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EXISTENCE THEORY FOR ψ-TYPE COMPLEX-ORDER IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This paper investigates the existence and uniqueness of solutions for a class of impulsive differential equations using ψ-type complex-order derivative. The results are obtained by using fixed point principles. An example to illustrate the results is included.

1. Introduction

Differential equations with fractional order have recently proved to be strong tools in the modeling of many phenomena in various fields of engineering, physics and economics. As a consequence there was an intensive development of the theory of differential equations of fractional order. One can see the monographs of Kilbas *et al.* [19], Lakshmikantham *et al.* [20], Miller and Ross [23], Podlubny [25] and Samko *et al.* [27] and the references therein. Though the concepts and the calculus of fractional derivative are few centuries old, it is realized only recently that these derivatives form an excellent framework for modelling real world problems. In the literature, there are several studies on distinct operators such as the Riemann-Liouville the Caputo, the Hilfer, the Erdelyi-Kober and the Hadamard, for example [18, 19]. As mentioned above, a problem in the advancement of fractional calculus studies is the wide variety of definitions of fractional derivative operators. One way to overcome this problem is to consider ever more general definitions of fractional derivative operators, from which fractional derivative operators can be obtained as particular cases. In this sense, the study of Caputo fractional derivative of a function with respect to another function has been initiated by Almeida [3]. An important application of more general definitions was done by Colombaro *et al.* [12]. We can also highlight the work done by Almeida [4, 5] on ψ-fractional derivative and integrals. However, most of the work done in the field so far has been based on the use of real order fractional derivative and integrals. It is worth to indicate that there are several authors who also applied complex-order fractional derivative. This in turn led to the sustained study of the theory of complex-order differential equations [21]. In 1977, Ross [26] considered a use for a derivative of complex order

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in the fractional calculus. Some basic theory for fractional differential equation with complex order was investigated by Neamaty *et al.* [24]. In [6, 13, 29, 30, 31], the authors have proved the existence of solutions of some kinds of complex-order differential equations by using fixed point techniques. Subsequently Harikrishnan *et al.* [16] have discussed the problem for ψ-Hilfer fractional differential equation with complex-order.

This paper is devoted to the study of the existence and uniqueness of solutions of ψ-type complex-order initial value problems for impulsive differential equations (IDEs for short) of the form

$$\left({}^{c}\mathscr{D}^{\theta;\psi}u\right)(t) = \mathscr{F}(t,u(t)), \quad t \in J = [0,T],\ t \neq t_k, \tag{1}$$

$$\Delta u|_{t=t_k} = I_k(u(t_k^-)), \tag{2}$$

$$u(0) = u_0, \tag{3}$$

where $k = 1,\ldots,n$, ${}^{c}\mathscr{D}^{\theta;\psi}$ is the ψ-type Caputo fractional derivative of order $\theta \in \mathbb{C}$, $\theta = m + i\alpha$, $\alpha \in \mathbb{R}^+$, $0 < \alpha < 1$, $m \in (0,1]$. Here, $\mathscr{F} : J\times\mathbb{R} \to \mathbb{R}$ is a given function, $I_k : \mathbb{R} \to \mathbb{R}$, $u_0 \in \mathbb{R}$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $u(t_k^+) = \lim_{h\to 0^+} u(t_k + h)$ and $u(t_k^-) = \lim_{h\to 0^-} u(t_k + h)$ represent the right and left limits of $u(t)$ at $t = t_k$.

IDEs have become increasingly important in recent years as mathematical models of real-world processes and phenomena studied in control theory, physics, chemistry, population dynamics, biotechnology, and economics. There has been a significant development in impulse theory and this has been especially true in the area of IDEs with fixed moments; see, for example, the monographs of Bainov and Simeonov [8], Lakshmikantham *et al.* [22], and Samoilenko and Perestyuk [28], Benchohra *et al.* [9] as well as the papers of Agur et al. [2], Ballinger and Liu [7], Benchohra *et al.* [10, 11], Franco *et al.* [14], and the references contained therein. This work initiates new avenues for obtaining existence and uniqueness of solutions of IDEs involving ψ-type complex-order derivative.

