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UNIQUENESS OF CERTAIN POWER OF A MEROMORPHIC FUNCTION SHARING A SET WITH ITS DIFFERENTIAL POLYNOMIAL

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ABSTRACT. In this paper we deal with the uniqueness problem of meromorphic function that share a set of small functions with its differential polynomial and obtain some results which improve and generalize the recent results due to [4].

1. INTRODUCTION

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane $\mathbb{C}$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [9, 18, 19]. In particular, for a meromorphic function f, $S(f)$ denotes the family of all meromorphic functions w such that $T(r, w) = S(r, f) = o(T(r, f))$, where $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. For convenience, we agree that $S(f)$ includes all constant functions and $\tilde{S}(f) := S(f) \cup \{\infty\}$.

For a meromorphic function f and a set $S \subset \mathbb{C}$, we define

$$E(S,\, f) = \bigcup_{a\in S}\{z \mid f(z) - a = 0, \text{counting multiplicities}\},$$

$$\overline{E}(S,\, f) = \bigcup_{a\in S}\{z \mid f(z) - a = 0, \text{ignoring multiplicities}\}.$$

We say that f and g share a set S CM, resp. IM, provided that $E(S,\, f) = E(S,\, g)$, resp. $\overline{E}(S,\, f) = \overline{E}(S,\, g)$. As a special case, let $S = \{a\}$, where $a \in \tilde{\mathbb{C}}$. If $E(S,\, f) = E(S,\, g)$, resp. $\overline{E}(S,\, f) = \overline{E}(S,\, g)$, we say that f and g share the value a CM, resp. IM.

Many research works on entire and meromorphic function f and its derivative $f^{(k)}$ have been done by many mathematicians in the world (see [2], [7], [10], [17], [21], [24], [26]). Recently, there have been an increasing interest in studying entire and meromorphic functions sharing a set of small functions with their derivative. In this direction we need the following definitions.

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Definition 1.1 (see [11, 12]). Let p be a non-negative integer or infinity. For $c \in \mathbb{C} \cup \{\infty\}$, we denote by $E_f(a,p)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq p$ and $p+1$ times if $m > p$. If $E_f(a,p) = E_g(a,p)$, we say that f, g share the value a with weight p.

We write f, g share (a, p) to mean that f, g share the value a with weight p. Clearly if f, g share (a, p), then f, g share (a, q) for all integer $q\,(0 \leq q < p)$. Also, we note that f, g share a value a IM or CM if and only if share $(a,0)$ or (a,∞) respectively.

Let S be a subset of $S(f) \cup \{\infty\}$, we can get the definition of $E_f(S,p)$ as

$$E_f(S,p) = \bigcup_{a \in S} E_f(a,p).$$

Definition 1.2 (see [2, 23]). When f and g share 1 IM, we denote by $N_L(r,1;f)$ the counting function of the 1-points of f whose multiplicities are greater than 1-points of g; Similarly, we have $N_L(r,1;g)$. Let z_0 be a zero of $f-1$ of multiplicity p and a zero of $g-1$ of multiplicity q, we also denote by $N_{11}(r,1;f)$ the counting function of those 1-points of f where $p=q=1$; $\overline{N}_E^{(2}(r,1;f)$ denotes the counting function of those 1-points of f where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way, one can define $N_{11}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$.

In 1996, the following conjecture was proposed by R. Brück [5].

Conjecture 1.1. Let f be non-constant entire function and $\rho_1(f)$ is not a positive integer or infinite. If f and $f^{'}$ share one finite value a CM, then

$$\frac{f^{'} - a}{f - a} = c,$$

for some non-zero constant c, where $\rho_1(f)$ is the first iterated order of f defined by

$$\rho_1(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r,f)}{\log r}.$$

In 1996, R. Brück [5] proved that the conjecture is true if $a = 0$ or $N(r,0;f^{'}) = S(r,f)$. In 1998, G. G. Gundersen and L. Z. Yang [8] proved that the conjecture is true if f is of finite order and fails, in general, for meromorphic functions. In 2004, Z. X. Chen and K. H. Shon [6] proved that the conjecture is true for entire function of order $\rho_1(f) < \frac{1}{2}$. In 2005, A. Al-khaladi [1] proved that the conjecture is true for meromorphic function f when $N(r,0;f^{'}) = S(r,f)$.

In 2008, L. Z. Yang and J. L. Zhang [20] obtained the following results.

Theorem A. Let f be a non-constant entire function, $n \geq 7$ be an integer. Denote $\mathcal{F} = f^n$. If $\mathcal{F}$ and $\mathcal{F}^{'}$ share 1 CM, then $\mathcal{F} \equiv \mathcal{F}^{'}$ and f assumes the form

$$f(z) = ce^{\frac{z}{n}},$$

where c is a nonzero constant.

Theorem B. Let f be a non-constant meromorphic function and $n \geq 12$ be an integer. Denote $\mathcal{F} = f^n$. If $\mathcal{F}$ and $\mathcal{F}^{'}$ share 1 CM, then $\mathcal{F} \equiv \mathcal{F}^{'}$ and f assumes the form

$$f(z) = ce^{\frac{z}{n}},$$

where c is a nonzero constant.

In 2009, J. L. Zhang and L. Z. Yang [25] improved Theorems A and B to a large extent and obtained the following results.

Theorem C. Let f be a non-constant entire function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and $n > k+1$, then $f^n \equiv (f^n)^{(k)}$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^k = 1$.

