Journal of Fractional Calculus and Applications
Vol. 11(2) July 2020, pp. 238-251.
ISSN: 2090-5858.
http://math-frac.oreg/Journals/JFCA/

BOUNDS ASSOCIATED TO HADAMARD INEQUALITY VIA GENERALIZED INTEGRAL OPERATORS AND APPLICATIONS FOR CONFORMABLE AND FRACTIONAL INTEGRALS

GHULAM FARID, MUHAMMAD RAEES, MATLOOB ANWAR

Abstract. Error bounds of Hadamard type inequalities have been studied extensively in the literature. This work is dedicated to a chain of Hadamard inequalities for various kinds of integral operators. In this study error bound for a version of the Hadamard inequality for a generalized integral operator is established. Some particular cases are discussed which have connection with already known results.

1. Fractional and Conformable Integrals

The study of fractional order derivatives and integrals received more attention after the formulation of electrochemiclal problems. In recent years the subject is studied extensively due to its applications in different areas of natural sciences such as: quantum mechanical calculations, chemical analysis of aqueous solutions, design of heat flux meters, transmission line theory etc. see [19]. For the detailed mathematical study of fractional integral and derivative operators, see [10, 11].
The classical fractional integral operator known as Riemann-Liouville fractional integral is defined as follows [10, 11]:
Definition 1. Let $f \in L_1[c,d]$. Then Riemann-Liouville fractional integral operators of order $\beta > 0$ with $c \geq 0$ are defined as follows:

$$^{\beta}J_{c^+}f\left(x\right) = \frac{1}{\Gamma\left(\beta\right)}\int_c^x (x-\tau)^{\beta-1}f(\tau)d\tau, \quad x > c, \tag{1}$$

$$^{\beta}J_{d^-}f\left(x\right) = \frac{1}{\Gamma\left(\beta\right)}\int_x^d \left(\tau-x\right)^{\beta-1}f(\tau)d\tau, \quad x < d. \tag{2}$$

Mubeen et al. [18] gave the k-analogue of Riemann-Liouville integrals.
Definition 2. Let $f \in L_1\left[c,d\right]$. Then the k-fractional integrals of order $\beta, k > 0$

2010 *Mathematics Subject Classification.* 26A33, 26D10, 26D15, 33B20.
Key words and phrases. Convex function, Generalized integral operator, Hadamard inequality, Riemann-Liouville fractional integrals, Generalized k-fractional integrals, Fractional integrals with exponential kernel.
Submitted Dec. 2, 2019. Revised Dec. 14, 2019.

with $c \geq 0$ are defined as follows:

$$^{\beta}J_{c^+}^{k}f\left(x\right)=\frac{1}{k\Gamma_k\left(\beta\right)}\int_c^x\left(x-\tau\right)^{\frac{\beta}{k}-1}f(\tau)d\tau,\quad x>c, \tag{3}$$

$$^{\beta}J_{d^-}^{k}f(x)=\frac{1}{k\Gamma_k\left(\beta\right)}\int_x^d\left(\tau-x\right)^{\frac{\beta}{k}-1}f(\tau)d\tau,\quad x<d. \tag{4}$$

Sarikaya et al. [22] introduced the notion of (k,s)-Riemann-Liouville fractional integrals as follows:

Definition 3. Let $f \in L_1[c,d]$. Then (k,s)-Riemann-Liouville fractional integral operators of order $\beta > 0$ with $c \geq 0$ are defined by:

$$^{\beta}_{s}J_{c^+}^{k}f\left(x\right)=\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_c^x\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s f(\tau)d\tau,\quad x>c, \tag{5}$$

$$^{\beta}_{s}J_{d^-}^{k}f\left(x\right)=\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_x^d\left(\tau^{s+1}-x^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s f(\tau)d\tau,\quad x<d, \tag{6}$$

where $k > 0$, $s \in \mathbb{R} - \{-1\}$.

Khalil et al. [14] gave conformable fractional integrals as follows:

Definition 4. Let $\beta \in (0,1)$. A function $f : [c,d] \to \mathbb{R}$ is β-fractional integrable on $[c,d]$ if the integral

$$I_c^{\beta}(f)(x)=\int_c^x f(\tau)d_{\beta}(\tau)=\int_c^x f(\tau)\tau^{\beta-1}d\tau,\quad x>c, \tag{7}$$

exists and is finite. $L_{\beta}^1([c,d])$ is the class of β-fractional integrable functions on $[c,d]$.

Recently Khan et al. [15] defined a generalized conformable integral operator as follows:

Definition 5. Let f be a conformable integrable function on the interval $[c,d] \subseteq [0,\infty)$. The left and right-sided genersalized conformable fractional integrals of order $\beta > 0$ with $r \in \mathbb{R}$, $\gamma \in (0,1]$, $r+\gamma \neq 0$ are defined by

$$^{\beta}_{r}J_{c^+}^{\gamma}f\left(x\right)=\frac{(r+\gamma)^{1-\beta}}{\Gamma(\beta)}\int_c^x\left(x^{r+\gamma}-\tau^{r+\gamma}\right)^{\beta-1}\tau^r f(\tau)d_{\gamma}\tau,\quad x>c, \tag{8}$$

$$^{\beta}_{r}J_{d^-}^{\gamma}f\left(x\right)=\frac{(r+\gamma)^{1-\beta}}{\Gamma(\beta)}\int_x^d\left(\tau^{r+\gamma}-x^{r+\gamma}\right)^{\beta-1}\tau^r f(\tau)d_{\gamma}\tau,\quad x<d. \tag{9}$$

A compact form of aforementioned fractional integral operator is defined as follows [10, 11]:

Definition 6. Let $f : [c,d] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(c,d]$, having continuous derivative g' on (c,d). The left and right-sided fractional integrals of a function f with respect to another function g on $[c,d]$ of order $\beta > 0$ are defined as:

$$^{\beta}_{g}J_{c^+}f\left(x\right)=\frac{1}{\Gamma\left(\beta\right)}\int_c^x\left[g(x)-g(\tau)\right]^{\beta-1}g'(\tau)f(\tau)d\tau,\quad x>c, \tag{10}$$

$$^{\beta}_{g}J_{d^-}f\left(x\right)=\frac{1}{\Gamma\left(\beta\right)}\int_x^d\left[g(\tau)-g(x)\right]^{\beta-1}g'(\tau)f(\tau)d\tau,\quad x<d. \tag{11}$$

Kwun [13] et al. gave the k-fractional analogue of integrals (10) and (11) as follows:

Definition 7. Let $g:[c,d]\to\mathbb{R}$ be an increasing and positive monotone function on $[c,d]$, having a continuous derivative g' on (c,d). The left and right-sided fractional integrals of a function f with respect to another function g on $[c,d]$ of order $\beta, k>0$ are defined by:

$$ {}_g^{\beta}J_{c^+}^{k} f\left(x\right)=\frac{1}{k\Gamma_k\left(\beta\right)}\int_c^x\left[g(x)-g(\tau)\right]^{\frac{\beta}{k}-1}g'(\tau)f(\tau)d\tau,\quad x>c, \tag{12}$$

$$ {}_g^{\beta}J_{d^-}^{k} f\left(x\right)=\frac{1}{k\Gamma_k\left(\beta\right)}\int_x^d\left[g(\tau)-g(x)\right]^{\frac{\beta}{k}-1}g'(\tau)f(\tau)d\tau,\quad x<d, \tag{13}$$

where $\Gamma_k(.)$ is the k-gamma function.

Raina [20], gave the following fractional integral operator by using special functions as follows:

Definition 8. Let $f\in L_1\left[c,d\right]$. The left and right-sided integrals with special functions are denoted and defined by

$$ {}_{\rho}^{\sigma}\zeta_{\mu,c^+} f\left(x\right)=\int_c^x\frac{\mathcal{F}_{\rho.\mu}^{\sigma}\left(w(x-\tau)^{\rho}\right)}{(x-\tau)^{1-\mu}}f(\tau)d\tau,\quad x>c, \tag{14}$$

$$ {}_{\rho}^{\sigma}\zeta_{\mu,d^-} f\left(x\right)=\int_x^d\frac{\mathcal{F}_{\rho.\mu}^{\sigma}\left(w(\tau-x)^{\rho}\right)}{(\tau-x)^{1-\mu}}f(\tau)d\tau,\quad x<d, \tag{15}$$

where $\rho,\mu>0$, coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers and

$$\mathcal{F}_{\rho.\mu}^{\sigma}\left(x\right)=\sum_{m=0}^{\infty}\frac{\sigma(m)}{\Gamma(\rho m+\mu)}x^m,\quad |x|<R,\ \text{with } R>0. \tag{16}$$

Tunc et al. [25], generalize the operator of Raina as follows:

Definition 9. For $k>0$, let $g:[c,d]\to\mathbb{R}$ be an increaing and positive monotone function having a continuous derivative g' on (c,d). The left and right sided generalized k-fractional integrals of f with respect to the function g on $[c,d]$ are respectively defined as follows:

$$ {}_{\rho}^{\sigma}\zeta_{\mu,c^+;w}^{k,g} f\left(x\right)=\int\limits_c^x\frac{\mathcal{F}_{\rho.\mu}^{\sigma,k}\left(w\left(g(x)-g(\tau)\right)^{\rho}\right)}{\left(g(x)-g(\tau)\right)^{1-\frac{\mu}{k}}}g'(\tau)f(\tau)d\tau,\quad x>c, \tag{17}$$

$$ {}_{\rho}^{\sigma}\zeta_{\mu,d^-;w}^{k,g} f\left(x\right)=\int\limits_x^d\frac{\mathcal{F}_{\rho.\mu}^{\sigma,k}\left(w\left(g(\tau)-g(x)\right)^{\rho}\right)}{\left(g(\tau)-g(x)\right)^{1-\frac{\mu}{k}}}g'(\tau)f(\tau)d\tau,\quad x<d, \tag{18}$$

with the coefficients $\sigma(n)$ $(n\in\mathbb{N}\cup\{0\})$ form a bounded sequence of positive real numbers and

$$\mathcal{F}_{\rho.\mu}^{\sigma,k}\left(x\right):=\sum_{n=0}^{\infty}\frac{\sigma(n)}{k\Gamma_k(\rho k n+\mu)}x^n,\quad (\rho,\mu>0;|x|<R)\ \text{with } R>0. \tag{19}$$

Recently Farid [3] introduced a generalized integral operator as follows:

Definition 10. Let $f,g:[c,d]\to\mathbb{R}, 0<c<d$, be the functions such that $f\in L_1[c,d]$ be positive and g be differentiable and increasing. Also let $\frac{\varphi}{x}$ be increasing function on $[0,\infty)$. Then for $x\in[c,d]$, the left and right integral operators are defined by

$$F_{c^+}^{\varphi,g}f(x)=\int_c^x\frac{\varphi\left(g(x)-g(\tau)\right)}{g(x)-g(\tau)}g'(\tau)f(\tau)d\tau,\quad x>c, \tag{20}$$

$$F_{d^-}^{\varphi,g}f(x)=\int_x^d\frac{\varphi\left(g(\tau)-g(x)\right)}{g(\tau)-g(x)}g'(\tau)f(\tau)d\tau,\quad x<d. \tag{21}$$

Following remark gives the summary of conformable and fractional integral operators which can be deduced from the last definition by different settings of φ and g.

Remark 1.