2. Preliminaries

In this section, we introduce notations and definitions that are used throughout the remainder of this paper. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|u\|_\infty := \sup\{|u(t)| :\ t \in J\}.$$

Definition 2.1. [3] *Let $\alpha > 0$, $\mathscr{F}$ an integrable function defined on J and $\psi \in C^n(J)$ an increasing differentiable function such that $\psi\,'(t) \neq 0$ for all $t \in J$. The left-sided ψ-Riemann-Liouville fractional order α of a function $\mathscr{F}$ is given by*

$$\mathscr{I}^{\alpha;\psi} = \frac{1}{\Gamma(\alpha)}\int_0^t \psi\,'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathscr{F}(s)ds,$$

where Γ is the gamma function.

Definition 2.2. [16] *The ψ-type Riemann-Liouville fractional integral of order $\theta \in \mathbb{C}$, $(\Re(\theta) > 0)$ of a function $\mathscr{F} : (0,\infty) \to \mathbb{R}$ is*

$$\mathscr{I}^{\theta;\psi}\mathscr{F}(t) = \frac{1}{\Gamma(\theta)}\int_0^t \psi\,'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}\mathscr{F}(s)ds.$$

Definition 2.3. [16] *For a function $\mathscr{F}$ given on the interval J, the ψ-type Caputo fractional-order $\theta \in \mathbb{C}$, $(\Re(\theta) > 0)$, is defined by*

$$\left(\mathscr{D}^{\theta;\psi}\mathscr{F}\right)(t) = \frac{1}{\Gamma(n-\theta)}\int_0^t \psi\,'(s)\left(\psi(t)-\psi(s)\right)^{n-\theta-1}\mathscr{F}(s)ds,$$

when $n = [\Re(\theta)] + 1$ and $[\Re(\theta)]$ denotes the integral part of the real number θ.

Definition 2.4. [19] *The Stirling asymptotic formula of the gamma function for $z \in \mathbb{C}$ is following*

$$\Gamma(z) = (2\pi)^{\frac{1}{2}} z^{\frac{z-1}{2}} e^{-z}\left[1 + O\left(\frac{1}{z}\right)\right], \quad (|arg(z)| < \pi; |z| \to \infty),$$

and its results for $|\Gamma(u+iv)|$, $(u, v \in \mathbb{R})$ is

$$|\Gamma(u+iv)| = (2\pi)^{\frac{1}{2}} |v|^{u-\frac{1}{2}} e^{-u-\pi|v|/2}\left[1 + O\left(\frac{1}{v}\right)\right], \quad (v \to \infty).$$

Lemma 2.5. [1] *Let $\theta \in \mathbb{C}$, $(\mathbb{R} > 0)$ and $\mathscr{F} : [a,b] \to \mathbb{R}$. The following holds:*

1. *If $\mathscr{F} \in C[a,b]$, then $\mathscr{D}^{\theta;\psi}\mathscr{I}^{\theta;\psi}\mathscr{F}(t) = \mathscr{F}(t)$.*
2. *If $\mathscr{F} \in C^{n-1}[a,b]$, then $\mathscr{I}^{\theta;\psi}\mathscr{D}^{\theta;\psi}\mathscr{F}(t) = \mathscr{F}(t) - \sum_{k=0}^{n-1} c_k\left[\psi(t) - \psi(a)\right]^k$, where, $c_k = \frac{\mathscr{F}_\psi^{[k]}(a)}{k!}$.*

The following lemma concerns a linear variant of the problem (1)-(3).