Theorem D. Let f be a non-constant meromorphic function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and $n > k + 1 + \sqrt{k+1}$, then $f^n \equiv (f^n)^{(k)}$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^k = 1$.

Regarding Theorems A-D, one may ask the following questions.

Question 1.1. Can the nature of sharing 1 or $a(z)$ CM be further relaxed in Theorems A and C?

Question 1.2. What will happen when 1 or $a(z)$ are replaced by a set $S_m = \{a(z), a(z)\omega, ..., a(z)\omega^{m-1}\}$ in Theorems A-D, where $\omega = cos\frac{2\pi}{m} + isin\frac{2\pi}{m}$ and m is a positive integer?

In 2016, H. Y. Xu, C. F. Yi and H. Wang [16] with the idea of weighted sharing of values, the solution of the above questions was investigated and obtained the following results.

Theorem E. Let f be a non-constant entire function, n, k, p, m be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $E_{f^n}(S_m, p) = E_{(f^n)^{(k)}}(S_m, p)$ and

$$n > \max\left\{k+1, k + \frac{\eta}{pm}\right\},$$

where $\eta = k + p + 2$, then $f^n \equiv t(f^n)^{(k)}$ with $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^{km} = 1$.

Theorem F. Let f be a non-constant meromorphic function, n, k, p, m be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $E_{f^n}(S_m, p) = E_{(f^n)^{(k)}}(S_m, p)$ and

$$n > \max\left\{k+1, \frac{p(m+1)k + 2\eta}{2pm} + \frac{\sqrt{4\eta(\eta + pk) + (m-1)^2p^2k^2}}{2pm}\right\},$$

where $\eta = k + p + 2$, then $f^n \equiv t(f^n)^{(k)}$ with $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a nonzero constant and $\lambda^{km} = 1$.

Next we recall the following definition.

Definition 1.3 (see [9]) Let $n_{0j}, n_{1j}, ..., n_{kj}$ be nonnegative integers and $g = f^n$. The expression $M_j[g] = (g)^{n_{0j}}(g^{(1)})^{n_{1j}}...(g^{(k)})^{n_{kj}}$ is called a differential monomial generated by g of degree $d_{Mj} = d(M_j) = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{Mj} = \sum_{i=0}^{k}(i+1)n_{ij}$.

The sum $\mathcal{P}[g] = \sum_{j=1}^{s} b_j M_j[g]$ is called a differential polynomial generated by g of degree $\bar{d}(\mathcal{P}) = \max\{d(M_j) : 1 \leq j \leq s\}$ and weight $\Gamma_{\mathcal{P}} = \max\{\Gamma_{Mj} : 1 \leq j \leq s\}$, where $T(r, b_j) = S(r, g)$ for $j = 1, 2, ..., s$.

The numbers $\underline{d}(\mathcal{P}) = \min\{d(M_j) : 1 \leq j \leq s\}$ and k (the highest order of the derivative of g in $\mathcal{P}[g]$) are called respectively the lower degree and order of $\mathcal{P}(g)$.

$\mathcal{P}[g]$ is said to be homogeneous if $\bar{d}(\mathcal{P}) = \underline{d}(\mathcal{P})$. $\mathcal{P}(g)$ is called a linear differential polynomial generated by g if $\bar{d}(\mathcal{P}) = 1$. Otherwise $\mathcal{P}[g]$ is called a non-linear differential polynomial.

We denote by $Q = \max\{\Gamma_{Mj} - d(M_j) : 1 \leq j \leq s\} = \max\{n_{1j} + 2n_{2j} + ... + kn_{kj} : 1 \leq j \leq s\}$.

Also for the sake of convenience for a differential monomial $M[g]$ we denote by $d_M = d(M)$ and $Q_M = \Gamma_M - d_M$.

Since derivative's natural extension is a differential monomial, it will be interesting to see whether Theorems E and F can remain true when $(f^n)^{(k)}$ is replaced by $M[f^n]$. In this direction, very recently Banerjee-Majumader [4] have improved Theorems E and F in the following way which in turn improve a recent result of Zhang and Yang [20, 25] as well.

Theorem G. Let f be a non-constant meromorphic function, n, k, p, m be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $E_{f^{nd_M}}(S_m, p) = E_{M[f^n]}(S_m, p)$ and if

1. $p \geq 2$ and $n > \frac{\gamma_m^p + \gamma_1^p + \sqrt{(\gamma_m^p - \gamma_1^p)^2 + 4C}}{2md_M}$, or if
2. $p = 0$ and $n > \frac{\alpha + \beta + \sqrt{(\alpha - \beta)^2 + 4D}}{2md_M}$,

where $C = \frac{(p+1)(p(k+1)d_M + 1)}{p^2}$ and $D = (Q_M + 3)(2(k+1)d_M + 1)$,

then $f^{nd_M} \equiv tM[f^n]$ with $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant with $\lambda^{mQ_M} = 1$.

Theorem H. Let f be a non-constant entire function, n, k, p, m be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $E_{f^{nd_M}}(S_m, p) = E_{M[f^n]}(S_m, p)$ and if

1. $p \geq 2$ and $n > \frac{pmQ_M + p + 1}{pmd_M}$, or if
2. $p = 0$ and $n > \frac{mQ_M + (k+1)d_M + 2}{md_M}$,

then $f^{nd_M} \equiv tM[f^n]$ with $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant with $\lambda^{mQ_M} = 1$.

In the same paper the following questions was asked:

Question 1.3. Can we replace f^n by a general linear expression $P(f)$ in anyway in Theorem G and Theorem H to get the same specific form of the function?

