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HANKEL DETERMINANT FOR BI-BAZELEVIC FUNCTION INVOLVING ERROR AND SIGMOID FUNCTION DEFINED BY DERIVATIVE CALCULUS VIA CHEBYSHEV POLYNOMIALS

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Abstract. In this investigation, a new subclass of bi-univalent function was introduce by means of differential operator involving error function and modified activation function via Chebyshev polynomials. The coefficients bounds and second Hankel determinant of this class were established using subordination principle.

1. Introduction

Let H denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $E = \{z : z \in C, |z| < 1\}$, which satisfying normalized condition $f(0) = f'(0) - 1 = 0$.
A function f is said to be univalent in a domain E if it is injective in E and it is call a starlike domain with respect to origin denoted by $f \in S^*$. A function $f \in S$ convex function is one which map unit disk comformally onto a convex domain denoted by K. For a given $0 \leq \gamma < 1$, a function $f \in A$ is call a starlike of order γ, class of such function denoted by $S^*(\gamma)$ if and only if $Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma$ and $z \in E$. Furthermore, for $0 \leq \gamma < 1$, a function $f \in A$ is called convex function of order γ, class of such function denoted by $K(\gamma)$ if $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma$.
Fractional calculus is considered one of the important branches of mathematical analysis and enormous progress has taken place recently in the study of fractional integral and differential operators because various operators provides tools of considerable importance in geometric function theory. Different types of fractional integral and differential operators were introduced by different researchers such as Salagean operator, Ruscheweyh operator, Strivastava operator, Abdunaby el al.

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operator,e.t.c [see (3,9,12,30,37,39)]. For details about the fractional calculus operator see [2,22,47].
Also, in recent time, several authors studies family of analytic function involving special functions such as activation, error function , Bessel function defined by A to find different conditions such that the member of such functions have certain geometric properties like univalency, starlikeness and convexity. But concern of this paper is that of error function which was by normalized by Ramanchandran [38] as

$$Erf(z) = \frac{\sqrt{\pi z}}{2} erf(\sqrt{z}) = z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{(2k-1)(k-1)!} z^k \tag{2}$$

Originally before normalization the error function is given by [4] as

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) dt = \frac{2}{\sqrt{\pi}} \sum_{k=2}^{\infty} \frac{(-1)^{k-1} z^{2k+1}}{(2k+1)k!} \tag{3}$$

The function (3)is an important in estimate the probability of observing a particle in a specified region [3]. Error function also present in which is a part of transport phenomena, significance in many discipline of physics, chemistry,biology, thermo mechanics and mass flow [4]. See [13] and [5] for properties and inequalities of error function and [16] for the properties of complementary error functions.
It is well-known that every function $f \in S$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z, \quad (z \in E)$$
$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}\right),$$

It is easily seen from above that where

$$f^{-1}(f(z))^{\alpha} = z^{\alpha}, \quad (z \in E)$$
$$f(f^{-1}(w))^{\alpha} = w^{\alpha}, \quad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}\right),$$

Let $\sum$ denote the class of bi-univalent in E by the relation (1). The idea of bi-univalent was conceptualized by Lewin [26] in 1967 and he confirmed that the second coefficient $|a_2| < 1.51$. However, Netanyahu [27] later proved that $max_f\,|a_2| = \frac{4}{3}$. The coefficient problem for each of coefficients $|a_n|$ is still an open problem see ([11],[27], [28]) for details. For more work on bi-univalent one can see [8,17,21,30,31,32,41,43,45,46,48].
Now, Noorman and Thormas [29] stated the q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$

$$H_2(2) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q+2} \\ .. & ... & \cdots & ... \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix} \tag{4}$$

where $a_m's$ are coefficient of various power of z in $f(z)$. Later, this determinant has been repeatedly investigated by several authors and researchers [15,19,23,37,38] among others. In particular, Fekete and Szego consider the Hankel determinant of $f \in A$ for $q = 2$ and $n = 1$

$$H_2(2) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \tag{5}$$

Then they later generalized the estimate as $\left|a_3 - \mu a_2^2\right|$ where μ is real and $f \in S$. Recently, Fadipe et al.[19] considered a function involving modified Sigmoid function

$$f_\gamma(z) = z + \sum_{k=2}^{\infty} a_k \gamma(s) z^k \tag{6}$$

where $\gamma(s) = \frac{2}{1+e^{-s}}$, $\quad s \geq 0$ and s is real. Furthermore, if f and g are analytic in E, and f is as defined (1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ then

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{7}$$

From (1) and (6) using (7) we define

$$Erf_\gamma(z) = z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{(2k-1)(k-1)!} \gamma(s) a_k z^k \tag{8}$$

where all parameters are as earlier defined.
From (8) we have

$$Erf_\gamma(z)^\alpha = \left(z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{(2k-1)(k-1)!} \gamma(s) a_k z^k\right)^\alpha \tag{9}$$

