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POSITIVE SOLUTIONS OF HIGHER ORDER NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL INITIAL CONDITIONS AT RESONANCE

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Abstract. By Leggett-Williams norm-type theorem for coincidences due to O'Regan and Zima, we present a new result on the existence of positive solutions for a class of differential equation of fractional order with nonlocal initial conditions at resonance. Moreover, an example is given to illustrate the efficiency of the main theorems.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order. During the last ten years, initial and boundary value problems of fractional differential equations have been studied extensively due to their significant applications in various sciences such as physics, mechanics, chemistry, phenomena arising in engineering (see [1], [3], [5], [8], [10], [13-15], [19-20]). Recently, fractional boundary value problems at resonance have been studied by a number of authors. For some recent works on the topic, see [2], [4], [6], [7], [9] and references therein.

Meanwhile, some attention has been paid to the existence of positive solutions to fractional boundary value problems at resonance, see [16-18].

In [18], Yang and Wang considered the positive solutions of the following two-point fractional boundary value problems at resonance

$$\begin{cases} D_{0+}^{\alpha} u(t) = f\big(t, u(t)\big), \quad 0 < t < 1, \\ u(0) = 0, \ u'(0) = u'(1), \end{cases}$$

where $1 < \alpha \leq 2$, D_{0+}^{α} denotes the Caputo fractional derivative, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ satisfies the L^1-Carathéodory conditions.

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In [17], a multi-point boundary value problem for the following fractional differential equations at resonance

$$\begin{cases} D_{0+}^{\alpha}u(t) + f\big(t, u(t)\big) = 0, \quad 0 < t < 1, \\ u'(0) = u'(1), \ u(0) = \sum_{i=1}^{m-2} \mu_i u(\xi_i), \end{cases}$$

is investigated, where $1 < \alpha \leq 2$, $m > 2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\mu_i \geq 0$, $i = 1, 2, \cdots, m-2$, $\sum_{i=1}^{m-2} \mu_i = 1$, D_{0+}^{α} denotes the Caputo fractional derivative, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

In a recent paper [16], Wu and Liu study a m-point boundary value problem for a class of fractional differential equations at resonance

$$\begin{cases} -D_{0+}^{\alpha}u(t) = f\big(t, u(t)\big), \quad 0 < t < 1, \\ u'(0) = 0, \ u(1) = \sum_{i=1}^{m-2} \beta_i u(\eta_i), \end{cases}$$

where $1 < \alpha \leq 2$, D_{0+}^{α} denotes the Caputo fractional derivative, $\beta_i \in \mathbb{R}^+$, $\sum_{i=1}^{m-2} \beta_i = 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Although the existing literature on the existence of solutions to resonant conditions of fractional differential equations is quite wide, only a few papers deal with the existence of positive solutions to fractional boundary value problems at resonance. To the best of our knowledge, the fractional differential equations with nonlocal initial conditions at resonance has yet to be initiated. To fill this gap, we investigate the existence of positive solutions of higher order fractional differential equation with nonlocal initial conditions of the form:

$$\begin{cases} D_{0+}^{\alpha}u(t) = f\big(t, u(t)\big), \quad 0 < t < 1, \\ u^{(i)}(0) = 0, \ u(0) = \int_0^1 u(t)dt, \end{cases} \tag{1}$$

where $n-1 < \alpha < n, n \geq 2, i = 1, 2, \cdots, n-1$, D_{0+}^{α} denotes the Caputo fractional derivative, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the L^1-Carathéodory conditions.

The rest of this paper is organized as follows. Section 2, we give some necessary notations, definitions and lemmas. In Section 3, we obtain the existence of positive solutions of (1) by Theorem 3.1. Finally, an example is given to illustrate our results in Section 4.

2. Preliminaries

First of all, we present the necessary definitions and lemmas from fractional calculus theory. For more details see [8].

Definition 2.1 ([8])**.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 ([8])**.** The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

where $n-1<\alpha\le n$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.1 ([8])**.** The fractional differential equation

$$D_{0+}^{\alpha}y(t)=0$$

has solution $y(t)=c_0+c_1t+\cdots+c_{n-1}t^{n-1}$, $c_i\in\mathbb{R}$, $i=0,1,\cdots,n-1$, $n=[\alpha]+1$. Furthermore, for $y\in AC^n[0,1]$,

$$\big(I_{0+}^{\alpha}D_{0+}^{\alpha}y\big)(t)=y(t)-\sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}t^k$$

and

$$\big(D_{0+}^{\alpha}I_{0+}^{\alpha}y\big)(t)=y(t).$$

Lemma 2.2 ([8])**.** The relation

$$I_{a+}^{\alpha}I_{a+}^{\beta}f(x)=I_{a+}^{\alpha+\beta}f(x),$$

is valid in following case $\beta>0,\ \alpha+\beta>0,\ f(x)\in L_1(a,b)$.