Question 1.4. Can we replace the differential monomial $M[f^n]$ by a differential polynomial $\mathcal{P}[f^n]$ in anyway in Theorem G and Theorem H to get the same specific form of the function?

Question 1.5. Can the lower bound of n be further reduced in Theorem G and Theorem H to get the same conclusions?

Our main intension of writing this paper is to find out the possible affirmative answer of all the above questions such that Theorems A - H can be accommodated under a single theorem which extends and improves all of them. Henceforth we need the following notations throughout the paper for the sake of convenience.

Let

$\alpha = 2Q + 3$, $\beta = mQ + (k+1)\bar{d}(\mathcal{P}) + 2$, $\gamma_m^p = mQ + 1 + \frac{1}{p}$ and $\gamma_1^p = Q + 1 + \frac{1}{p}$,

where p, m and k are three positive integers.

The following two theorems are the main results of this paper which gives an affirmative answer of the questions of Banerjee-Majumder [4] in a more convenient way.

Theorem 1.1. Let f be a non-constant meromorphic function, n, k, p, m be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $E_{f^{nd_{M_1}}+f^{nd_{M_2}}+\ldots+f^{nd_{M_s}}}(S_m, p) = E_{\mathcal{P}[f^n]}(S_m, p)$ and if

1. $p \geq 2$ and $n > \frac{\gamma_m^p + \gamma_1^p + \sqrt{(\gamma_m^p - \gamma_1^p)^2 + 4C}}{2m\bar{d}(\mathcal{P})}$, or if
2. $p = 0$ and $n > \frac{\alpha + \beta + \sqrt{(\alpha - \beta)^2 + 4D}}{2m\bar{d}(\mathcal{P})}$,

where $C = \frac{(p+1)(p(k+1)\bar{d}(\mathcal{P})+1)}{p^2}$ and $D = (Q+3)(2(k+1)\bar{d}(\mathcal{P}) + 1)$, then

$f^{nd_{M_1}} + f^{nd_{M_2}} + \ldots + f^{nd_{M_s}} \equiv t\mathcal{P}[f^n]$ with $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant with $\lambda^{m(\Gamma_j - d_{M_j})} = 1$.

Theorem 1.2. Let f be a non-constant entire function, n, k, p, m be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $E_{f^{nd_{M_1}}+f^{nd_{M_2}}+\ldots+f^{nd_{M_s}}}(S_m, p) = E_{\mathcal{P}[f^n]}(S_m, p)$ and if

1. $p \geq 2$ and $n > \frac{pmQ+p+1}{pm\bar{d}(\mathcal{P})}$, or if
2. $p = 0$ and $n > \frac{mQ+(k+1)\bar{d}(\mathcal{P})+2}{m\bar{d}(\mathcal{P})}$,

then $f^{nd_{M_1}}+f^{nd_{M_2}}+...+f^{nd_{M_s}}\equiv t\mathcal{P}[f^n]$ with $t^m=1$ and f assumes the form

$$f(z)=ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant with $\lambda^{m(\Gamma_j-d_{M_j})}=1$.

2. Some Corollaries

In Theorem 1.1 and Theorem 1.2, if we take $\mathcal{P}[f^n]=(f^n)^{(k)}$, where $n>k$, then it is clear that $\bar{d}(\mathcal{P})=1$, $Q=k$. Let $s=1$ then we get $f^{nd_{M_1}}$ and we take $d_{M_1}=1$. The following are some corollaries of the main results of this paper. What worth noticing here is that the lower bound of n is reduced as compare to Theorem E and Theorem F.

Corollary 1. Let f be a non-constant meromorphic function and n, m, p, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z)\not\equiv 0,\infty$. If $E_{f^n}(S_m,p)=E_{(f^n)^{(k)}}(S_m,p)$ and if

1. $p\geq 2$ and $n>\frac{2p+p(m+1)k+2}{2pm}+\frac{\sqrt{4(p+1)(pk+p+1)+(m-1)^2p^2k^2}}{2pm}$,
2. $p=0$ and $n>\frac{(m+3)k+6}{2m}+\frac{\sqrt{4(k+3)(2k+3)+(m-1)^2k^2}}{2m}$,

then $f^n\equiv t(f^n)^{(k)}$ where $t^m=1$ and f assumes the form

$$f(z)=ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{mk}=1$.

Corollary 2. Let f be a non-constant entire function and n, m, p, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z)\not\equiv 0,\infty$. If $E_{f^n}(S_m,p)=E_{(f^n)^{(k)}}(S_m,p)$ and if

1. $p\geq 2$ and $n>k+\frac{p+1}{pm}$, or if
2. $p=0$ and $n>k+\frac{k+3}{m}$,

then $f^n\equiv t(f^n)^{(k)}$ where $t^m=1$ and f assumes the form

$$f(z)=ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{mk}=1$.

3. Examples

The following example shows that conditions 1. and 2. in Corollary 1 and Corollary 2 are essential in order to get the conclusions.