Expand (9) binomially we get

$$Erf_\gamma(z)^\alpha = z^\alpha - \frac{\alpha}{3}\gamma(s)a_2 z^{\alpha+1} + \left[\frac{\alpha}{10}\gamma(s)a_3 + \frac{\alpha(\alpha-1)}{18}\gamma^2(s)a_2^2\right] z^{\alpha+2}$$

$$\left[-\frac{\alpha}{42}\gamma(s)a_4 - \frac{\alpha(\alpha-1)}{30}\gamma^2(s)a_2a_3 - \frac{\alpha(\alpha-1)(\alpha-2)}{162}\gamma^3(s)a_2^3\right] z^{\alpha+3} \tag{10}$$

$$Erg_\gamma(w)^\alpha = w^\alpha + \frac{\alpha}{3}\gamma(s)a_2 w^{\alpha+1} + \left[+\frac{\alpha(\alpha+1)}{18}\gamma^2(s)a_2^2 - \frac{\alpha}{10}\gamma(s)a_3\right] z^{\alpha+2}$$

$$\left[\frac{\alpha}{42}\gamma(s)a_4 - \frac{\alpha(\alpha+1)}{30}\gamma^2(s)a_2a_3 + \frac{\alpha(\alpha+1)(\alpha+2)}{162}\gamma^3(s)a_2^3\right] z^{\alpha+3} \tag{11}$$

Applying Salagean derivative operator [41] on (10) and (11) we get

$$\frac{D^n Erf_\gamma(z)^\alpha}{\alpha^n z^\alpha} = z^\alpha - \frac{\alpha}{3}\left(\frac{\alpha+1}{\alpha}\right)^n \gamma(s)a_2 z^{\alpha+1} + \left[\frac{\alpha}{10}\gamma(s)a_3 + \frac{\alpha(\alpha-1)}{18}\gamma^2(s)a_2^2\right]\left(\frac{\alpha+2}{\alpha}\right)^n z^{\alpha+2}$$

$$\left[-\frac{\alpha}{42}\gamma(s)a_4 - \frac{\alpha(\alpha-1)}{30}\gamma^2(s)a_2a_3 - \frac{\alpha(\alpha-1)(\alpha-2)}{162}\gamma^3(s)a_2^3\right]\left(\frac{\alpha+3}{\alpha}\right)^n z^{\alpha+3} \tag{12}$$

$$\frac{D^n Erg_\gamma(w)^\alpha}{\alpha^n w^\alpha} = w^\alpha + \frac{\alpha}{3}\left(\frac{\alpha+1}{\alpha}\right)^n \gamma(s)a_2 w^{\alpha+1} + \left[+\frac{\alpha(\alpha+1)}{18}\gamma^2(s)a_2^2 - \frac{\alpha}{10}\gamma(s)a_3\right]\left(\frac{\alpha+2}{\alpha}\right)^n z^{\alpha+2}$$

$$\left[\frac{\alpha}{42}\gamma(s)a_4 - \frac{\alpha(\alpha+1)}{30}\gamma^2(s)a_2a_3 + \frac{\alpha(\alpha+1)(\alpha+2)}{162}\gamma^3(s)a_2^3\right]\left(\frac{\alpha+3}{\alpha}\right)^n z^{\alpha+3} \tag{13}$$

Researchers like [1], [20], [32], [33], [34], [42] and the likes have used (10) to define several classes of analytic functions and obtained interesting results which are too numerous to discuss.
The Chebyshev polynomials are a sequence of othogonal polynomials which are

practically related to De Moivres formular and which are defined recursively. Chebyshev polynomials play an important role in numerical analysis. For details about Chebyshev polynomials see ([6],[7],[10],[19]). There are four kinds of Chebyshev polynomials, we shall only concern ourself with Chebyshev polynomials of second kind. Chebyshev polynomials of second is given as

$$U_k(t) = \frac{sin(k+1)\alpha}{sin\alpha} \quad t \in [-1, 1], \tag{14}$$

where k denotes the degree of the polynomial and $t = cos\alpha$. The Chebyshev polynomials of the second kind $U_k(t)$, $t \in [-1, 1]$ have the generating function of the form

$$H(z,t) = \frac{1}{1 - 2tz + z^2} = 1 + \sum_{k=1}^{\infty} \frac{sin(k+1)\alpha}{sin\alpha} z^n. \tag{15}$$

Note that if $t = cos\alpha$, $\alpha \in (\frac{-\pi}{3}, \frac{\pi}{3})$.
Thus

$$H(z,t) = \frac{1}{1 - 2cos\alpha z + z^2} = 1 + \sum_{k=1}^{\infty} \frac{sin(k+1)\alpha}{sin\alpha} z^n.$$