In the following, let us recall some definitions on Fredholm operators and cones in Banach space (see [11]).

Let X,Y be real Banach spaces. Consider a linear mapping $L:\operatorname{dom}L\subset X\to Y$ and a nonlinear operator $N:X\to Y$. Assume that

(A1) L is a Fredholm operator of index zero; that is, $\operatorname{Im}L$ is closed and $\dim\ker L=\operatorname{codim}\operatorname{Im}L<\infty$.

This assumption implies that there exist continuous projections $P:X\to X$ and $Q:Y\to Y$ such that $\operatorname{Im}P=\ker L$ and $\ker Q=\operatorname{Im}L$. Moreover, since $\dim\operatorname{Im}Q=\operatorname{codim}\operatorname{Im}L$, there exists an isomorphism $J:\operatorname{Im}Q\to\ker L$. Denote by L_p the restriction of L to $\ker P\cap\operatorname{dom}L$. Clearly, L_p is an isomorphism from $\ker P\cap\operatorname{dom}L$ to $\operatorname{Im}L$, we denote its inverse by $K_P:\operatorname{Im}L\to\ker P\cap\operatorname{dom}L$. It is known that the coincidence equation $Lx=Nx$ is equivalent to

$$x=(P+JQN)x+K_P(I-Q)Nx.$$

Let C be a cone in X such that

(i) $\mu x\in C$ for all $x\in C$ and $\mu\ge0$,
(ii) $x,-x\in C$ implies $x=\theta$.

It is well known that C induces a partial order in X by

$$x\preceq y\quad\text{if and only if}\quad y-x\in C.$$

The following property is valid for every cone in a Banach space X.

Lemma 2.3 ([12])**.** *Let C be a cone in X. Then for every $u\in C\setminus\{0\}$ there exists a positive number $\sigma(u)$ such that*

$$\|x+u\|\ge\sigma(u)\|u\|\quad\text{for all } x\in C.$$

Let $\gamma:X\to C$ be a retraction; that is, a continuous mapping such that $\gamma(x)=x$ for all $x\in C$. Set

$$\Psi:=P+JQN+K_P(I-Q)N\quad\text{and}\quad\Psi_\gamma:=\Psi\circ\gamma.$$

We use the following result due to O'Regan and Zima.

Theorem 2.1 ([12]). *Let C be a cone in X and let Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume (A1) and the following assumptions hold:*

(A2) *$QN : X \to Y$ is continuous and bounded and $K_P(I-Q)N : X \to X$ be compact on every bounded subset of X,*
(A3) *$Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap \mathrm{Im}L$ and $\lambda \in (0,1)$,*
(A4) *γ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C,*
(A5) *Topological degree : $\deg\{[I-(P+JQN)\gamma]|_{\ker L}, \ker L \cap \Omega_2, 0\} \neq 0$,*
(A6) *there exists $u_0 \in C \setminus \{0\}$ such that $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for $x \in C(u_0) \cap \partial\Omega_1$, where $C(u_0) = \{x \in C : \mu u_0 \preceq x \text{ for some } \mu > 0\}$ and $\sigma(u_0)$ such that $\|x + u_0\| \geq \sigma(u_0)\|x\|$ for every $x \in C$,*
(A7) *$(P+JQN)\gamma(\partial\Omega_2) \subset C$,*
(A8) *$\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$.*

Then the equation $Lx = Nx$ has a solution in the set $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Existence result

Definition 3.1. The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the L^1-Carathéodory conditions, if

(i) for each $u \in \mathbb{R}$, the mapping $t \mapsto f(t,u)$ is Lebesgue measurable on $[0,1]$,
(ii) for a.e. $t \in [0,1]$, the mapping $u \mapsto f(t,u)$ is continuous on $\mathbb{R}$,
(iii) for each $r > 0$, there exists $\alpha_r \in L^1[0,1]$ satisfying $\alpha_r(t) > 0$ on $[0,1]$ such that
$$|u| \leq r \text{ implies } |f(t,u)| \leq \alpha_r(t).$$