Example 3.1. For $n\geq 2$, let the principal branch of f be given by $f(z)=\left(e^{\theta z}+2a\right)^{\frac{1}{n}}$, where $a\neq 0$ is a constant and θ is a root of the equation $z^n+1=0$. Let $S_m=\{a\}$ and $\mathcal{P}[f^n]=(f^n)^{(n)}$. Clearly $f^n=e^{\theta z}+2a$ and $\mathcal{P}[f^n]=-e^{\theta z}$ and let $s=1$ then we get $f^{nd_{M_1}}$. Here we take $d_{M_1}=1$. Therefore we see that $E_{f^{nd_{M_1}}+f^{nd_{M_2}}+...+f^{nd_{M_s}}}(S_m,\infty)=E_{\mathcal{P}[f^n]}(S_m,\infty)$ and

$$n\leq \min\left\{k+\frac{p+1}{pm},k+\frac{k+3}{m}\right\}=\min\{n+1,2n+3\}=n+1.$$

Here it is clear that

$$f^n\not\equiv t\mathcal{P}[f^n]$$

with $t^m = 1$. Also we see that f does not assume the form

$$f(z) = ce^{\frac{\lambda}{n}}z$$

with $\lambda^{m(\Gamma_{M_j} - d_{M_j})} = 1$.

The following example shows that conditions 1. and 2. used in Corollary 1 and Corollary 2 are not necessary but sufficient.

Example 3.2. Let $S_m = \{-1, 1, -i, i\}$ and f be given by $f(z) = e^{\frac{\lambda}{4}z}$, where λ is a root of the equation $z^4 + 1 = 0$. Let $\mathcal{P}[f^4] = (f^4)^{(4)}$. It is clear that $f^4(z) = e^{\lambda z}$ and $\mathcal{P}[f^4] = -e^{\lambda(z)}$. Also $E_{f^{nd_{M_1}} + f^{nd_{M_2}} + ... + f^{nd_{M_s}}}(S_m, \infty) = E_{\mathcal{P}[f^n]}(S_m, \infty)$ and

$$n \le \min\left\{k + \frac{p+1}{pm}, k + \frac{k+3}{m}\right\} = \min\{\frac{17}{4}, \frac{23}{4}\} = \frac{17}{4}.$$

But we see that $f^4 \equiv t\mathcal{P}[f^4]$ with $t^m = (-1)^4 = 1$. Also here f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}}z,$$

where $c = 1$ and $\lambda^{m(\Gamma_{M_j} - d_{M_j})} = \lambda^{16} = 1$.

The following examples show that if the conditions of Theorem 1.1 and Theorem 1.2 are satisfied, then the conclusions hold.

Example 3.3. Let $S_m = \{-1, 1, -i, i\}$ and f be given by $f(z) = 2^{\frac{1}{5}}e^{\frac{\lambda}{5}z}$, where λ is a root of the equation $z^3+1 = 0$. Let $\mathcal{P}[f^n] = 2(f^n)^{(k)}$. It is clear that $f^n(z) = 2e^{\lambda z}$ and $\mathcal{P}[f^n] = -2e^{\lambda(z)}$ with $n = 5$, $k = 3$, $m = 4$ and let $s = 1$ then we get $f^{nd_{M_1}}$. Here we take $d_{M_1} = 1$. Also $E_{f^{nd_{M_1}} + f^{nd_{M_2}} + ... + f^{nd_{M_s}}}(S_m, \infty) = E_{\mathcal{P}[f^n]}(S_m, \infty)$ and

$$n > \max\left\{k + \frac{p+1}{pm}, k + \frac{k+3}{m}\right\} = \max\{\frac{13}{4}, \frac{9}{2}\} = \frac{9}{2}.$$

Here we see that $f^{nd_{M_1}} + f^{nd_{M_2}} + ... + f^{nd_{M_s}} \equiv t\mathcal{P}[f^n]$ with $t^m = (-1)^4 = 1$. Also here f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}}z,$$

where $c = 1$ and $\lambda^{m(\Gamma_{M_j} - d_{M_j})} = \lambda^{16} = 1$.

Example 3.4. For a non-zero complex number a, let $S_m = \{a, aw, aw^2\}$, where w is the non-real cube root of unity and f is given by $f(z) = 2^{\frac{1}{n}}e^{w^{\frac{1}{nk}}z}$. It is clear that $f^n(z) = 2e^{w^{\frac{1}{k}}z}$ and $\mathcal{P}[f^n] = 2we^{w^{\frac{1}{k}}(z)}$, where $\mathcal{P}[f^n] = 2(f^n)^{(k)}$ with $n = 8$, $k = 6$, $m = 3$ and let $s = 1$ then we get $f^{nd_{M_1}}$. Here we take $d_{M_1} = 1$. Also $E_{f^{nd_{M_1}} + f^{nd_{M_2}} + ... + f^{nd_{M_s}}}(S_m, \infty) = E_{\mathcal{P}[f^n]}(S_m, \infty)$ and

$$n > \max\left\{k + \frac{p+1}{pm}, k + \frac{k+3}{m}\right\} = \max\{\frac{19}{3}, 9\} = 9.$$

Here we see that $f^{nd_{M_1}} + f^{nd_{M_2}} + ... + f^{nd_{M_s}} \equiv t\mathcal{P}[f^n]$ with $t^m = (\frac{1}{w})^3 = 1$. Also here f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}}z,$$

where $c = 1$ and $\lambda^{m(\Gamma_{M_j} - d_{M_j})} = \lambda^{18} = 1$.