Using Fadipe et al [19], we state that

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + ... \qquad (z \in E, \ \ t \in [-1, 1]),$$

where

$$U_{k-1} = \frac{sin(karccost)}{\sqrt{1-t^2}}, \qquad k \in N$$

are Chebyshev polynomial of the second kind.
It well known that

$$U_k(t) = 2tU_{k-1}(t) - U_{k-2}(t)$$

so that

$$U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t. \tag{16}$$

2. Lemma and Definitions

Lemma 1 [14]: If $w(z) = b_1 z + b_2 z^2 + ..., b_1 \neq 0$ is analytic and satisfies $|w(z)| < 1$ in the unit disk E, then for each $0 < r < 1$, $|w'(z)| < 1$ and $\left|w(re^{i\theta})\right|$ unless $w(z) = e^{i\theta} z$ for some number θ.
Lemma 2 [42]: The power series for $p(z) = 1 + \sum_{k=2}^{\infty} k^n c_k z^k$.function Let the $f \in B_n$ be given by (1) then

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

for some x,$|x| < 1$ and

$$4c_2 = c_1^3 + 2x(4 - c_1^2)c_1 - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z$$

for some x,$|x| < 1$.
Definition 1: A function Erf^{α} given by (5)is said to be in the class $TH^n_{\sum}(\alpha, \lambda, \gamma(s))$, $\alpha > 0$, $\ s \geq 0$ if it satisfies the following

$$(1-\lambda)\frac{D^n Erf_{\gamma}(z)^{\alpha}}{\alpha^n z^{\alpha}} + \lambda \frac{D^{n+1} Erf_{\gamma}(z)^{\alpha}}{\alpha^{n+1} z^{\alpha}} \prec H(z,t) \tag{17}$$

and

$$(1-\lambda)\frac{D^n Erg_{\gamma}(w)^{\alpha}}{\alpha^n w^{\alpha}} + \lambda \frac{D^{n+1} Erg_{\gamma}(w)^{\alpha}}{\alpha^{n+1} w^{\alpha}} \prec H(w,t) \tag{18}$$

where $\gamma(s)=\frac{2}{1+e^{-s}}$

Theorem 1: Let $f\in TH^{n}_{\sum}(\alpha,\lambda,\gamma(s))$. Then

$$|a_2|\le\frac{6t\sqrt{2t}}{\sqrt{\left|2\alpha t^2\left(\frac{\alpha+2}{\alpha}\right)^n[\alpha+2\lambda]-(4t^2-1)\left(\frac{\alpha+2}{\alpha}\right)^n[\alpha+\lambda]^2\right|\gamma^2(s)}} \tag{19}$$

$$|a_3|\le\frac{20t^2}{\left(\frac{\alpha+1}{\alpha}\right)^2n[\alpha+\lambda]\gamma(s)}+\frac{20t}{\left(\frac{\alpha+2}{\alpha}\right)^n[\alpha+2\lambda]\gamma(s)} \tag{20}$$

$$|a_4|\le\frac{168t^3}{\left(\frac{\alpha+1}{\alpha}\right)^2n[\alpha+\lambda]^2\gamma(s)}+\frac{168t^3}{\left(\frac{\alpha+1}{\alpha}\right)^n\left(\frac{\alpha+2}{\alpha}\right)^n[\alpha+\lambda][\alpha+2\lambda]\gamma(s)}-\frac{28t^3(\alpha^2+4)}{\left(\frac{\alpha+1}{\alpha}\right)^3n[\alpha+\lambda]^3\gamma(s)}$$

$$-\frac{84t}{\left(\frac{\alpha+1}{\alpha}\right)^n[\alpha+3\lambda]\gamma(s)}-\frac{84(4t^2-1)}{\left(\frac{\alpha+1}{\alpha}\right)^n[\alpha+3\lambda]\gamma(s)}-\frac{42(8t^2-4t)}{\left(\frac{\alpha+1}{\alpha}\right)^n[\alpha+3\lambda]\gamma(s)} \tag{21}$$

Proof: It is evident from (17) and (18) that

$$-\frac{1}{3}\left(\frac{\alpha+1}{\alpha}\right)^n[\alpha+\lambda]\gamma(s)a_2=U_1(t)c_1 \tag{22}$$

$$\frac{1}{10}\left(\frac{\alpha+2}{\alpha}\right)^n[\alpha+2\lambda]a_3\gamma(s)+\frac{(\alpha-1)}{18}\left(\frac{\alpha+2}{\alpha}\right)^n[\alpha+2\lambda]\gamma^2(s)a_2^2=c_2U_1(t)+c_2^2U_2(t) \tag{23}$$

$$-\frac{1}{42}\left(\frac{\alpha+3}{\alpha}\right)^n[\alpha+2\lambda]a_4\gamma(s)-\frac{(\alpha-1)}{30}\left(\frac{\alpha+3}{\alpha}\right)^n[\alpha+3\lambda]\gamma^2(s)a_2a_3$$

$$-\frac{(\alpha-1)(\alpha-2)}{162}\left(\frac{\alpha+3}{\alpha}\right)^n[\alpha+3\lambda]\gamma^3(s)a_2^3=c_3U_1(t)+2c_1c_2U_2(t)+c_1^3U_3(t) \tag{24}$$

$$\frac{1}{3}\left(\frac{\alpha+1}{\alpha}\right)^n[\alpha+\lambda]\gamma(s)a_2=U_1(t)d_1 \tag{25}$$