(i) If $\varphi(x)=x^{\frac{\lambda}{k}}\mathcal{F}_{\rho,\lambda}^{\sigma,k}(w(x)^\rho)$ in (20) and (21), then fractional integrals operators (17) and (18) will be re-captured.

(ii) If $g(x)=x$ in (17) and (18), then fractional integral operators (14) and (15) will be obtained.

(iii) If $\varphi(x)=x^{\lambda}\mathcal{F}_{\rho,\lambda}^{\sigma}(w(x)^\rho)$ in (20) and (21), then fractional integral operators defined by Dragomir in [1] are obtained.

(iv) If $\varphi(x)=x^{\frac{\lambda}{k}}\mathcal{F}_{\rho,\lambda}^{\sigma,k}(w(x)^\rho)$ in (20) and (21) and $g(x)=x$, then k-analogue of fractional integral operators (14) and (15) defined by Tunc et al in [25] will be re-captured.

(v) If $\varphi(x)=x^{\frac{\lambda}{k}}\mathcal{F}_{\rho,\lambda}^{\sigma,k}(w(x)^\rho)$, then (20) and (21) will give the generalized Hadamard k-fractional integral operators defined in [25], subject to the condition that $g(x)=\ln x$.

(vi) If $\varphi(x)=x^{\frac{\lambda}{k}}\mathcal{F}_{\rho,\lambda}^{\sigma,k}(w(x)^\rho)$ and $g(x)=\frac{x^{s+1}}{(s+1)}, s\in\mathbb{R}-\{-1\}$, then (20) and (21) reduced to the (k,s)-fractional integral operators with special functions defined in [25].

(vii) If $\varphi(x)=\frac{1}{\beta}\exp(-Ax), A=\frac{1-\beta}{\beta},\beta>0$, then generalized fractional integral operators with exponential kernel will obtained defined in [1] as follows:

$${}_g^{\beta}E_{c^+}f(x)=\frac{1}{\beta}\int\limits_c^x exp\left(-\frac{1-\beta}{\beta}(g(x)-g(\tau))\right)f(\tau)d\tau,\ x>c, \tag{22}$$

$${}_g^{\beta}E_{d^-}f(x)=\frac{1}{\beta}\int\limits_x^d exp\left(-\frac{1-\beta}{\beta}(g(\tau)-g(x))\right)f(\tau)d\tau,\ x<d. \tag{23}$$

(viii) If $\varphi(x)=\frac{1}{\Gamma(\beta)}x^{\beta}$, and $g(x)=-x^{-1}$, then Harmonic fractional integral operators will be obtained defined in [1] as follows:

$${}^{\beta}R_{c^+}f(x)=\frac{x^{1-\beta}}{\Gamma(\beta)}\int\limits_c^x(x-\tau)^{\beta-1}\frac{f(\tau)}{\tau^{\beta+1}}d\tau,\ x>c, \tag{24}$$

$${}^{\beta}R_{d^-}f(x)=\frac{x^{1-\beta}}{\Gamma(\beta)}\int\limits_x^d(\tau-x)^{\beta-1}\frac{f(\tau)}{\tau^{\beta+1}}d\tau,\ x<d. \tag{25}$$

(ix) If $\varphi(x)=\frac{1}{\Gamma(\mu)}x^{\mu}$, and $g(x)=\exp(\beta x)$,then β-Exponential fractional integral operators with order $\mu>0$ will be obtained [1]:

$${}^{\beta}\Im_{c^+}f(x)=\frac{\beta}{\Gamma(\mu)}\int\limits_c^x(\exp(\beta x)-\exp(\beta\tau))^{\mu-1}\exp(\beta\tau)f(\tau)d\tau,\ x>c, \tag{26}$$

$$
{}^{\beta}\Im_{d^-} f(x) = \frac{\beta}{\Gamma(\mu)} \int_x^d (\exp(\beta\tau) - \exp(\beta x))^{\mu-1} \exp(\beta\tau) f(\tau) d\tau, \ x < d. \tag{27}
$$

(x) If $\varphi(x) = x^{\beta} \ln x$, then left and right-sided lograthmic fractional integrals which were introduced in [1] will be obtained:

(xi) If $\varphi(x) = \frac{1}{k\Gamma_k(\beta)} x^{\frac{\beta}{k}}$ and $g(x) = \frac{1}{1+s} x^{1+s}, 1+s \neq 0$, then (5) and (5) will be obtained.

(xii) If $\varphi(x) = \frac{1}{\Gamma(\beta)} x^{\beta}$ and $g(x) = \ln x$, then Hadamard fractional integral operators will be obtained [10]. In recent past the researchers have utilized various kinds of integral operators espacially fractional and conformable integral operators to establish the well known Hadamard inequality for example see [2],[4], [5], [6], [7], [8], [16],[17], [23], [24], [25] and the references therein. Recently Hadamard inequality via generalized integral operators (20) and (21) is established in [21]. The aim of this paper is to study the error bounds of the Hadamard inequality for integral operators (20) and (21). These error bounds have interesting consequences for estimation of Hadamard inequalities for conformable and fractional integral operators.

The paper is organized as follows. In Section 2 an identity is established by using integral operators (20) and (21). By using this identity error bounds of the Hadamard inequality for integral operators (20) and (21) are established. In Section 3, by considering appropriate settings of functions several error bounds for corresponding Hadamard inequalities for fractional and conformable integral operators are obtained.