4. **Preliminary Lemmas**

In this section we present some lemmas which will be needed in the sequel. Let $\mathcal{F}$, $\mathcal{G}$ be two non-constant meromorphic functions. Henceforth we shall denote by $\mathcal{H}$ the following function

$$\mathcal{H} = \left(\frac{\mathcal{F}^{''}}{\mathcal{F}^{'}} - \frac{2\mathcal{F}^{'}}{\mathcal{F}-1}\right) - \left(\frac{\mathcal{G}^{''}}{\mathcal{G}^{'}} - \frac{2\mathcal{G}^{'}}{\mathcal{G}-1}\right), \tag{1}$$

$$\mathcal{V} = \left(\frac{\mathcal{F}^{'}}{\mathcal{F}-1} - \frac{\mathcal{F}^{'}}{\mathcal{F}}\right) - \left(\frac{\mathcal{G}^{'}}{\mathcal{G}-1} - \frac{\mathcal{G}^{'}}{\mathcal{G}}\right), \tag{2}$$

and

$$\mathcal{U} = \frac{\mathcal{F}^{'}}{\mathcal{F}-1} - \frac{\mathcal{G}^{'}}{\mathcal{G}-1}. \tag{3}$$

Lemma 1 ([16]). Let f be a non-constant meromorphic function and k, p are positive integers. Then

$$N_p(r,0;f^{(k)}) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,0;f) + S(r,f).$$
$$N_p(r,0;f^{(k)}) \leq k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f).$$

Lemma 2 ([22]). Let f be a non-constant meromorphic function and $P(f) = a_n f^n + a_{n-1}f^{n-1} + ... + a_0$, where $a_0, a_1, ..., a_n$ are constants with $a_n \neq 0$. Then

$$T(r,P(f)) = nT(r,f) + S(r,f).$$

Lemma 3 ([3]). For any two non-constant meromorphic functions f_1 and f_2,

$$N_p(r,f_1 f_2) \leq N_p(r,f_1) + N_p(r,f_2).$$

Lemma 4 ([22]). Let H be given by (1), $\mathcal{F}$ and $\mathcal{G}$ be two non-constant meromorphic functions. If $H \not\equiv 0$, then

$$N_{11}(r,1;F) \leq N(r,H) + S(r,\mathcal{F}) + S(r,\mathcal{G}).$$

Lemma 5. For the differential polynomial $\mathcal{P}[f^n]$,

$$N_p(r,0;\mathcal{P}[f^n]) \leq \bar{d}(\mathcal{P})N_{p+k}\left(r,\frac{1}{f^n}\right) + Q\overline{N}(r,f) + S(r,f).$$

Proof. Clearly for any non-constant meromorphic function f, $N_p(r,f) \leq N_q(r,f)$ if $p \leq q$ and $b_1 = b_2 = ... = b_s = 1$. Now by using the above fact and Lemma 1, Lemma 3, we get

$$
\begin{aligned}
N_p(r,0;\mathcal{P}[f^n]) &\leq \sum_{j=1}^{s} N_p\left(r,\frac{1}{M_j[f^n]}\right)+S(r,f)\\
&= N_p\left(r,\frac{1}{M_1[f^n]}\right)+N_p\left(r,\frac{1}{M_2[f^n]}\right)+...+N_p\left(r,\frac{1}{M_j[f^n]}\right)\\
&+S(r,f)\\
&= N_p\left(r,\frac{1}{(f^n)^{n_{01}}+(f^{n^{(1)}})^{n_{11}}...(f^{n^{(k)}})^{n_{k1}}}\right)\\
&+N_p\left(r,\frac{1}{(f^n)^{n_{02}}+(f^{n^{(1)}})^{n_{12}}...(f^{n^{(k)}})^{n_{k2}}}\right)+...\\
&+N_p\left(r,\frac{1}{(f^n)^{n_{0s}}+(f^{n^{(1)}})^{n_{1s}}...(f^{n^{(k)}})^{n_{ks}}}\right)+S(r,f)\\
&= N_p\left(r,\frac{1}{\prod_{i=0}^{k}(f^{n^{(i)}})^{n_{i1}}}\right)+N_p\left(r,\frac{1}{\prod_{i=0}^{k}(f^{n^{(i)}})^{n_{i2}}}\right)+...\\
&+N_p\left(r,\frac{1}{\prod_{i=0}^{k}(f^{n^{(i)}})^{n_{is}}}\right)+S(r,f)\\
&= \sum_{i=0}^{k} n_{i1}N_p\left(r,\frac{1}{(f^n)^{(i)}}\right)+\sum_{i=0}^{k} n_{i2}N_p\left(r,\frac{1}{(f^n)^{(i)}}\right)+...\\
&+\sum_{i=0}^{k} n_{is}N_p\left(r,\frac{1}{(f^n)^{(i)}}\right)+S(r,f)\\
&= \sum_{i=0}^{k}\left[(n_{i1}+n_{i2}+...+n_{is})N_p\left(r,\frac{1}{(f^n)^{(i)}}\right)\right]+S(r,f)\\
&\leq \sum_{i=0}^{k}\left[(n_{i1}+n_{i2}+...+n_{is})\left\{N_{p+i}\left(r,\frac{1}{f^n}\right)+i\overline{N}(r,f)\right\}\right]\\
&+S(r,f)\\
&\leq \max_{1\leq j\leq s}\left\{\sum_{i=0}^{k} n_{ij}N_{p+k}\left(r,\frac{1}{f^n}\right)\right\}+\max_{1\leq j\leq s}\{\sum_{i=0}^{k} in_{ij}\overline{N}(r,f)\}+S(r,f)\\
&\leq n\bar{d}(\mathcal{P})N_{p+k}\left(r,\frac{1}{f}\right)+Q\overline{N}(r,f)+S(r,f).
\end{aligned}
$$

Lemma 6. Let f be a non-constant meromorphic function and $a \equiv a(z)$ be a small meromorphic functions of f such that $a(z) \not\equiv 0, \infty$ and let $\mathcal{F}_1 = \frac{f^{nd_{M_1}}+f^{nd_{M_2}}+...+f^{nd_{M_s}}}{a}$ and $\mathcal{G}_1 = \frac{\mathcal{P}[f^n]}{a}$. Let $\mathcal{V}$ be given by (2) and $\mathcal{F} = \mathcal{F}_1^m$ and $\mathcal{G} = \mathcal{G}_1^m$. If n, m, and k are positive integers such that $n > k$ and $\mathcal{V} \equiv 0$, then $f^{nd_{M_1}} + f^{nd_{M_2}} + ... + f^{nd_{M_s}} \equiv t\mathcal{P}[f^n]$, where $t^m = 1$ and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{m(\Gamma_{M_j}-d_{M_j})} = 1$.