$$\frac{(\alpha+1)}{18}\left(\frac{\alpha+2}{\alpha}\right)^n[\alpha+2\lambda]\gamma^2(s)a_2^2-\frac{1}{10}\left(\frac{\alpha+2}{\alpha}\right)^n[\alpha+2\lambda]\gamma(s)a_3=d_2U_1(t)+d_2^2U_2(t) \tag{26}$$

and

$$\frac{1}{42}\left(\frac{\alpha+3}{\alpha}\right)^n[\alpha+2\lambda]\gamma(s)a_4-\frac{\alpha(\alpha+1)}{30}\left(\frac{\alpha+3}{\alpha}\right)^n[\alpha+3\lambda]\gamma^2(s)a_2a_3$$

$$-\frac{(\alpha+1)(\alpha+2)}{162}\left(\frac{\alpha+3}{\alpha}\right)^n[\alpha+3\lambda]\gamma^3(s)a_2^3=d_3U_1(t)+2d_1d_2U_2(t)+d_1^3U_3(t)' \tag{27}$$

From (22) and (25)

$$c_1=-d_1,\quad a_2=\frac{3U_1(t)c_1}{\left(\frac{\alpha+1}{\alpha}\right)^n[\alpha+\lambda]\gamma(s)}=-\frac{3U_1(t)d_1}{\left(\frac{\alpha+1}{\alpha}\right)^n[\alpha+\lambda]\gamma(s)} \tag{28}$$

and

$$\frac{2}{9}\left(\frac{\alpha+1}{\alpha}\right)^2n[\alpha+\lambda]^2\gamma^2(s)a_2^2=U_1^2(t)[c_1^2+d_1^2] \tag{29}$$

It is evident from (23), (26), (28) and (29) that

$$a_2^2 = \frac{9U_1^2(t)[c_2+d_2]}{[\alpha U_1^2(t)\left(\frac{\alpha+2}{\alpha}\right)^n[\alpha+2\lambda] - 2U_2(t)\left(\frac{\alpha+1}{\alpha}\right)^2 n[\alpha+\lambda]^2]\gamma^2(s)} \tag{30}$$

It is fairly well known [14] that if $|w(z)|<1$ and $|v(z)|<1$ then

$$|c_i| \leq 1, \quad and\, |d_i| \leq 1, \quad for \quad all \;\; i \in N \tag{31}$$

Applying Lemma 1 in conjunction with (31) and (16) on (30) readily yields

$$|a_2| \leq \frac{6t\sqrt{2t}}{\sqrt{\left|2\alpha t^2\left(\frac{\alpha+2}{\alpha}\right)^n[\alpha+2\lambda] - (4t^2-1)\left(\frac{\alpha+2}{\alpha}\right)^n[\alpha+\lambda]^2\right|\gamma^2(s)}}$$

Again, from (23) and (26) it is evident that

$$a_3 = \frac{5U_1^2(t)(c_1^2+d_1^2)}{2\left(\frac{\alpha+1}{\alpha}\right)^2 n[\alpha+\lambda]\gamma(s)} + \frac{5U_1(t)(c_2-d_2)}{2\left(\frac{\alpha+2}{\alpha}\right)^n[\alpha+2\lambda]\gamma(s)} \tag{32}$$

Applying Lemma 1 for the coefficients $c_1, \;\; c_2, \;\; d_1 \;\; and \;\; d_2$ and making uses of (16) and (31) we have

$$|a_3| \leq \frac{20t^2}{\left(\frac{\alpha+1}{\alpha}\right)^2 n[\alpha+\lambda]\gamma(s)} + \frac{20t}{\left(\frac{\alpha+2}{\alpha}\right)^n[\alpha+2\lambda]\gamma(s)} \tag{33}$$

Next, by (24) and (27) we notice that

$$a_4 = \frac{21(C_1^2+d_1^2)U_1^3(t)}{2\left(\frac{\alpha+1}{\alpha}\right)^n(\alpha+\lambda)^3\gamma(s)} + \frac{21(C_2-d_2)U_1^2(t)}{2\left(\frac{\alpha+1}{\alpha}\right)^n\left(\frac{\alpha+2}{\alpha}\right)^n(\alpha+\lambda)(\alpha+2\lambda)\gamma(s)} - \frac{7(\alpha^2+4)U_1^3(t)}{\left(\frac{\alpha+1}{\alpha}\right)^3 n(\alpha+\lambda)^3\gamma(s)}$$

$$-\frac{21(c_3-d_3)U_3(t)}{\left(\frac{\alpha+1}{\alpha}\right)^n(\alpha+3\lambda)\gamma(s)} - \frac{21(c_1c_2-d_1d_2)U_3(t)}{\left(\frac{\alpha+1}{\alpha}\right)^n(\alpha+3\lambda)\gamma(s)} - \frac{21(c_1^3-d_1^3)U_3(t)}{\left(\frac{\alpha+1}{\alpha}\right)^n(\alpha+3\lambda)\gamma(s)} \tag{34}$$