Now, we state our result on the existence of positive solutions for (1.1) under the following assumptions:

In this section, we state our result on the existence of positive solutions for (1). For simplicity of notation, we set

$$G(t,s) = \begin{cases} 1 - \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}(1-s)^\alpha - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(\alpha+2)}{\Gamma(2\alpha+2)}t^\alpha, & 0 \leq t < s \leq 1, \\ 1 - \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}(1-s)^\alpha - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(\alpha+2)}{\Gamma(2\alpha+2)}t^\alpha + & \\ \frac{\alpha}{\Gamma(\alpha+2)}\frac{(t-s)^{\alpha-1}}{(1-s)^\alpha}, & 0 \leq s < t \leq 1, \end{cases}$$

By means of inequality technique, we can prove $G(t,s) > 0,\ t,s \in [0,1]$. Furthermore, we let a constant κ satisfying

$$0 < \kappa \leq \min\left\{1,\ \frac{1}{\max_{t,s\in[0,1]} G(t,s)}\right\}. \tag{2}$$

Theorem 3.1. *Assume that:*

(H1) *the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the L^1-Carathéodory conditions;*
(H2) *there exist positive constants $b_1, b_2, b_3, c_1, c_2, B$ with*
$$B > \frac{c_2}{c_1} + \frac{3b_2c_2}{(\alpha+1)b_1c_1} + \frac{3b_3}{(\alpha+1)b_1}.$$
For all $t \in [0,1]$ and $x \in [0,B]$, one has
$$-\kappa x \leq f(t,x) \leq -c_1x + c_2 \text{ and } f(t,x) \leq -b_1|f(t,x)| + b_2x + b_3;$$

(H3) *there exist* $b \in (0, B)$, $t_0 \in [0, 1]$, $\rho \in (0, 1]$, $\delta \in (0, 1)$ *and* $q(t) \in L^1[0, 1]$, $q(t) \geq 0$ *on* $[0, 1]$, $h(x) \in C\big((0, b], \mathbb{R}^+\big)$ *such that* $f(t, u) \geq q(t)h(u)$ *for* $(t, u) \in [0, 1] \times (0, b]$. *Moreover,* $\frac{h(u)}{u^\rho}$ *is non-increasing on* $(0, b]$, *and*

$$(\alpha+1)\frac{h(b)}{b}\int_0^1 G(t_0, s)(1-s)^{\alpha-1}q(s)ds \geq \frac{1-\delta}{\delta^\rho}.$$

Then the problem (1) *has at least one positive solution on* $[0, 1]$.

Proof. Consider the Banach spaces $X = C[0, 1]$ with the supremum norm $\|x\| = \max_{t\in[0,1]} |x(t)|$ and $Y = L^1[0, 1]$ with the usual integral norm $\|y\| = \int_0^1 |y(t)|dt$.

Define $L : \mathrm{dom}L \to X$ and $N : X \to Y$ with

$$\mathrm{dom}L = \Big\{x \in X : D^\alpha_{0+}u(t) \in L^1[0, 1], u^{(i)}(0) = 0;\ u(0) = \int_0^1 u(t)dt;\ 1 \leq i \leq n-1\Big\}$$

by

$$Lu = D^\alpha_{0+}u$$

and

$$Nu(t) = f\big(t, u(t)\big).$$

Then the problem (1) can be written by $Lu = Nu, u \in \mathrm{dom}L$.

It is clear that Lemma 2.1 implies

$$\ker L = \big\{u \in \mathrm{dom}L : u(t) = c \in \mathbb{R} \text{ on } [0, 1]\big\}$$

and

$$\mathrm{Im}L = \left\{y \in Y : \int_0^1 (1-s)^\alpha y(s)ds = 0\right\}.$$

Consider the linear operator $P : X \to X$ defined by

$$Px(t) = (\alpha+1)\int_0^1 (1-s)^\alpha x(s)ds, \quad t \in [0, 1].$$

Next, we define the operator $Q : Y \to Y$ by

$$Qy(t) = (\alpha+1)\int_0^1 (1-s)^\alpha y(s)ds, \quad t \in [0, 1].$$

Obviously, $P^2 = P$ and $Q^2 = Q$. So we can show that $\mathrm{Im}P = \ker L$ and $\ker Q = \mathrm{Im}L$.

It follows from $\mathrm{Ind}\, L = \dim\ker L - \mathrm{codim}\,\mathrm{Im}\, L = 0$ that L is a Fredholm mapping of index zero. Then, (A1) holds.