Proof. From $\mathcal{V} \equiv 0$ and the definitions of F, G, we get

$$1 - \frac{1}{\mathcal{F}_1^m} \equiv \mathcal{A} - \frac{\mathcal{A}}{\mathcal{G}_1^m}, \tag{4}$$

where $\mathcal{A}$ is a non-zero constant. We now consider the following cases.

Case 1. Let $N(r,\infty;f) = S(r,f)$. If $\mathcal{A} \neq 1$, then from (4) we have

$$\overline{N}\left(r, \frac{1}{1-\mathcal{A}}; \mathcal{F}_1^m\right) = \overline{N}(r,\infty;\mathcal{G}_1^m) = S(r,f).$$

By the Second Fundamental Theorem and definitions of $\mathcal{F}_1$, $\mathcal{G}_1$, we have

$$T(r,\mathcal{F}_1^m) \leq \overline{N}(r,\infty;\mathcal{F}_1^m) + \overline{N}(r,0;\mathcal{F}_1^m) + \overline{N}\left(r, \frac{1}{1-\mathcal{A}}; \mathcal{F}_1^m\right) + S(r,f).$$

i.e.,

$$mn[d_{M_1} + d_{M_2} + ... + d_{M_s}]T(r,f) \leq \overline{N}(r,0;f) + S(r,f),$$

which is not possible.

Case 2. Let $N(r,\infty;f) \neq S(r,f)$. Then there exists a z_0 which is not a zero or pole of $a(z)$ such that $\frac{1}{f(z_0)} = 0$, so $\frac{1}{\mathcal{F}_1(z_0)} = \frac{1}{\mathcal{G}_1(z_0)} = 0$. Therefore, from (4) we get $\mathcal{A} = 1$.

Thus, by (4) and $\mathcal{A} = 1$, then $\mathcal{F}_1^m = \mathcal{G}_1^m$, i.e.,

$$f^{nd_{M_1}} + f^{nd_{M_2}} + ... + f^{nd_{M_s}} \equiv t\mathcal{P}[f^n], \tag{5}$$

where $t^m = 1$ and $b_1 = b_2 = ... = b_s = 1$. Now if z_0 be a zero of f with multiplicity p, then z_0 is a zero of $f^{nd_{M_1}} + f^{nd_{M_2}} + ... + f^{nd_{M_s}}$ with multiplicity $np\bar{d}(\mathcal{P})$ and a zero of $\mathcal{P}[f^n]$ with multiplicity $np\bar{d}(\mathcal{P}) - Q$. Therefore,

$$np\bar{d}(\mathcal{P}) = np\bar{d}(\mathcal{P}) - Q,$$

which is not possible. Thus it is obvious that 0 is a Picard exceptional value of f. Similarly we can get that ∞ is also a Picard exceptional value of f. Then from (5) we have

$$f(z) = ce^{\frac{\lambda}{n}}z,$$

where c is a non-zero constant and $\lambda^{m(\Gamma_{M_j} - d_{M_j})} = 1$.

Lemma 7. Let V be given by (2), and $\mathcal{F}, \mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 6 and n, m be positive integers. If $V \not\equiv 0$, then

$$(mn\bar{d}(\mathcal{P}) - 1)\overline{N}(r,\infty;f) \leq N(r,\infty;\mathcal{V}) + S(r,f).$$

Proof. Using the same arguments as in Lemma 8 [4], we can easily obtain Lemma 7.

Lemma 8. Let $\mathcal{U}$ be given by (3) and $\mathcal{F}, \mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 6. If n, m are positive integers such that $n > k$ and $U \equiv 0$, then

$$f^{nd_{M_1}} + f^{nd_{M_2}} + ... + f^{nd_{M_s}} \equiv t\mathcal{P}[f^n],$$

where $t^m = 1$, and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{m(\Gamma_j - d_{M_j})} = 1$.

Proof. Using the same arguments as in Lemma 9 [4], we can easily obtain Lemma

8.

Lemma 9. Let $\mathcal{U}$ be given by (3) and $\mathcal{F}$, $\mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 6. If n, m, k are positive integers such that $n > k$ and $\mathcal{U} \not\equiv 0$, then

$$[(n\bar{d}(\mathcal{P}) - Q)m - 1]\overline{N}(r, 0; f) \leq N(r, \infty; \mathcal{U}) + S(r, f).$$

Proof. Using the same arguments as in Lemma 10 [4], we can easily obtain Lemma 9.