2. Main Results

In this paper φ and g are same as defined in Definition 1. Following notations will be use frequently in this study:

$$
\begin{aligned}
\widetilde{u}(x) &:= u(c+d-x), \\
U(x) &:= u(x) + \widetilde{u}(x),
\end{aligned}
$$

$$
\Delta_0^t(\varphi, g) = \int_0^t \frac{\varphi(g(sd+(1-s)c) - g(c))}{g(sd+(1-s)c) - g(c)} g'(sd+(1-s)c) ds,
$$

and

$$
\nabla_0^t(\varphi, g) = \int_0^t \frac{\varphi(g(d) - g(sc+(1-s)d))}{(g(d) - g(sc+(1-s)d))} g'(sc+(1-s)d) ds.
$$

The following lemma is useful to establish the bounds of the Hadamard inequality for integral operators (20) and (21).

Lemma 1. Let $u : [c, d] \to \mathbb{R}$ be a differentiable mapping on (c, d) with $c < d$. If

$u' \in L_1[c,d]$, then the following equalities for integral operators (20) and (21) hold:

$$\begin{aligned}
&\frac{u(c)+u(d)}{2}-\frac{1}{2(d-c)\left(\Delta_0^1(\varphi,g)+\nabla_0^1(\varphi,g)\right)}\left[F_{c^+}^{\varphi,g}U(b)+F_{d^-}^{\varphi,g}U(a)\right]\\
=\;&\frac{d-c}{2\left(\Delta_0^1(\varphi,g)+\nabla_0^1(\varphi,g)\right)}\int_0^1\Omega_1(t)u'(tc+(1-t)d)]dt\\
=\;&\frac{d-c}{2\left(\Delta_0^1(\varphi,g)+\nabla_0^1(\varphi,g)\right)}\int_0^1\Omega_2(t)U'(td+(1-t)c)dt,
\end{aligned}$$

where

$$\Omega_1(t)=\left(\Delta_0^{1-t}(\varphi,g)-\Delta_0^t(\varphi,g)\right)+\left(\nabla_0^{1-t}(\varphi,g)-\nabla_0^t(\varphi,g)\right),$$

$$\Omega_2(t)=\Delta_0^t(\varphi,g)+\nabla_0^t(\varphi,g).$$

Proof. It is easy to see that,

$$\begin{aligned}
&\int_0^1\left[\Delta_0^{1-t}(\varphi,g)-\Delta_0^t(\varphi,g)\right]u'(tc+(1-t)d)dt\\
=\;&\int_0^1\Delta_0^t(\varphi,g)[u'(td+(1-t)c)-u'(tc+(1-t)d)]dt \qquad (28)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\left[\nabla_0^{1-t}(\varphi,g)-\nabla_0^t(\varphi,g)\right]u'(tc+(1-t)d)dt\\
=\;&\int_0^1\nabla_0^t(\varphi,g)[u'(td+(1-t)c)-u'(tc+(1-t)d)]dt. \qquad (29)
\end{aligned}$$

Clearly,

$$\begin{aligned}
\int_0^1\Omega_2(t)U'(td+(1-t)c)dt \;=\;& \int_0^1\Delta_0^t(\varphi,g)U'(td+(1-t)c)dt\\
&+\int_0^1\nabla_0^t(\varphi,g)U'(td+(1-t)c)dt. \qquad (30)
\end{aligned}$$

Let

$$I_c=\int_0^1\Delta_0^t(\varphi,g)U'(td+(1-t)c)dt \qquad (31)$$

and

$$I_d=\int_0^1\nabla_0^t(\varphi,g)U'(td+(1-t)c)dt. \qquad (32)$$

Then one can have

$$
\begin{aligned}
(d-c)I_c &= \Delta_0^1(\varphi,g)U(d) \\
&\quad -\int_0^1 \frac{\varphi(g(td+(1-t)c)-g(c))}{(g(td+(1-t)c)-g(c))} g'(td+(1-t)c)U(td+(1-t)c)dt.
\end{aligned}
$$

By suitable change of variables and applying definition (20) and (21) of generalized integral operators, we have

$$
(d-c)I_c = \Delta_0^1(\varphi,g)[u(c)+u(d)] - \frac{1}{(d-c)} F_{d^-}^{\varphi,g} U(c). \tag{33}
$$

Similarly,

$$
(d-c)I_d = \nabla_0^1(\varphi,g)[u(c)+u(d)] - \frac{1}{(d-c)} F_{c^+}^{\varphi,g} U(d), \tag{34}
$$

From (28), (29), (30), (33) and (34), the required equalities can be acheived.

Remark 2. The aforementioned lemma holds for all kinds of integral operators comprises in Remark 1. In particular one can obtain [2, Lemma 2.1], [6, Lemma 2.4], [5, Lemma 2.3], [16, Theorem 4.1], [23, Lemma 5], [24, Lemma 2] and [25, Lemma 1] etc. Furthermore, some new equalities can be obtain for operators (5), (6), (8), (9),(12), (13), (22), (23), (24), (25), (26), (27) by using approperiate settings of φ and g as given in Remark 1.

Theorem 1. Let $g : [c,d] \to \mathbb{R}$ be a positive monotone increasing function on $(c,d]$, having continuous derivatives g' on (c,d). Let $u : [c,d] \to \mathbb{R}$ be a differentiable mapping on (c,d). If $|u'|$ is convex on $[c,d]$, then the following inequality for generalized fractional integrals (20) and (21) hold:

$$
\begin{aligned}
&\left| \frac{u(c)+u(d)}{2} - \frac{1}{2(d-c)\left(\Delta_0^1(\varphi,g)+\nabla_0^1(\varphi,g)\right)} \left[F_{c^+}^{\varphi,g}U(d) + F_{d^-}^{\varphi,g}U(c)\right] \right| \\
&\leq \frac{d-c}{\left(\Delta_0^1(\varphi,g)+\nabla_0^1(\varphi,g)\right)} \left(\frac{|u'(c)|+|u'(d)|}{2} \right) \int_0^1 t\,|\Omega_1(t)|\,dt,
\end{aligned} \tag{35}
$$

where $\Omega_1(t)$ is same as defined in Lemma 2.