Again, applying Lemma 1 for the coefficients $c_1, \;\; c_2, \;\; d_1, \;\; d_2, \;\; c_3 \;\; and \;\; d_3$ and making uses of (16) and (31) we have

$$|a_4| \leq \frac{168t^3}{\left(\frac{\alpha+1}{\alpha}\right)^{2n}[\alpha+\lambda]^2\gamma(s)} + \frac{168t^3}{\left(\frac{\alpha+1}{\alpha}\right)^n\left(\frac{\alpha+2}{\alpha}\right)^n[\alpha+\lambda][\alpha+2\lambda]\gamma(s)} - \frac{28t^3(\alpha^2+4)}{\left(\frac{\alpha+1}{\alpha}\right)^3 n[\alpha+\lambda]^3\gamma(s)}$$

$$-\frac{84t}{\left(\frac{\alpha+1}{\alpha}\right)^n[\alpha+3\lambda]\gamma(s)} - \frac{84(4t^2-1)}{\left(\frac{\alpha+1}{\alpha}\right)^n[\alpha+3\lambda]\gamma(s)} - \frac{42(8t^2-4t)}{\left(\frac{\alpha+1}{\alpha}\right)^n[\alpha+3\lambda]\gamma(s)} \tag{35}$$

as asserted by (21) which complete the proof.

Corollary 2: Let $f \in TH_{\Sigma}^{n}(1, \lambda, \gamma(s))$. Then

$$|a_2| \leq \frac{6t\sqrt{2t}}{\sqrt{|2\alpha t^2 3^n[1+2\lambda] - (4t^2-1)3^n[1+\lambda]^2|\,\gamma^2(s)}} \tag{36}$$

$$|a_3| \leq \frac{20t^2}{2^{2n}[1+\lambda]\gamma(s)} + \frac{20t}{2^{2n}[1+2\lambda]\gamma(s)} \tag{37}$$

$$|a_4| \leq \frac{168t^3}{2^{3n}[1+\lambda]^2\gamma(s)} + \frac{168t^3}{2^n3^n[1+\lambda][\alpha+2\lambda]\gamma(s)} - \frac{28t^3(\alpha^2+4)}{2^{3n}[\alpha+\lambda]^3\gamma(s)}$$

$$-\frac{84t}{4^n[1+3\lambda]\gamma(s)} - \frac{84(4t^2-1)}{4^n[1+3\lambda]\gamma(s)} - \frac{42(8t^2-4t)}{4^n[1+3\lambda]\gamma(s)} \tag{38}$$

Theorem 2: Let $f \in TH_{\sum}^{n}(\alpha,\lambda,\gamma(s))$. Then

$$\left|a_2a_4 - a_3^2\right| \leq \theta\gamma^2 + \beta\gamma \tag{39}$$

where

$$\theta = A_1 - B_1 - B_2, \quad \gamma = \frac{-2(B_1+B_2)}{A_1-B_1-B_2}, \quad C = \left(\frac{\alpha+1}{\alpha}\right)^n \left(\frac{\alpha+1}{\alpha}\right)^{3n} (\alpha+3\lambda)^3$$

$$A_1 = \frac{[336t^4 - 168t^4\alpha^2]A - 126t^4B - [1008t^3 - 252t] - 100t^4D}{\left(\frac{\alpha+3}{\alpha}\right)^n \left(\frac{\alpha+1}{\alpha}\right)^{4n} (\alpha+3\lambda)^4(\alpha+3\lambda)\gamma^2(s)}, \quad \gamma = 4(B_1 + C_1)$$

$$B_1 = \frac{1008t^3 + 252t^2 - 252t}{\left(\frac{\alpha+3}{\alpha}\right)^n \left(\frac{\alpha+1}{\alpha}\right)^n (\alpha+\lambda)(\alpha+3\lambda)\gamma^2(s)}, \quad C_1 = \frac{126t^2}{\left(\frac{\alpha+3}{\alpha}\right)^n \left(\frac{\alpha+1}{\alpha}\right)^n (\alpha+\lambda)(\alpha+3\lambda)\gamma^2(s)}$$

$$A = \left(\frac{\alpha+3}{\alpha}\right)^n (\alpha+\lambda)^3(\alpha+3\lambda), \quad B = \left(\frac{\alpha+3}{\alpha}\right)^n (\alpha+\lambda)^3, \quad D = \left(\frac{\alpha+3}{\alpha}\right)^n (\alpha+3\lambda)$$