Lemma 10. Let $\mathcal{F}$, $\mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$ be as in Lemma 6 and $\mathcal{V}$ as in (2). Now if $n > k$ and $E_p(1, \mathcal{F}) = E_p(1, \mathcal{G})$ and $\mathcal{V} \not\equiv 0$, then the following holds:

1. When $p \geq 2$, then

$$\left\{mn\bar{d}(\mathcal{P}) - 1 - Q - \frac{1}{p}\right\}\overline{N}(r, \infty; f) \leq \left\{(k+1)\bar{d}(\mathcal{P}) + \frac{1}{p}\right\}\overline{N}(r, 0; f) + S(r, f). \tag{6}$$

2. When $p = 0$, then

$$\left\{mn\bar{d}(\mathcal{P}) - 1 - 2(Q+1)\right\}\overline{N}(r, \infty; f) \leq \left\{2(k+1)\bar{d}(\mathcal{P}) + 1\right\}\overline{N}(r, 0; f) + S(r, f). \tag{7}$$

Proof. Using the same arguments as in Lemma 11 [4], we can easily obtain Lemma 10.

Lemma 11. Let $\mathcal{F}$, $\mathcal{G}, \mathcal{F}_1$, $\mathcal{G}_1$ be as in Lemma 6 and $\mathcal{U}$ as in (3). Now if $n > k$ and $E_p(1, \mathcal{F}) = E_p(1, \mathcal{G})$ and $\mathcal{U} \not\equiv 0$, then the following holds:

1. When $p \geq 2$, then

$$\left\{(n\bar{d}(\mathcal{P}) - Q)m - 1 - \frac{1}{p}\right\}\overline{N}(r, 0; f) \leq \left\{1 + \frac{1}{p}\right\}\overline{N}(r, \infty; f) + S(r, f). \tag{8}$$

2. When $p = 0$, then

$$\left\{(n\bar{d}(\mathcal{P}) - Q)m - (k+1)\bar{d}(\mathcal{P}) - 2\right\}\overline{N}(r, 0; f) \leq \{Q + 3\}\overline{N}(r, \infty; f) + S(r, f). \tag{9}$$

Proof. Using the same arguments as in Lemma 12 [4], we can easily obtain Lemma 10.

Lemma 12 ([4]). Let $\mathcal{F}$ and $\mathcal{G}$ be two non-constant meromorphic functions such that $E_p(1, \mathcal{F}) = E_p(1, \mathcal{G})$ and $\mathcal{H} \not\equiv 0$ and $p = 0$, then

$$\begin{aligned} T(r, \mathcal{F}) + T(r, \mathcal{G}) \leq\ & 2N_2(r, 0; \mathcal{F}) + 2N_2(r, 0; \mathcal{G}) + 6\overline{N}(r, \infty; \mathcal{F}) + 3\overline{N}_L(r, 1; \mathcal{F}) \\ & + 3\overline{N}_L(r, 1; \mathcal{G}) + S(r, \mathcal{F}). \end{aligned}$$

Lemma 13 ([4]). Let $\mathcal{F}$ and $\mathcal{G}$ be two non-constant meromorphic functions such that $E_p(1, \mathcal{F}) = E_p(1, \mathcal{G})$ and $\mathcal{H} \not\equiv 0$ and $p \geq 2$, then

$$T(r, \mathcal{F}) + T(r, \mathcal{G}) \leq 2N_2(r, 0; \mathcal{F}) + 2N_2(r, 0; \mathcal{G}) + 6\overline{N}(r, \infty; \mathcal{F}) + S(r, \mathcal{F}).$$

Lemma 14. Let $\mathcal{H}$ be given by (1) and F, G, F_1 and G_1 be given by Lemma 6. If n, m and k are positive integers such that $n > k$ and $\overline{N}(r, \infty; f) = N(r, 0; f) = S(r, f)$ and $\mathcal{H} \equiv 0$, then

$$f^{nd_{M_1}} + f^{nd_{M_2}} + ... + f^{nd_{M_s}} \equiv t\mathcal{P}[f^n],$$

where $t^m = 1$, and f assumes the form

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^{m(\Gamma_{M_j} - d_{M_j})} = 1$.

Proof. Using the same arguments as in Lemma 15 [4], we can easily obtain Lemma 9.

5. Proofs of the Theorems

Proof of Theorem 1.1. Let $\mathcal{F}_1 = \frac{f^{ndM_1}+f^{ndM_2}+\ldots+f^{ndM_s}}{a}$ and $\mathcal{G}_1 = \frac{\mathcal{P}[f^n]}{a}$ and $\mathcal{F} = \mathcal{F}_1^m$, $\mathcal{G} = \mathcal{G}_1^m$, where f is a non-constant meromorphic function. Now we discuss the following cases.

Case 1. If $\mathcal{UV} \equiv 0$, then by using Lemma 6 and Lemma 8, we get the conclusions of the Theorem 1.1.

Case 2. If $\mathcal{UV} \not\equiv 0$, then from the assumption of Theorem 1.1, we see that $E_p(1,\mathcal{F}) = E_p(1,\mathcal{G})$.