Proof. By Lemma 2, and property of modulus, we have

$$
\begin{aligned}
&\left| \frac{u(c)+u(d)}{2} - \frac{1}{2(d-c)\left(\Delta_0^1(\varphi,g)+\nabla_0^1(\varphi,g)\right)} \left[F_{c^+}^{\varphi,g}U(d) + F_{d^-}^{\varphi,g}U(c)\right] \right| \\
&\leq \frac{d-c}{2\left(\Delta_0^1(\varphi,g)+\nabla_0^1(\varphi,g)\right)} \int_0^1 |\Omega_1(t)|\,|u'(tc+(1-t)d)|\,dt.
\end{aligned}
$$

By convexity of $|u'|$, we get

$$\left|\frac{u(c)+u(d)}{2}-\frac{1}{2(d-c)\left(\Delta_0^1(\varphi,g)+\nabla_0^1(\varphi,g)\right)}\left[F_{c+}^{\varphi,g}U(d)+F_{d-}^{\varphi,g}U(c)\right]\right|$$
$$\leq \frac{d-c}{2\left(\Delta_0^1(\varphi,g)+\nabla_0^1(\varphi,g)\right)}\left[|u'(c)|\int_0^1 t\,|\Omega_1(t)|\,dt+|u'(d)|\int_0^1 (1-t)\,|\Omega_1(t)|\,dt\right]$$
$$=\frac{d-c}{2\left(\Delta_0^1(\varphi,g)+\nabla_0^1(\varphi,g)\right)}\left[|u'(c)|\int_0^1 t\,|\Omega_1(t)|\,dt+|u'(d)|\int_0^1 t\,|\Omega_1(1-t)|\,dt\right].$$

Note that

$$\begin{aligned}\Omega_1(1-t) &= \Delta_0^t(\varphi,g)-\Delta_0^{1-t}(\varphi,g)+\nabla_0^t(\varphi,g)-\nabla_0^{1-t}(\varphi,g)\\ &= -\Omega_1(t).\end{aligned}$$

Using value of $\Omega_1(1-t)$ in above inequality, we get required inequality (35).

Remark 3. The aforementioned inequality gives error bounds of Hadamard inequalities of all kinds of integral operators comprises in Remark 1. In particular by using suitable settings of φ and g as given in Remark 1, one can obtain [2, Theorem 2.2], [6, Theorem 2.5], [5, Theorem 2.4], [16, Theorem 4.1], [23, Theorem 6], [23, Corollary 5], [24, Theorem 3] and [25, Theorem 2].

3. Error bounds associated to Hadamard inequalities via conformable and fractional integrals

In this section we construct error bounds of the Hadamard inequalities for various kinds of fractional and conformable integral operators.

Theorem 2. Let $g:[c,d]\to\mathbb{R}$ be a positive monotone increasing function on $(c,d]$, having continuous derivatives g' on (c,d). Let $u:[c,d]\to\mathbb{R}$ be a differentiable mapping on (c,d). If $|u'|$ is convex on $[c,d]$, then the following inequality for operators (12) and (13) holds:

$$\left|\frac{u(c)+u(d)}{2}-\frac{\Gamma_k(\beta+k)}{4\left[g(d)-g(c)\right]^{\frac{\beta}{k}}}\left[{}_g^\beta J_{c^+}^k U(d)+{}_g^\beta J_{d^-}^k U(c)\right]\right|$$
$$\leq \frac{{}_k^\beta A^g\,(c,d)}{4\left[g(d)-g(c)\right]^{\frac{\beta}{k}}(d-c)}(|u'(c)|+|u'(d)|), \tag{36}$$

where,

$${}_k^\beta A^g\,(c,d)={}_k^\beta\,\chi^g(d,d)+{}_k^\beta\,\chi^g(c,d)-{}_k^\beta\,\chi^g(d,c)-{}_k^\beta\,\chi^g(c,c), \tag{37}$$

and

$${}_k^\beta\chi^g(x,y):=\int_c^{\frac{c+d}{2}}|x-t|\,|g(y)-g(t)|^{\frac{\beta}{k}}\,dt-\int_{\frac{c+d}{2}}^{d}|x-t|\,|g(y)-g(t)|^{\frac{\beta}{k}}\,dt, \tag{38}$$

for all $x, y \in [c, d]$.

Proof. Let us define the function φ by $\varphi(t) = \frac{t^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$. Then we have

$$\Omega_1(t) = \frac{1}{(d-c)\Gamma_k(\beta+k)}[(g(tc+(1-t)d) - g(c))^{\frac{\beta}{k}} - (g(tc+(1-t)d) - g(c))^{\frac{\beta}{k}} + (g(tc+(1-t)d) - g(c))^{\frac{\beta}{k}} - (g(tc+(1-t)d) - g(c))^{\frac{\beta}{k}}]$$

and

$$\Delta_0^1(\varphi, g) + \nabla_0^1(\varphi, g) = \frac{2}{(d-c)\Gamma_k(\beta+k)}[(g(d) - g(c))^{\frac{\beta}{k}}]. \tag{39}$$

Also by change of variables we have

$$\int_0^1 t\,|\Omega_1(t)|\,dt = \frac{1}{(d-c)^3\Gamma_k(\beta+k)}\int_c^d (d-x)\,|\psi(x)|\,dt, \tag{40}$$

where

$$\psi(x) = (g(x) - g(c))^{\frac{\beta}{k}} - (g(c+d-x) - g(c))^{\frac{\beta}{k}} - (g(d) - g(c+d-x))^{\frac{\beta}{k}} - (g(d) - g(x))^{\frac{\beta}{k}}.$$