Proof: It is evident from (28), (32) and (34) that

$$a_2a_4 - a_3^2 = \frac{21(2-\alpha^2)U_1^4(t)c_1^4}{2\left(\frac{\alpha+3}{\alpha}\right)^{4n}(\alpha+\lambda)^4\gamma^2(s)} + \frac{63c_1^2(c_2-d_2)U_1^2(t)}{2\left(\frac{\alpha+1}{\alpha}\right)^{2n}\left(\frac{\alpha+2}{\alpha}\right)^n(\alpha+\lambda)^2(\alpha+2\lambda)\gamma^2(s)} - \frac{25c_1^4U_1^4(t)}{\left(\frac{\alpha+1}{\alpha}\right)^{4n}(\alpha+\lambda)^4\gamma^2(s)}$$

$$- \frac{252U_1^2(t)c_1^2c_2}{\left(\frac{\alpha+1}{\alpha}\right)^n(\alpha+\lambda)(\alpha+3\lambda)\gamma^2(s)} - \frac{126U_1(t)U_3(t)c_1^4}{\left(\frac{\alpha+1}{\alpha}\right)^n(\alpha+\lambda)(\alpha+3\lambda)\gamma^2(s)} - \frac{63[c_3-d_3]U_1^2(t)c_1}{\left(\frac{\alpha+1}{\alpha}\right)^n(\alpha+\lambda)(\alpha+3\lambda)\gamma^2(s)}$$

$$- \frac{25c_1^3(c_2-d_2)U_1^2(t)}{2\left(\frac{\alpha+1}{\alpha}\right)^{2n}\left(\frac{\alpha+2}{\alpha}\right)^n(\alpha+\lambda)^2(\alpha+2\lambda)\gamma^2(s)} - \frac{25U_1^3(t)[c_2-d_2]^2U_1^2(t)}{\left(\frac{\alpha+2}{\alpha}\right)^{2n}(\alpha+2\lambda)^2\gamma^2(s)} \tag{40}$$

According to Lemma 2 we have

$$2c_2 = c_1^2 + x(4-c_1^2) \quad and \quad 2d_2 = d_1^2 + x(4-d_1^2)$$

hence we have

$$c_2 = d_2$$

$$4c_3 = c_1^3 + 2x(4-c_1^2)c_1 - x^2(4-c_1^2)c_1 + 2(4-c_1^2)(1-|z|^2)$$

$$4d_3 = d_1^3 + 2x(4-d_1^2)d_1 - x^2(4-d_1^2)d_1 + 2(4-d_1^2)(1-|z|^2)$$

and

$$c_3 - d_3 = \frac{1}{2}c_1^3 + x(4-c_1^2)c_1 - \frac{1}{2}x^2(4-c_1^2)c_1.$$

Thus upon substitution and letting $c_1 = c$ we get

$$a_2a_4 - a_3^2 = \left(\frac{[336t^4 - 168t^4\alpha^2]A - 126t^4B - [1008t^3 - 252t] - 100t^4D}{\left(\frac{\alpha+3}{\alpha}\right)^n \left(\frac{\alpha+1}{\alpha}\right)^{4n} (\alpha+3\lambda)^4(\alpha+3\lambda)\gamma^2(s)}\right) c^4$$

$$- \left(\frac{1008t^3 + 252t^2 - 252t}{\left(\frac{\alpha+3}{\alpha}\right)^n \left(\frac{\alpha+1}{\alpha}\right)^n (\alpha+\lambda)(\alpha+3\lambda)\gamma^2(s)}\right) c^2(4-c^2)x + \left(\frac{126t^2}{\left(\frac{\alpha+3}{\alpha}\right)^n \left(\frac{\alpha+1}{\alpha}\right)^n (\alpha+\lambda)(\alpha+3\lambda)\gamma^2(s)}\right) c^2(4-c^2)x^2$$

where

$$A = \left(\frac{\alpha+3}{\alpha}\right)^n (\alpha+\lambda)^3(\alpha+3\lambda), \quad B = \left(\frac{\alpha+3}{\alpha}\right)^n (\alpha+\lambda)^3,$$

$$C = \left(\frac{\alpha+1}{\alpha}\right)^n \left(\frac{\alpha+1}{\alpha}\right)^{3n} (\alpha+3\lambda)^3, \quad D = \left(\frac{\alpha+3}{\alpha}\right)^n (\alpha+3\lambda)$$

Since $|c| \leq 1$ by using Lemma 2, we may assume without restriction $c \in [0,1]$. Thus using the triangle inequality, with $\rho = |x|$ we obtain

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| &\leq A_1c^4 + B_1c^2(4-p^2)\rho + C_1c^2(4-c^2)\rho^2 \\ &= F(c,\rho), \end{aligned}$$

where

$$A_1 = \frac{[336t^4 - 168t^4\alpha^2]A - 126t^4B - [1008t^3 - 252t] - 100t^4D}{\left(\frac{\alpha+3}{\alpha}\right)^n\left(\frac{\alpha+1}{\alpha}\right)^{4n}(\alpha+3\lambda)^4(\alpha+3\lambda)\gamma^2(s)}$$

$$B_1 = \frac{1008t^3 + 252t^2 - 252t}{\left(\frac{\alpha+3}{\alpha}\right)^n\left(\frac{\alpha+1}{\alpha}\right)^n(\alpha+\lambda)(\alpha+3\lambda)\gamma^2(s)}, \quad C_1 = \frac{126t^2}{\left(\frac{\alpha+3}{\alpha}\right)^n\left(\frac{\alpha+1}{\alpha}\right)^n(\alpha+\lambda)(\alpha+3\lambda)\gamma^2(s)}$$