We consider the mapping $K_P : \mathrm{Im}\, L \to \mathrm{dom}\, L \cap \ker P$ by

$$K_Py(t) = \int_0^1 k(t, s)y(s)ds,$$

where

$$k(t, s) := \begin{cases} \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} - \frac{\Gamma(\alpha+2)}{\Gamma(2\alpha+1)}(1-s)^{2\alpha}, & 0 \leq s \leq t \leq 1, \\ -\frac{\Gamma(\alpha+2)}{\Gamma(2\alpha+1)}(1-s)^{2\alpha}, & 0 \leq t < s \leq 1, \end{cases}$$

It is easy to see that

$$|k(t, s)| \leq 3(1-s)^\alpha, s, t \in [0, 1]. \tag{3}$$

Consider the cone

$$C = \{x \in X : x(t) \geq 0,\ t \in [0, 1]\}.$$

It is clear that (H1) implies (A2).

Let

$$\Omega_1 = \big\{x \in X : \delta\|x\| < |x(t)| < b,\ t \in [0,1]\big\},$$

$$\Omega_2 = \big\{x \in X : |x(t)| < B,\ t \in [0,1]\big\}.$$

Obviously, Ω_1 and Ω_2 are bounded and

$$\overline{\Omega}_1 = \big\{x \in X : \delta\|x\| \le |x(t)| \le b,\ t \in [0,1]\big\} \subset \Omega_2.$$

Moreover, $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Let $J = I$ and $(\gamma x)(t) = |x(t)|$ for $x \in X$, then γ is a retraction and maps subsets of $\overline{\Omega}_2$ into bounded subsets of C, which means that (A4) holds.

Next, we will show (A3) holds. Suppose that there exist $u_0 \in \partial\Omega_2 \cap C \cap \operatorname{dom} L$ and $\lambda_0 \in (0,1)$ such that $Lu_0 = \lambda_0 N u_0$, then $D^{\alpha}_{0+}u_0(t) = \lambda_0 f(t, u_0(t))$ for all $t \in [0,1]$. In view of (H2), we get

$$\begin{aligned} D^{\alpha}_{0+}u_0(t) = \lambda_0 f(t,u_0(t)) &\le -\lambda_0 b_1 |f(t,u_0(t))| + \lambda_0 b_2 u_0(t) + \lambda_0 b_3 \\ &= -b_1 |D^{\alpha}_{0+}u_0(t)| + \lambda_0 b_2 u_0(t) + \lambda_0 b_3 \\ &\le -b_1 |D^{\alpha}_{0+}u_0(t)| + b_2 u_0(t) + b_3 \end{aligned} \tag{4}$$

and

$$D^{\alpha}_{0+}u_0(t) = \lambda_0 f(t,u_0(t)) \le -\lambda_0 c_1 u_0(t) + \lambda_0 c_2. \tag{5}$$

From (4), we obtain

$$\begin{aligned} 0 &= u_0(0) - \int_0^1 u_0(t)dt = I^{\alpha+1}_{0+} D^{\alpha}_{0+}u_0(t)|_{t=1} \\ &\le -\frac{b_1}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} \left|D^{\alpha}_{0+}u_0(s)\right| ds + \frac{b_2}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} u_0(s) ds \\ &+ \frac{b_3}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} ds, \end{aligned}$$

which gives

$$\int_0^1 (1-s)^{\alpha} \left|D^{\alpha}_{0+}u_0(s)\right| ds \le \frac{b_2}{b_1} \int_0^1 (1-s)^{\alpha} u_0(s) ds + \frac{b_3}{(\alpha+1)b_1}.$$