Observe that ψ is a non-decreasing function on $[c, d]$. We have indeed,

$$\psi(c) = 2\,(g(c))^{\beta} - 2\,(g(d))^{\beta} < 0,$$
$$\psi(\frac{c+d}{2}) = 0$$

and

$$\psi(d) = 2\,(g(d))^{\beta} - 2\,(g(c))^{\beta} > 0.$$

Hence we have,

$$\int_c^d (d-x)\,|\psi(x)|\,dx = I_1 + I_2 + I_3 + I_4, \tag{41}$$

where

$$I_1 = \int_c^{\frac{c+d}{2}} (d-x)[g(c+d-x) - g(c)]^{\frac{\beta}{k}}dx - \int_{\frac{c+d}{2}}^d (d-x)[g(c+d-x) - g(c)]^{\frac{\beta}{k}}dx,$$

$$I_2 = \int_c^{\frac{c+d}{2}} (d-x)[g(d) - g(x)]^{\frac{\beta}{k}}dx - \int_{\frac{c+d}{2}}^d (d-x)[g(d) - g(x)]^{\frac{\beta}{k}}dx,$$

$$I_3 = -\int_c^{\frac{c+d}{2}} (d-x)[g(d) - g(c+d-x)]^{\frac{\beta}{k}}dx + \int_{\frac{c+d}{2}}^d (d-x)[g(d) - g(c+d-x)]^{\frac{\beta}{k}}dx,$$

$$I_4 = -\int_c^{\frac{c+d}{2}} (d-x)[g(x) - g(c)]^{\frac{\beta}{k}}dx + \int_{\frac{c+d}{2}}^d (d-x)[g(x) - g(c)]^{\frac{\beta}{k}}.$$

By (38),

$$I_2 =^{\beta}_{k} \chi^g(b,b), I_4 = -^{\beta}_{k}\chi^g(b,a), \tag{42}$$

and by suitable change of variables,

$$I_1 = -^{\beta}_{k}\chi^g(c,c), I_3 =^{\beta}_{k} \chi^g(c,d), \tag{43}$$

Using (37), (41), (42) and (43) in (40), one have

$$\int_0^1 t\,|\Omega_1(t)|\,dt = \frac{{}^{\beta}_{k}A^g(a,b)}{(b-a)^3\Gamma_k(\beta+k)} \tag{44}$$

Thus inequality (35) along with (39) and (44) reduced to the required inequality (36).

Corollary 1. Let $u : [c,d] \to \mathbb{R}$ be a differentiable mapping on (c,d). If $|u'|$ is convex on $[c,d]$, then the following inequality for operators (8) and (9) hold:

$$\begin{aligned} &\left|\frac{u(c)+u(d)}{2} - \frac{(r+\alpha)^{\beta}\Gamma(\beta+1)}{4\left[d^{r+\alpha}-c^{r+\alpha}\right]^{\beta}}\left[{}^{\beta}_{r}J^{\alpha}_{c^+}U(d) +^{\beta}_{r} J^{\alpha}_{d^-}U(c)\right]\right| \\ &\le \frac{{}^{\beta}_{r}B^{\alpha}\,(c,d)}{4\left[d^{r+\alpha}-c^{r+\alpha}\right]^{\beta}(d-c)}(|u'(c)|+|u'(d)|), \end{aligned} \tag{45}$$

where,

$${}^{\beta}_{r}B^{\alpha}(c,d) =^{\beta}_{r} \varrho^{\alpha}(d,d) +^{\beta}_{r} \varrho^{\alpha}(c,d) -^{\beta}_{r} \varrho^{\alpha}(d,c) -^{\beta}_{r} \varrho^{\alpha}(c,c),$$

and

$${}^{\beta}_{r}\varrho^{\alpha}(x,y) := \int\limits_{c}^{\frac{c+d}{2}} |x-t|\left|y^{r+\alpha}-t^{r+\alpha}\right|^{\beta} dt - \int\limits_{\frac{c+d}{2}}^{d} |x-t|\left|y^{r+\alpha}-t^{r+\alpha}\right|^{\beta} dt,$$

for all $x, y \in [c,d]$.

Proof. Considering $g(t) = \frac{t^{r+\alpha}}{r+\alpha}$, $r+\alpha \neq 0, \alpha \in (0,1)$ and $k=1$, in inequality (36), one gets the required result.

Corollary 2. Let $u : [c,d] \to \mathbb{R}$ be a differentiable mapping on (c,d). If $|u'|$ is convex on $[c,d]$, then the following inequality for operators (5) and (6) hold:

$$\begin{aligned} &\left|\frac{u(c)+u(d)}{2} - \frac{(1+s)^{\frac{\beta}{k}}\Gamma(\beta+1)}{4\left[d^{1+s}-c^{1+s}\right]^{\beta}}\left[{}^{\beta}_{s}J^{k}_{c^+}U(d) +^{\beta}_{s} J^{k}_{d^-}U(c)\right]\right| \\ &\le \frac{{}^{\beta}_{s}C^{k}\,(c,d)}{4\left[d^{1+s}-c^{1+s}\right]^{\frac{\beta}{k}}(d-c)}(|u'(c)|+|u'(d)|), \end{aligned} \tag{46}$$

where,

$${}^{\beta}_{s}C^{k}(c,d) =^{\beta}_{s} \omega^{k}(d,d) +^{\beta}_{s} \omega^{k}(c,d) -^{\beta}_{s} \omega^{k}(d,c) -^{\beta}_{s} \omega^{k}(c,c),$$

and

$${}^{\beta}_{s}\omega^{k}(x,y) := \int\limits_{c}^{\frac{c+d}{2}} |x-t|\left|y^{1+s}-t^{1+s}\right|^{\frac{\beta}{k}} dt - \int\limits_{\frac{c+d}{2}}^{d} |x-t|\left|y^{1+s}-t^{1+s}\right|^{\frac{\beta}{k}} dt,$$

for all $x, y \in [c, d]$.