Then,

$$\frac{\partial F}{\partial \rho} = B_1c^2(4-c^2) + 2C_1c^2(4-c^2)\rho$$

It is clear $\frac{\partial F}{\partial \rho} > 0$ which shows that $F(c,\rho)$ is an increasing function on the closed interval $[0,1]$. This shows that maximum occurs at $\rho = 1$. Therefore $maxF(c,\rho) = F(c,1) = G(c)$

Now

$$\begin{aligned} F(c,1) = G(c) &= A_1c^4 + B_1c^2(4-c^2) + C_1c^2(4-c^2) \\ G(c) &= (A_1 - B_1 - C_1)c^4 + 4(B_1 + C_1)c^2 \end{aligned}$$

$$G'(c) = 4(A_1 - B_1 - C_1)c^3 + 8(B_1 + C_1)c \tag{41}$$

$$G''(c) = 12(A_1 - B_1 - C_1)c^2 + 8(B_1 + C_1) < 0. \tag{42}$$

For optimum value of $G(c)$, consider $G'(c) = 0$. From (41) we get

$$c^2 = \frac{-2(B_1 + c_1)}{A_1 - B_1 - B_2} = \gamma \tag{43}$$

Upon substitution for the value of c^2 from (43) in (42), it can be shown that

$$G''(c) = 12(A_1 - B_1 - C_1)c^2 + 8(B_1 + C_1)$$

Therefore, by using second derivative test $G(c)$ has the maximum value at c, where c^2 is given by (43). Substituting the obtained value of c^2 in the expression which gives maximum of $G(c)$ as

$$\left|a_2a_4 - a_3^2\right| \leq \theta\gamma^2 + \beta\gamma$$

References

1. S. Abdulhalim, On the a class of analytic involving the Salagean differential operator, TamkangJournal of Mathematics, vol. 23, no. 51-58
2. Z.E. Abdulnaby, W. R.Ibrahim, On a fractional class of analytic function defined by using a new operator, arxivpreprint arxiv: 1602.07682 (2016).
3. Adem Kilicman, R.W.Ibrahim and Z.E. Abdulnaby, On a generalized fractional Integral operator in a complex domain. Appl.Math.Inf.Sci. 10, No.3 (2016) 1053-1059.
4. M. Abramowitz and I.A. Stegun, (Eds), Handbook of Mathematical functions with Formulas, raphs, and Mathematical Tables, Dover Publications Inc., New York, 1965.
5. H. Alzer, Error function inequalities, Adv. Comput. Math., 33(2010), no. 3, 349-379. https://doi.org//101007.

