] = \int_0^\infty e^{-st} \int_0^t \frac{(t-\tau)^{\tau^2-1}}{\Gamma(\tau^2)} \tau^\lambda d\tau dt$$

$$= \int_0^\infty e^{-st} \int_\tau^\infty \frac{(t-\tau)^{\tau^2-1}}{\Gamma(\tau^2)} \tau^\lambda dt d\tau$$

$$= \int_0^\infty e^{-s(\tau+r)} \int_0^\infty \frac{r^{\tau^2-1}}{\Gamma(\tau^2)} \tau^\lambda dr d\tau$$

$$= \int_0^\infty \frac{e^{-s\tau}}{\Gamma(\tau^2)} \tau^\lambda L[r^{\tau^2-1};s]d\tau$$

$$= \int_0^\infty e^{-s\tau} s^{-\tau^2} \tau^\lambda d\tau. \tag{3.15}$$

By (3.13) and (3.15), we could assure that there is no explicit connection between $L[x(t);s]$ and $L[I_{0+}^{p(t,x(t))}x(t);s]$.

In view of Lemmas 3.2-3.3 and the connection between Laplace transforms of function $x(t)$ and its derivative $x'(t)$, we couldn't obtain the Laplace transform formula for variable-order fractional derivatives (3.11) and (3.12).

Lemma 3.5. Let $x(t), p(t, x(t))$ be real functions, assume that variable-order fractional derivatives $D_{0+}^{p(t,x(t))}x(t)$ defined by $(3.11), (3.12)$ exist, and the Laplace transform $L[x(t); s]$ and $L[D_{0+}^{p(t,x(t))}x(t); s]$ exist. There is no obvious connection between $L[D_{0+}^{p(t,x(t))}x(t); s]$ and $L[x(t); s]$.

REFERENCES

[1] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier B. V., Amsterdam, 2006.

[2] A. Razminia, A.F. Dizaji, V.J. Majd, Solution existence for non-autonomous variable-order fractional differential equations, Mathematical and Computer Modelling, 55(2012) 1106-1117.

[3] S.G.Samko, Fractional integration and differentiation of variable order, Analysis Mathematica, 21(1995) 213-236.

[4] S.G.Samko, B.Boss, Integration and differentiation to a variable fractional order, Integral Transforms and Special Functions, 1(4)(1993) 277-300.

[5] C.C.Tseng, Series expansion design for variable fractional order integrator and Differentiator using logarithm, Signal Processing, 88(2008) 278-2292.

[6] C.F.M. Coimbra, Mechanics with variable-order differential operators, Annalen der Physik, 12(2003) 692-703.

[7] C.F. Lorenzo, T.T. Hartley, Variabe order and distributed order fractional operators, Nonlinear Dynamics, 29(2002) 57-98.

[8] C.H. Chan, J.J. Shyu, R. H.H. Yang, An iterative method for the design of variable fractional-order FIR differintegrators, Signal Processing, 89(2009) 320-327.

[9] C.H. Chan, J.J. Shyu, R.H.H. Yang, A new structure for the design of wideband variable fractional-order FIR differentiator, Signal Processing, 90(2010) 2594-2604.

[10] D Valerio, J.S. Costa, Variable-order fractional derivatives and their numerical approximations. Signal Processing, 91(2011)470-483.

[11] C.H. Chan, J.J.Shyu, R.H.H. Yang , Iterative design of variable fractional-order IIR differintegrators, Signal Processing, 90(2010) 670-678.

[12] H.G. Sun, W. Chen, Y.Q. Chen, Variable-order fractional differential operators in anomalous diffusion modeling, Physica A, 388(2009) 4586-4592.

[13] H.G. Sun, W. Chen, H. Wei, Y.Q. Chen, A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems, The European Physical Journal Special Topics, 193(2011) 185-192.

[14] H. Sheng, H.G. Sun, Y.Q. Chen, T. Qiu, Synthesis ofmultifractional Gaussian noises based on variable-order fractional operators, Signal Processing, 91(2011) 1645-1650.

[15] H. Sheng, H.G. Sun, C. Coopmans, Y.Q. Chen, G.W. Bohannan, Physical experimental study of variable-order fractional integrator and differentiator, in: Proceedings of the 4th IFAC Workshop on Fractional Differentiation and its Applications(FDA'10), 2010.

[16] R. Lin, F. Liu, V. Anh, I. Turner, Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation, Applied Mathematics and Computation, 2(2009) 435-445.

[17] X.Y. Li, B. Wu, A numerical technique for variable fractional functional boundary value probems, Applied Mathematics Letters, 43(2015) 108-113.

[18] Y.M. Chen, Y.Q. Wei, D.Y, Liu, H.Yu, Numerical solution for a class of nonlinear variable order fractional differential equations with Legendre Wavalets, Applied Mathematics Letters, 46(2015) 83-88.

[19] Y. Jia, M. Xu, Y.Z. Lin, A numberical solution for variable order fractional functional differential equations, Applied Mathematics Letters, 64(2017) 125-130.

[20] X. Yang, J. A. Tenreiro Machado, A new fractional operator of variable order: Application in the description of anomalous diffusion, Physica A, 481(2017) 276-283.

[21] G.Pang, P.Perdikaris, W.Cai, G.E.Karniadakis, Discovering variable fractional orders of advection-dispersion equations from field date using multi-fidelity Bayesan optimization, Journal of Computational Physicis, 348(2017) 694-714.

[22] H.Sun, X.Liu, Y.Zhang, G.Pang, R.Garrard, A fast semi-discrete Kansa method to solve the two-dimensional spatiotemporal fractional diffusion equation, Journal of Computational Physicis, 345(2017) 74-90.

[23] J. Yang, H. Yao, B. Wu, An efficient numberical method for variable order fractional functional differential equation, Applied Mathematics Letters, 76(2018) 221-226.

[24] J. Vanterler da C.Sousa, E. Capelas de Oliverira, Two new fractional derivatives of variable order with non-singular kernel and fractional differential equation, Computational and Applied Mathematics, 37(4) (2018) 5375-5394.

[25] Y. Kian, E. Sorsi, M. Yamamoto, On time-fractional diffusion equations with space-dependent variable order, Annales Henri Poincaré, 19 (2018) 3855-3881.

[26] W. Malesza, M. Macias, D. Sierociuk, Analysitical solution of fractional variable order differential equations, Journal of Computational and Applied Mathematics, 348 (2019) 214-236.

[27] M. Hajipour, A. Jajarmi, D. Baleanu, H. Sun, On an accurate discretization of a variable-order fractional reaction-diffusion equation, Commun Nonlinear Sci Numer Simulat, 69(2019) 119-133.

[28] S. Zhang, S. Li, L. Hu, The existence and uniqueness result of solutions to initial value problems of nonlinear diffusion equations involving with the conformable variable derivative, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, (2018) https://doi.org/10.1007/s13398-018-0572-2.

[29] S. Zhang, The uniqueness result of solutions to initial value problems of differential equations of variable-order, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 112(2018) 407-423.

SHUQIN ZHANG

SCHOOL OF SCIENCE, CHINA UNIVERSITY OF MINING AND TECHNOLOGY, BEIJING, CHINA
  E-mail address: `zsqjk@163.com`

LEI HU

SCHOOL OF SCIENCE, SHANDONG JIAOTONG UNIVERSITY, JINAN, 250357, CHINA
  E-mail address: `huleimath@163.com`