Proof. By using $g(t) = \frac{t^{1+s}}{1+s}$, $s \in \mathbb{R} - \{-1\}$, in inequality (36), one gets the required inequality.

Corollary 3. Let $u : [c, d] \to \mathbb{R}$ be a differentiable mapping on (c, d). If $|u'|$ is convex on $[c, d]$, then the following inequality for β-exponential fractional integrals (26) and (27) of order μ hold:

$$\left| \frac{u(c) + u(d)}{2} - \frac{\Gamma(\mu+1)}{4\left[\exp(\beta d) - \exp(\beta c)\right]^{\mu}} \left[{}^{\mu}\Im_{c^+} U(d) +^{\mu} \Im_{d^-} U(c) \right] \right| \leq \frac{{}^{\beta}_{\mu}N\left(c, d\right)}{4\left[\exp(\beta d) - \exp(\beta c)\right]^{\mu}(d-c)} (|u'(c)| + |u'(d)|), \tag{47}$$

where,

$$ {}^{\beta}_{\mu}N(c, d) = {}^{\beta}_{\mu}\Lambda(d, d) + {}^{\beta}_{\mu}\Lambda(c, d) - {}^{\beta}_{\mu}\Lambda(d, c) - {}^{\beta}_{\mu}\Lambda(c, c),$$

and

$$ {}^{\beta}_{\mu}\Lambda(x, y) := \int_{c}^{\frac{c+d}{2}} |x - t| \left|\exp(\beta y) - \exp(\beta t)\right|^{\mu} dt - \int_{\frac{c+d}{2}}^{d} |x - t| \left|\exp(\beta y) - \exp(\beta t)\right|^{\mu} dt,$$

for all $x, y \in [c, d]$.

Proof. Replacing β by μ and using $g(t) = \exp(\beta t)$, $\beta > 0, k = 1$, in inequality (36), one gets the required inequality.

Corollary 4. Let $u : [c, d] \to \mathbb{R}$ be a differentiable mapping on (c, d). If $|u'|$ is convex on $[c, d]$, then the following inequality for operators (24) and (25) holds:

$$\left| \frac{u(c) + u(d)}{2} - \frac{(cd)^{\beta}\Gamma(\beta+1)}{4(d-c)^{\beta}} \left[{}^{\beta}R_{c^+} U(d) +^{\beta} R_{d^-} U(c) \right] \right| \leq \frac{{}^{\beta}L\left(c, d\right)}{4(d-c)^{\beta+1}} (|u'(c)| + |u'(d)|), \tag{48}$$

where,

$$ {}^{\beta}_{\nu}L(c, d) = c^{\beta} \left[{}^{\beta}\Phi(d, d) +^{\beta} \Phi(c, d) \right] - d^{\beta} \left[{}^{\beta}\Phi(c, c) -^{\beta} \Phi(d, c) \right],$$

and

$$ {}^{\beta}\Phi(x, y) := \int_{c}^{\frac{c+d}{2}} |x - t| \frac{|y - t|^{\beta}}{t^{\beta}} dt - \int_{\frac{c+d}{2}}^{d} |x - t| \frac{|y - t|^{\beta}}{t^{\beta}} dt,$$

for all $x, y \in [c, d]$.

Proof. Inequality (36), reduced to the required inequality subject to the condition that $g(t) = -t^{-1}$, $t > 0, k = 1$.

Corollary 5. Let $u : [c, d] \to \mathbb{R}$ be a differentiable mapping on (c, d). If $|u'|$ is convex on $[c,d]$, then the following inequality for Riemann integrals hold:

$$\left|\frac{u(c)+u(d)}{2}-\frac{1}{2(g(d)-g(c))}\int_c^d g'(t)U(t)dt\right| \\ \leq \frac{1}{(d-c)(g(d)-g(c))}\left(\frac{|u'(c)|+|u'(d)|}{2}\right)\left[\Theta^{c,g}(x)-\Theta^{d,g}(x)\right], \quad (49)$$

where

$$\Theta^{y,g}(x)=\int_c^{\frac{c+d}{2}}|y-x|\,g(x)dx-\int_{\frac{c+d}{2}}^d|y-x|\,g(x)dx.$$

Proof. By taking φ as identity function in inequality (35) and using the same lines as adopted in Theorem (3), we come to the desired inequality.

Theorem 3. Let $g:[c,d]\to\mathbb{R}$ be a positive monotone increasing function on $(c,d]$, having continuous derivatives g' on (c,d). Let $u:[c,d]\to\mathbb{R}$ be a differentiable mapping on (c,d). If $|u'|$ is convex on $[c,d]$, then the following inequality for operators (22) and (23) holds:

$$\left|\frac{u(c)+u(d)}{2}-\frac{1-\alpha}{4(1-\exp(-B))}\left[{}_g^{\beta}E_{c^+}U(d)+{}_g^{\beta}E_{d^-}U(c)\right]\right| \\ \leq \frac{1}{4(1-\exp(-B))}\frac{N(c,d)}{(d-c)}, \quad (50)$$

where

$$N(c,d)=\xi(d,c)-\xi(c,d)+\xi(c,c)-\xi(d,d),$$

$$\xi(x,y) := \int_c^{\frac{c+d}{2}}|x-u|\exp\left(-A\left(|g(u)-g(y)|\right)\right)du \\ -\int_{\frac{c+d}{2}}^d|x-u|\exp\left(-A\left(|g(u)-g(y)|\right)\right)du,$$

$A=\frac{1-\beta}{\beta}$ and $B=A(g(d)-g(c))$. bf ProofIf we use $\varphi(u)=\frac{u}{\beta}\exp(-Au)$, where $A=\frac{1-\beta}{\beta}$, $\beta\in(0,1)$, then by same lines as followed in the proof of Theorem 3, required inequality (50) is obtained.