6. S. Altinkaya and S. Yalcin, Second Hankel determinant for a general subclasses of Bi-univalent function TWNS Journal of Pure and Applied Mathematics 7(1)(2016), 98-104.
7. S. Altinkaya and S. Yalcin, On the chebyshev polynomial bounds for class of univalent functions, Khayyam J. Math; 2(2016) n. 1, 1-5.
8. I.T. Awolere and S. Ibrahim-Tiamiyu, new classes of Bi-univalent pseudo Starlike function using Alhindi-Darus generalized hypergeometric function, Journal of Nig. Of Mathematical Physics, vol. 40, (2017), 65-72.
9. M. Biernacki, Sur lint egrale des fonctions univalentes, Bull. Acad. Polon. Sci, 8, 29"34 (1960).
10. S. Bulut and M. Magesh, On the sharp bounds for comprehensive class of analytic and univalent functions by means of Chebyshev polynomials, Khayyam J. Math; 2(2016) no. 2, 194-200.
11. D.A. Brannan and S.T. Taha, On some classes of bi-univalent functions, Studia Univ. Babes-Bolyai Math. 31(2) (1986), 70-77.
12. B.C. Carlson, D. B. Shaffer, Starlike and prestarlike hypergeometric functions. SIAM J. Math. Anal., 15, 737" 745 (1984).
13. D. Coman, The radius of starlikeness for error function, Stu. Uni. Babes-Bolyai math., 36(1991), no. 2, 13-16.
14. P.L. Duren, Univalent in Grundlehren der Mathematischen Wissenschaften, Heidelberg and Tokyo, (1983).
15. R. Ehrenborg, Hankel determinant of exponential polynomial. American Mathematical Monthly, 107(2000), 557-560.
16. A. Elbert and A. Laforgia, The zeros of the complementary error function, Numer. Algorithms, 49(2008), no. 1-4, 153-157. http:/doi.org/s/1075-008-9186-7.
17. B.A. Frasin and M.K. Aouf, New subclasses of bi-univalent functions, Appl..Math..Lett.24(2011) 1569-1573.
18. O.A. Fadipe-Joseph, Bilikis B. Kadir, Sunday E. Akinwumi and Esther O. Adeniran, Polynomial bounds for a class of univalent function involving Sigmoid function, Khayyam J. Math; 4(2018) no. 2, 88-101.
19. J.O. Hamzat, and A.A. Oni, Hankel determinant for a certain subclasses of analytic functions. Asian Research Journal of Mathematics, 5(2), 2017, 1-10.
20. J.O. Hamzat, Coefficients bounds for Bazilevic functions associated with modifies Sigmoid function, Asian Research Journal of Mathematics, 2456-477x. Vol. 5, (2017), issue 1-10.
21. T. Hayami and S. Owa, Coefficient bounds for bi-univalent functions, Pan Amer. Math, J. 22(4) (2012) 15-26.
22. R.W.Ibrahim, On a class of analytic functions generated by fractional integral operator. Concrete operators 4. 1(2017) 1-6.
23. A. Janteng, S.A. Halim and M. Darus, Hankel determinant for starlike and convex functions, Int. J. of Math. Anal., 1(13) (2007), 619-625.
24. R.W.Ibrahim,Fractional complex transforms for fractional differential equations Advances in difference equations,2012.1 (2012):192.
25. J.W. Layman, The Hankel transform and some of its properties, J. of Integer, Sequences, 4(2001), 1-11.
26. M. Lewin, On the coefficient problem for bi-univalent functions, Proceeding of the American Mathematical Society, vol. 18, pp 63-68, 1967.
27. E. Netanyahu, The minimal distance of the image boundary from origin and second coefficient of a univalent function in 1,Arch Ration Mec., Anal, vol. 32 pp 100-112, 1969.
28. J.W. Noonan and D.K. Thomas, On the second Hankel of a really P-valent function, Trans. American Math. Soc., 223(2) (1976), 237-346.
29. K.I.Noor, and M. A. Noor, On certain classes of analytic functions defined by noor integral operator. Jour. Math. Anal. Appl., 281(1), 244"252 (2003).
30. G. Murugurusundarymoorthy and T. Janani, Bi-starlike function of complex order associated with hypergeometric functions, Miskolc amsth. Notes, 16, no. 1, (2015).
31. G. Murugurusundarymoorthy, Subclasses of bi-univalent function of complex order based on subordination conditions involving Wright hypergeometric functions, J. Math.Fund.Sci., vol. 47, No. 1, 2015, 60-75.
32. A.T. Oladipo, New subclasses of bi-univalent Bazilevic functions of type alpha involving Salagean derivative operator, Acta Universitatis Apulensis No. 43 (2015) 33-43.
33. A.T. Oladipo and D. Breaz, On the family of Bazilevic functions, Acta Universitatis Apulensis (2010), no. 24.

34. T.O. Opoola, on a new subclass of univalent functions, Mathematical, Vol. 36, (1994), no. 2 tomme (36, 195-200).
35. S. Owa, On the distortion theorems. I, Kyungpook Math. J., 18, 53"59 (1978).
36. S. Owa, H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, Canad. J. Math., 39, 1057"1077 (1987).
37. K. Rajya Laxmi and R. Bharavi Sharma, Second Hankel determinant for some subclasses of bi-univalent functions associated with pseudo-starlike functions, Journal of Complex Analysis.
38. C. Ramachandran, K. Dhanalakshmi and L. Vanitha, Hankel determinant for a subclass of analytic functions associated with error functions bounded by conical regions, Int. J. of Math.Anal., vol. 11, (2017), no. 12, 571-581.
39. S. Ruscheweyh, New criteria for univalent functions, Proc.Amer. Math. Soc., 109"115 (1975).
40. Salagean Gr..St (1981): Subclasses of univalent functions, Complex Analysis " Fifth Romania Finish Sem part Proc. Bucharest Lecture Notes in Notes in Math 1013 362-372.
41. L. Shuhai, M. Lina and T. Huo, Coefficient estimates for psuedo " Bazzilevic functions of bi-univalent functions. GeneralMath.Notes, No. 1 (2015), 6-21.
42. M. Sidiq and K.O. Babalola, Coefficient bounds for certain Bazievic maps, Mathematical, 59 (82), (2017) No. 1-2, 124-133.
43. H.M. Strivastava, A.K. Mishira and P. Gochayat, certain subclasses of analytic and bi-univalent functions, Applied Mathematics Letters, vol. 23, no. 10 (2010).
44. H.M. Strivastava, M.Darus and R.W. Ibrahim, Integral transform and special function, Volume 22 Issue 1 pages 17-28, P11 928795140 (2011).
45. H.M. Strivastava, G. Murugurusundarymoorthy and K. Vijaya, Coefficient estimates for some families of bi-univalent functions of the Ma-Minda type involving the Hohlov operators, Journal of Classical Analysis, vol.2 no 2 (2013), 167-181.
46. H.M. Strivastava, A.K. Mishira and P. Gochhayat, certain subclasses of analytic and bi-univalent functions, Applied Mathematics Letters, vol 23, no. 10 (2010).
47. Srivastava, H., An appliction of the fractional derivative.Math. Japon., 29, 383"389 (1984).
48. T.S. Taha, Topics in univalent function theory, Ph.D Thesis, University of London, 1981.

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