Lemma 2.6. *Let $\theta \in \mathbb{C}$ and $h : J \to \mathbb{R}$ be continuous. A function u is a solution of the integral equation*

$$u(t) = \begin{cases} u_0 + \frac{1}{\Gamma(\theta)}\int_0^t \psi\,'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}h(s)ds, & \textit{if } t \in [0, t_1], \\ u_0 + \frac{1}{\Gamma(\theta)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\psi\,'(s)\left(\psi(t_k)-\psi(s)\right)^{\theta-1}h(s)ds & \\ +\frac{1}{\Gamma(\theta)}\int_{t_k}^t \psi\,'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}h(s)ds & \\ +\sum_{0<t_k<t} I_k(u(t_k^-)), & \textit{if } t \in (t_k, t_{k+1}]. \end{cases} \tag{4}$$

where $k = 1, \ldots, m$, if and only if u is a solution of the ψ-type complex-order IDE

$$\left({}^c\mathscr{D}^{\theta;\psi}u\right)(t) = h(t), \quad t \in J = [0,T],\ t \neq t_k, \tag{5}$$

$$\Delta u|_{t=t_k} = I_k(u(t_k^-)), \tag{6}$$

$$u(0) = u_0. \tag{7}$$

Proof. Assume u satisfies (5)-(7). If $t \in [0, t_1]$ then

$$\mathscr{D}^{\theta;\psi}u(t) = h(t).$$

Lemma 2.5 implies

$$u(t) = u_0 + \frac{1}{\Gamma(\theta)}\int_0^t \psi\,'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}h(s)ds.$$

If $t \in (t_1, t_2]$, then Lemma 2.5 implies

$$\begin{aligned} u(t) &= u(t_1^+) + \frac{1}{\Gamma(\theta)} \int_{t_1}^{t} \psi\,'(s) \left(\psi(t) - \psi(s)\right)^{\theta-1} h(s) ds \\ &= \Delta u|_{t=t_1} + u(t_1^-) + \frac{1}{\Gamma(\theta)} \int_{t_1}^{t} \psi\,'(s) \left(\psi(t) - \psi(s)\right)^{\theta-1} h(s) ds \\ &= I_1(u(t_1^-)) + u_0 + \frac{1}{\Gamma(\theta)} \int_{0}^{t_1} \psi\,'(s) \left(\psi(t_1) - \psi(s)\right)^{\theta-1} h(s) ds \\ &\quad + \frac{1}{\Gamma(\theta)} \int_{t_1}^{t} \psi\,'(s) \left(\psi(t) - \psi(s)\right)^{\theta-1} h(s) ds. \end{aligned}$$

If $t \in (t_2, t_3]$ then from Lemma 2.5, we get

$$\begin{aligned} u(t) &= u(t_2^+) + \frac{1}{\Gamma(\theta)} \int_{t_2}^{t} \psi\,'(s) \left(\psi(t) - \psi(s)\right)^{\theta-1} h(s) ds \\ &= \Delta u|_{t=t_2} + u(t_2^-) + \frac{1}{\Gamma(\theta)} \int_{t_2}^{t} \psi\,'(s) \left(\psi(t) - \psi(s)\right)^{\theta-1} h(s) ds \\ &= I_2(u(t_2^-)) + I_1(u(t_1^-)) + u_0 + \frac{1}{\Gamma(\theta)} \int_{0}^{t_1} \psi\,'(s) \left(\psi(t_1) - \psi(s)\right)^{\theta-1} h(s) ds \\ &\quad + \frac{1}{\Gamma(\theta)} \int_{t_1}^{t_2} \psi\,'(s) \left(\psi(t_2) - \psi(s)\right)^{\theta-1} h(s) ds \\ &\quad + \frac{1}{\Gamma(\theta)} \int_{t_2}^{t} \psi\,'(s) \left(\psi(t) - \psi(s)\right)^{\theta-1} h(s) ds. \end{aligned}$$

If $t \in (t_k, t_{k+1}]$ then again from Lemma 2.5 we get (4). The converse follows by direct computation. This completes the proof. □

In order to define the solution of (1)-(3), we shall consider the space (the Filippov-Wazewski theorem, see [17])

$$\begin{aligned} PC(J, \mathbb{R}) = \{u : J \to \mathbb{R} :\ u : C\left((t_k, t_{k+1}], \mathbb{R}\right),\ k = 0, \dots, m \quad \text{and there exist} \\ u(t_k^-) \quad \text{and } u(t_k^+),\ k = 1, \dots, n \text{ with } u(t_k^-) = u(t_k)\big\}. \end{aligned}$$

The space $PC(J, \mathbb{R})$ is a Banach space with the norm

$$\|u\|_{PC} = \sup_{t \in J} |u(t)|.$$

Set $J' := [0, T] \setminus \{t_1, t_2, \dots, t_n\}$.