4. Concluding remarks

This study establishes a very general version of an inequality associated to bounds of Hadamard inequalities. The inequality (35) provides bounds of almost all the Hadamard inequalities via conformable and fractional integral operators available in the literature. Inequalities (36), (45), (46), (47), (48) and (50) gives error bounds of generalized fractional and conformable integral operators. More results can be deduce from inequality (35) by different settings of φ and g. We feel that some generalize inequalities may be obatin by using different kinds of convex functions via integral operators (20) and (21). We hope that this work will attract the attention of researchers working in fractional calculus, mathematical analysis and other related fields.

Funding. Not applicable.

Competing interests. The authors do not have any competing interest.

Authors contributions. All authors contributed equally in this paper. All authors read and approved the final manuscript.

References

[1] S. S. Dragomir, Inequalities of Jensen's type for generalized k-g-Fractional integrals of functions for which the composite $f \circ g^{-1}$ is convex, RGMIA Res. Rep. Coll. 20, 133, 2017.
[2] S. S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. lett., 11, 5, 91-95, 1998.
[3] G. Farid, Existence of an integral operator and its consequences in fractional calculus, Open J. Math. Sci., 3, 210-216, 2019.
[4] G. Farid, K. A. Khan, N. Latif, A. U. Rehman and S. Mehmood, General fractional integral inequalities for convex and m-convex functions via an extended generalized Mittag-Leffler function, J. Inequal. Appl., 2018:243, pp-12, 2018.
[5] G. Farid, A. Ur Rehman and M. Zahra, On Hadamard inequalities for k-fractional integrals, Nonlinear Funct. Anal. Appl., 21, 3, 463-478, 2016.
[6] M. Jleli and B. Samet, On Hermite-Hadamard type inequalities via fractional integral of a function with respect to another function, J. Nonlinear Sci. Appl., 9, 1252-1260, 2016.
[7] S. M. Kang, G. Farid, W. Nazeer and S. Mehmood, $(h-m)$-convex functions and associated fractional Hadamard and Fejér–Hadamard inequalities via an extended generalized Mittag-Leffler function, J. Inequal. Appl., 2019:78, pp-10, 2019.
[8] S. M. Kang, G. Farid, W. Nazeer and B. Tariq, Hadamard and Fejér-Hadamard inequalities for extended generalized fractional integrals involving special functions, J. Inequal. Appl., 2018:119, 2018.
[9] A. Kashuri, M. Anwar, G. Farid and M. Raees, Hadamard inequality at midpoint via generalized integral operators and corresponding fractional and conformable integral inequalities, (submitted).
[10] A. A. Kilbas, O. I. Marichev and S. G. Samko, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach, Switzerland, 1993.
[11] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B.V., Amsterdam, 2006.
[12] U. N. Katugampola, New approach to a generalized fractional integral, Appl. Math. Comput. 218, 3, 860-865, 2011.
[13] Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah and S. M. Kang, Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard Inequalities, IEEE access, 6, 64946-64953, 2018.
[14] R. Khalil, M, Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivatives, J. Comput. Appli. Math., 264, 65-70, 2014.
[15] T. U. Khan and M. A. Khan, Generalized conformable fractional operators, J. Comput. Appl. Math., 346, 378-389, 2019.
[16] M. Kirane and B.T. Torebek, Hermite-Hadamard, Hermite-Hadamard-Fejer, Dragomir-Agarwal and Pachpatti type inequalities for convex functions via fractional integrals, arXive:1701.00092.
[17] N. Mehreen and M. Anwar, Hermite-Hadamard type inequalities for exponentially p-convex functions the second sense with applications, J. Inequal. Appl, 2019:92, pp-17, 2019.
[18] S. Mubeen and G. M. Habibullah, k-Fractional integrals and application, Int. J. Contemp. Math. Sciences, 7, 2, 89-94, 2012.
[19] K. B. Oldham and J. Spanier, The fractional caculus: Theory and application of differentiation and integration to arbitrary order, Dover Publications, Inc, New York, 2006.

[20] R. K. Raina, On generalized Wright's Hypergeometric functions and fractional calculus operators, East Asian Math. J. 21, 2, 191-203, 2005.

[21] M. Raees, G. Farid, M. Anwar and A. Kashuri, Hadamard inequality for a generalized integral operator and associated results for fractional and conformable integrals, Submitted.

[22] M. Z. Sarikaya, M. Dahmani, M. E. Kiris and F. Ahmad, (k, s)-Riemann-Liouville fractional integral and applications, Hacet. J. Math. Stat., 45, 1, 77-89, 2016. doi:10.15672/HJMS.20164512484

[23] M. Z. Sarikaya and F. Ertugral, On the generalised Hermite-Hadamard inequalities, https://www.researchgate.net/publication/321760443.

[24] M. Z. Sarikaya, E. Set, H. Yaldiz and N. Basak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Math. Comput. modell., 57, 2403-2407, 2013.

[25] T, Tunc, H. Budak, F. Usta and M. Z. Sarikaya, On new generalized fractional integral operators and related fractional inequalities, at: https://www.researchgate.net/publication/313650587.

Ghulam Farid
COMSATS University Islamabad, Attock Campus, Attock, Pakistan
E-mail address: faridphdsms@hotmail.com, ghlmfarid@ciit-attock.edu.pk

Muhammad Raees
School of Natural Sciences, National University of Sciences and Technology, Islamabad, Pakistan
E-mail address: muhammad.raees@sns.nust.edu.pk,raeesqau1@gmail.com

Matloob Anwar
School of Natural Sciences, National University of Sciences and Technology, Islamabad, Pakistan
E-mail address: manwar@sns.nust.edu.pk