From (5), we obtain

$$\int_0^1 (1-s)^{\alpha} u_0(s) ds \le \frac{c_2}{(\alpha+1)c_1}.$$

From (3) and the equation

$$u_0 = (I-P)u_0 + Pu_0 = K_P L(I-P)u_0 + Pu_0 = Pu_0 + K_P L u_0,$$

we can get

$$\begin{aligned}
u_0 &= (\alpha+1)\int_0^1 (1-s)^{\alpha} u_0(s)ds + \int_0^1 k(t,s) D_{0+}^{\alpha} u_0(s)ds \\
&\le \frac{c_2}{c_1} + \int_0^1 |k(t,s)| \cdot \left| D_{0+}^{\alpha} u_0(s) \right| ds \\
&= \frac{c_2}{c_1} + \int_0^1 \frac{|k(t,s)|}{(1-s)^{\alpha}} \cdot (1-s)^{\alpha} \left| D_{0+}^{\alpha} u_0(s) \right| ds \\
&\le \frac{c_2}{c_1} + 3\int_0^1 (1-s)^{\alpha} \left| D_{0+}^{\alpha} u_0(s) \right| ds \\
&\le \frac{c_2}{c_1} + 3\left[\frac{b_2}{b_1} \int_0^1 (1-s)^{\alpha} u_0(s)ds + \frac{b_3}{(\alpha+1)b_1} \right] \\
&\le \frac{c_2}{c_1} + \frac{3b_2c_2}{(\alpha+1)b_1c_1} + \frac{3b_3}{(\alpha+1)b_1}.
\end{aligned}$$

Then, we have

$$B = \|u_0\| \le \frac{c_2}{c_1} + \frac{3b_2c_2}{(\alpha+1)b_1c_1} + \frac{3b_3}{(\alpha+1)b_1},$$

which contradicts (H2). Hence (A3) holds.

To prove (A5), consider $x \in \ker L \cap \overline{\Omega}_2$, then $x(t) \equiv c$ on $[0,1]$. For $c \in [-B,B]$ and $\lambda \in [0,1]$, we have

$$\begin{aligned}
H(c,\lambda) &= [I - \lambda(P + JQN)\gamma]c \\
&= c - \lambda(\alpha+1)\int_0^1 (1-s)^{\alpha}|c|ds - \lambda(\alpha+1)\int_0^1 (1-s)^{\alpha+1} f(s,|c|)ds \\
&= c - \lambda|c| - \lambda(\alpha+1)\int_0^1 (1-s)^{\alpha} f(s,|c|)ds.
\end{aligned}$$

It is easy to show that $H(c,\lambda)=0$ implies $c \ge 0$. Suppose $H(B,\lambda)=0$ for some $\lambda \in (0,1]$, then we have

$$0 = B - \lambda B - \lambda(\alpha+1)\int_0^1 (1-s)^{\alpha} f(s,B)ds.$$

According to (H2), we have

$$0 \le B(1-\lambda) = \lambda(\alpha+1)\int_0^1 (1-s)^{\alpha} f(s,B)ds \le \lambda(-c_1B + c_2) < 0,$$

which is a contradiction. In addition, if $\lambda = 0$, then $B = 0$, which is impossible. As a result, for $x \in \ker L \cap \partial\Omega_2$ and $\lambda \in [0,1]$, we have $H(x,\lambda) \neq 0$. Thus,

$$\begin{aligned}
&\deg\{[I - (P+JQN)\gamma]_{\ker L}, \ker L \cap \Omega_2, 0\} \\
&\qquad = \deg\{H(\cdot,1), \ker L \cap \Omega_2, 0\} \\
&\qquad = \deg\{H(\cdot,0), \ker L \cap \Omega_2, 0\} \\
&\qquad = \deg\{I, \ker L \cap \Omega_2, 0\} = 1 \neq 0.
\end{aligned}$$

So (A5) holds.

Next, we prove (A6). Let $u_0(t) \equiv 1, t \in [0,1]$, then $u_0 \in C \setminus \{0\}$, $C(u_0) = \{x \in C : x(t) > 0 \text{ on } [0,1]\}$. We take $\sigma(u_0) = 1$. Let $x \in C(u_0) \cap \partial\Omega_1$, then $0 < \|x\| \le b$ and $x(t) \ge \delta\|x\|$ on $[0,1]$.