Subcase 2.1. When $p \geq 2$, then by using Lemma 10 and Lemma 11, we get

$$\begin{aligned}&\left\{mn\bar{d}(\mathcal{P}) - 1 - Q - \frac{1}{p}\right\}\left\{(n\bar{d}(\mathcal{P}) - Q)m - 1 - \frac{1}{p}\right\}\overline{N}(r,\infty;f)\\ &\leq \left\{(k+1)\bar{d}(\mathcal{P}) + \frac{1}{p}\right\}\left\{1 + \frac{1}{p}\right\}\overline{N}(r,\infty;f) + S(r,f). \qquad (10)\end{aligned}$$

and

$$\begin{aligned}&\left\{mn\bar{d}(\mathcal{P}) - 1 - Q - \frac{1}{p}\right\}\left\{(n\bar{d}(\mathcal{P}) - Q)m - 1 - \frac{1}{p}\right\}\overline{N}(r,0;f)\\ &\leq \left\{(k+1)\bar{d}(\mathcal{P}) + \frac{1}{p}\right\}\left\{1 + \frac{1}{p}\right\}\overline{N}(r,0;f) + S(r,f). \qquad (11)\end{aligned}$$

Now from the equations (10) and (11), we get

$$\left\{(mn\bar{d}(\mathcal{P}) - \gamma_1^p)(mn\bar{d}(\mathcal{P}) - \gamma_m^p) - C\right\}\overline{N}(r,\infty;f) \leq S(r,f) \qquad (12)$$

and

$$\left\{(mn\bar{d}(\mathcal{P}) - \gamma_1^p)(mn\bar{d}(\mathcal{P}) - \gamma_m^p) - C\right\}\overline{N}(r,0;f) \leq S(r,f), \qquad (13)$$

where $\gamma_1^p = Q + 1 + \frac{1}{p}$, $\gamma_m^p = mQ + 1 + \frac{1}{p}$ and $C = \left\{(k+1)\bar{d}(\mathcal{P}) + \frac{1}{p}\right\}\left\{1 + \frac{1}{p}\right\}$.

Since

$$\begin{aligned}&\left\{mn\bar{d}(\mathcal{P}) - \gamma_1^p\right\}\left\{mn\bar{d}(\mathcal{P}) - \gamma_m^p\right\} - C\\ &= m^2(\bar{d}(\mathcal{P}))^2n^2 - mn\bar{d}(\mathcal{P})\{\gamma_1^p + \gamma_m^p\} + \{\gamma_1^p\gamma_m^p - C\}\\ &= m^2(\bar{d}(\mathcal{P}))^2\left\{n - \frac{\gamma_m^p + \gamma_1^p + \sqrt{(\gamma_m^p - \gamma_1^p)^2 + 4C}}{2m\bar{d}(\mathcal{P})}\right\}\left\{n - \frac{\gamma_m^p + \gamma_1^p - \sqrt{(\gamma_m^p - \gamma_1^p)^2 + 4C}}{2m\bar{d}(\mathcal{P})}\right\},\end{aligned}$$

in view of the assumptions of Theorem 1.1, we get a contradiction from (12) and (13).

Thus we obtained from above

$$\overline{N}(r,0;f) = S(r,f) = \overline{N}(r,\infty;f). \qquad (14)$$

We now consider the following two cases:

Case 2.1.1. Let $\mathcal{H} \not\equiv 0$. Using Lemma 12 and Lemma 13 and (14), we get $T(r,f) = S(r,f)$, which is a contradiction.

Case 2.1.2. Let $\mathcal{H} \equiv 0$. Then from Lemma 14, we get the conclusion of Theorem 1.1.

Subcase 2.2. When $p = 0$, using Lemma 10 and Lemma 11, we get

$$\begin{aligned}&\left\{mn\bar{d}(\mathcal{P}) - 1 - 2(Q+1)\right\}\left\{(n\bar{d}(\mathcal{P}) - Q)m - (k+1)\bar{d}(\mathcal{P}) - 2\right\}\overline{N}(r,\infty;f)\\ &\leq \left\{2(k+1)\bar{d}(\mathcal{P}) + 1\right\}\{Q+3\}\overline{N}(r,\infty;f) + S(r,f)\end{aligned} \tag{15}$$

and

$$\begin{aligned}&\left\{mn\bar{d}(\mathcal{P}) - 1 - 2(Q+1)\right\}\left\{(n\bar{d}(\mathcal{P}) - Q)m - (k+1)\bar{d}(\mathcal{P}) - 2\right\}\overline{N}(r,0;f)\\ &\leq \left\{2(k+1)\bar{d}(\mathcal{P}) + 1\right\}\{Q+3\}\overline{N}(r,0;f) + S(r,f).\end{aligned} \tag{16}$$

Now using equations (15) and (16) and proceeding the same way as done in Subcase 2.1, the rest of the proof can be carried out. So we omit the detail.

Proof of Theorem 1.2. Since f is an entire function, we have $N(r,\infty;f) = S(r,f)$. Now if $\mathcal{U} \equiv 0$, then using Lemma 8, we get the conclusion of Theorem 1.2.

If $\mathcal{U} \not\equiv 0$, then using Lemma 9 for $p \geq 2$ we get from (11) that

$$(mn\bar{d}(\mathcal{P}) - \gamma_1^p)(mn\bar{d}(\mathcal{P}) - \gamma_m^p)\overline{N}(r,0;f) \leq S(r,f).$$

Since $n > \frac{pmQ+p+1}{pm\bar{d}(\mathcal{P})}$, we get a contradiction.

Again when $p = 0$, using Lemma 9 we get from (16)

$$\left\{mn\bar{d}(\mathcal{P}) - [2Q+3]\right\}\left\{(mn\bar{d}(\mathcal{P}) - [mQ + (k+1)\bar{d}(\mathcal{P}) + 2]\right\}\overline{N}(r,0;f) \leq S(r,f),$$

which is a contradiction since $n > \frac{mQ+(k+1)\bar{d}(\mathcal{P})+2}{m\bar{d}(\mathcal{P})}$.

Therefore $\overline{N}(r,0;f) = S(r,f)$. Now the rest of the proof follows Case 1 and Case 2 of the proof of Theorem 1.2.

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