We assume the following conditions to prove the existence of solution of the problem (1)-(3).

- (A1) The function $\mathscr{F} : J \times \mathbb{R} \to \mathbb{R}$ is continuous.
- (A2) There exists a constant $l > 0$ such that $|\mathscr{F}(t, x) - \mathscr{F}(t, y)| \leq l\,|x - y|$, for each $t \in J$, and each $x, y \in \mathbb{R}$.
- (A3) There exists a constant $l^* > 0$ such that $|I_k(x) - I_k(y)| \leq l^*\,|x - y|$, for each $x, y \in \mathbb{R}$ and $k = 1, \dots, n$.
- (A4) There exists a constant $M > 0$ such that $|\mathscr{F}(t, x)| \leq M$ for each $t \in J$ and each $x \in \mathbb{R}$.
- (A5) The functions $I_k : \mathbb{R} \to \mathbb{R}$ are continuous and there exists a constant $M^* > 0$ such that $|I_k(x)| \leq M^*$ for each $x \in \mathbb{R}$, $k = 1, \dots, n$.

(A6) There exists $p \in C(J, \mathbb{R}^+)$ and $\varphi : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that $|\mathscr{F}(t, x)| \leq p(t)\varphi(|x|)$ for all $t \in J$, $x \in \mathbb{R}$.

(A7) There exists $\varphi^* : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that $|I_k(x)| \leq \varphi^*(|x|)$ for all $x \in \mathbb{R}$.

(A8) There exists an number $\overline{M} > 0$ such that

$$\frac{\overline{M}}{|u_0| + \varphi(\overline{M})\frac{n(\psi(T))^m \|p\|_\infty}{m|\Gamma(\theta)|} + \varphi(\overline{M})\frac{(\psi(T))^m \|p\|_\infty}{m|\Gamma(\theta)|} + n\varphi^*(\overline{M})} > 1.$$

Theorem 2.7. *Assume that (A1)-(A3) are satisfied. If*

$$\left[\frac{l(n+1)(\psi(T))^m}{m\,|\Gamma(\theta)|} + nl^*\right] < 1, \tag{8}$$

then problem (1)-(3) *has a unique solution on* J.

Proof. In view of Lemma 2.6, we define the mapping $\mathscr{N} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ by

$$(\mathscr{N}u)(t) = u_0 + \frac{1}{\Gamma(\theta)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \psi\,'(s)\left(\psi(t_k) - \psi(s)\right)^{\theta-1} \mathscr{F}(s, u(s))ds$$
$$+ \frac{1}{\Gamma(\theta)} \int_{t_k}^{t} \psi\,'(s)\left(\psi(t) - \psi(s)\right)^{\theta-1} \mathscr{F}(s, u(s))ds + \sum_{0<t_k<t} I_k(u(t_k^-))$$

and we have to show that $\mathscr{N}$ has a fixed point. This fixed point is then a solution of the problem (1)-(3). Now, for $u, v \in PC(J, \mathbb{R})$, we have