By (H3), for every $x \in C(u_0) \cap \partial\Omega_1$, we have

$$\begin{aligned}
(\Psi)v(t_0) &= (\alpha+1)\int_0^1 (1-s)^\alpha x(s)ds + (\alpha+1)\int_0^1 G(t_0,s)(1-s)^\alpha f\big(s,x(s)\big)ds \\
&\geq \delta\|x\| + (\alpha+1)\int_0^1 G(t_0,s)(1-s)^\alpha q(s)h\big(x(s)\big)ds \\
&= \delta\|x\| + (\alpha+1)\int_0^1 G(t_0,s)(1-s)^\alpha q(s)\cdot\frac{h\big(x(s)\big)}{x^\rho(s)}x^\rho(s)ds \\
&\geq \delta\|x\| + \delta^\rho\|x\|^\rho(\alpha+1)\int_0^1 G(t_0,s)(1-s)^\alpha q(s)\cdot\frac{h(b)}{b^\rho}ds \\
&= \delta\|x\| + \delta^\rho\|x\|\cdot\frac{b^{1-\rho}}{\|x\|^{1-\rho}}(\alpha+1)\int_0^1 G(t_0,s)(1-s)^\alpha q(s)\frac{h(b)}{b}ds \\
&\geq \delta\|x\| + \delta^\rho\|x\|\cdot(\alpha+1)\int_0^1 G(t_0,s)(1-s)^\alpha q(s)\frac{h(b)}{b}ds \\
&\geq \delta\|x\| + \delta^\rho\|x\|\cdot\frac{1-\delta}{\delta^\rho} \\
&= \|x\|.
\end{aligned}$$

Thus, for all $x \in C(u_0) \cap \partial\Omega_1$, we have $\|x\| \leq \sigma(u_0)\|\Psi x\|$, i.e. (A6) holds.

For $x \in \partial\Omega_2$, from (H1) we have

$$\begin{aligned}
(P+JQN)\gamma x &= (\alpha+1)\int_0^1 (1-s)^\alpha|x(s)|ds + (\alpha+1)\int_0^1 (1-s)^\alpha f(s,|x(s)|)ds \\
&\geq (\alpha+1)\int_0^1 (1-s)^\alpha(1-\kappa)|x(s)|ds \\
&\geq 0.
\end{aligned}$$

Thus, $(P+JQN)\gamma x \subset C$ for $x \in \partial\Omega_2$. Then (A7) holds.

Next, we prove (A8). For $x(t) \in \overline{\Omega}_2 \setminus \Omega_1$ and $t \in [0,1]$, we have

$$\begin{aligned}
\Psi_\gamma x(t) &= \big(P + JQN + K_P(I-Q)N\big)|x(t)| \\
&= (\alpha+1)\int_0^1 (1-s)^\alpha|x|ds + (\alpha+1)\int_0^1 G(t,s)(1-s)^\alpha f(s,|x|)ds \\
&> (\alpha+1)\int_0^1 (1-s)^\alpha|x(s)|\big(1-\kappa G(t,s)\big)ds \\
&\geq 0.
\end{aligned}$$

Hence, $\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$; i.e., (A8) holds.

Hence, applying Theorem 2.1, BVP (1) has a positive solution u^* on $[0,1]$ with $b \leq \|u^*\| \leq B$. This completes the proof. □

4. Example

To illustrate how our main result can be used in practice, we present here an example.

Let us consider the following fractional differential equation at resonance

$$\begin{cases} D_{0^+}^{1.5}u(t) = f(t,u), \quad 0<t<1, \\ u'(0) = 0, u(0) = \int_0^1 u(t)dt, \end{cases} \tag{6}$$

where

$$f(t,u) = \frac{1}{300}(1+2t-2t^2)(u^2-4u+3)u.$$

Corresponding to BVP (1), we have that $\alpha = 1.5$ and

$$G(t,s) = \begin{cases} 1 - \frac{\Gamma(2.5)}{\Gamma(4)}(1-s)^{1.5} - \frac{t^{1.5}}{\Gamma(2.5)} + \frac{\Gamma(3.5)}{\Gamma(5)}t^{1.5}, \quad 0 \le t < s \le 1, \\ 1 - \frac{\Gamma(2.5)}{\Gamma(4)}(1-s)^{1.5} - \frac{t^{1.5}}{\Gamma(2.5)} + \frac{\Gamma(3.5)}{\Gamma(5)}t^{1.5} + \\ \frac{1.5}{\Gamma(3.5)}\frac{(t-s)^{0.5}}{(1-s)^{1.5}}, \quad 0 \le s < t \le 1, \end{cases}$$

Obviously, $G(t,s) \ge 0$ for $t,s \in [0,1]$. We let $\kappa = \frac{1}{30}$ and choose $B = 1.2$ $c_1 = \frac{1}{300}$, $c_2 = \frac{1}{300}$, $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{300}$, $b_3 = \frac{1}{600}$, and $b = 1/2$, $t_0 = 0$, $\rho = 1$, $\delta = 0.9995$, $q(t) = \frac{1}{2}(1+2t-t^2)$, $h(x) = x$. We can check that the conditions (H_1)-(H_3) of Theorem 3.1 are satisfied, then BVP (6) has a positive solution on $[0,1]$.

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