$$\begin{aligned}
&|(\mathscr{N}u)(t) - (\mathscr{N}v)(t)| \\
&\leq \frac{1}{|\Gamma(\theta)|} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \left|\psi\,'(s)\left(\psi(t_k) - \psi(s)\right)^{\theta-1}\right| |\mathscr{F}(s, u(s)) - \mathscr{F}(s, v(s))|\, ds \\
&+ \frac{1}{|\Gamma(\theta)|} \int_{t_k}^{t} \left|\psi\,'(s)\left(\psi(t) - \psi(s)\right)^{\theta-1}\right| |\mathscr{F}(s, u(s)) - \mathscr{F}(s, v(s))|\, ds \\
&+ \sum_{0<t_k<t} \left|I_k(u(t_k^-)) - I_k(v(t_k^-))\right| \\
&\leq \frac{l}{|\Gamma(\theta)|} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left|\psi\,'(s)\left(\psi(t_k) - \psi(s)\right)^{\theta-1}\right| |u(s) - v(s)|\, ds \\
&+ \frac{l}{|\Gamma(\theta)|} \int_{t_k}^{t} \left|\psi\,'(s)\left(\psi(t) - \psi(s)\right)^{\theta-1}\right| |u(s) - v(s)|\, ds + \sum_{k=1}^{n} l^* \left|u(t_k^-) - v(t_k^-)\right| \\
&\leq \frac{nl(\psi(T))^m}{m\,|\Gamma(\theta)|} \|u - v\|_\infty + \frac{l(\psi(T))^m}{m\,|\Gamma(\theta)|} \|u - v\|_\infty + nl^* \|u - v\|_\infty .
\end{aligned}$$

Therefore,

$$\|(\mathscr{N}u) - (\mathscr{N}v)\|_\infty \leq \left[\frac{l(n+1)(\psi(T))^m}{m\,|\Gamma(\theta)|} + nl^*\right] \|u - v\|_\infty .$$

Consequently by Eq. (8), $\mathscr{N}$ is a contraction. As a result of Banach fixed point theorem, we deduce that $\mathscr{N}$ has a fixed point which is a solution of problem (1)-(3). □

The following discussion is based on Schaefer's fixed point theorem.

Theorem 2.8. *If the assumptions (A1), (A4)-(A5) are satisfied, then problem* (1)-(3) *has at least one solution on J.*

Proof. We need to prove that the operator $\mathscr{N}$ has a fixed point. We will split the proof in four steps.

Claim 1. $\mathscr{N}$ is continuous.

Let $\{u_p\}$ be a sequence such that $u_p \to u$ in $PC(J,\mathbb{R})$. Then for each $t \in J$

$$\begin{aligned}
&|(\mathscr{N}u_p)(t) - (\mathscr{N}u)(t)| \\
&\leq \frac{1}{|\Gamma(\theta)|}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\left|\psi\,'(s)\left(\psi(t_k)-\psi(s)\right)^{\theta-1}\right||\mathscr{F}(s,u_p(s)) - \mathscr{F}(t,u(s))|\,ds \\
&+ \frac{1}{|\Gamma(\theta)|}\int_{t_k}^{t}\left|\psi\,'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}\right||\mathscr{F}(s,u_p(s)) - \mathscr{F}(t,u(s))|\,ds \\
&+ \sum_{0<t_k<t}\left|I_k(u_p(t_k^-)) - I_k(u(t_k^-))\right|.
\end{aligned}$$

Since $\mathscr{F}$ and I_k, $k = 1, \ldots, n$ are continuous functions, we have

$$\|(\mathscr{N}u_p) - (\mathscr{N}u)\|_\infty \to 0 \quad \text{as } p \to \infty.$$

Claim 2. $\mathscr{N}$ maps bounded sets into bounded sets in $PC(J,\mathbb{R})$.

In fact, it is sufficient to show that for any $\eta^* > 0$, there exists a positive constant ζ such that for each $u \in B_{\eta^*} = \{u \in PC(J,\mathbb{R}) : \|u\|_\infty \leq \eta^*\}$, we have $\|(\mathscr{F}u)\|_\infty \leq \zeta$. By (A4) and (A5) we have for each $t \in J$

$$\begin{aligned}
|(\mathscr{N}u)(t)| &\leq |u_0| + \frac{1}{|\Gamma(\theta)|}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\left|\psi\,'(s)\left(\psi(t_k)-\psi(s)\right)^{\theta-1}\right||\mathscr{F}(s,u(s))|\,ds \\
&+ \frac{1}{|\Gamma(\theta)|}\int_{t_k}^{t}\left|\psi\,'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}\right||\mathscr{F}(s,u(s))|\,ds + \sum_{0<t_k<t}\left|I_k(u(t_k^-))\right| \\
&\leq |u_0| + \frac{nM(\psi(T))^m}{m\,|\Gamma(\theta)|} + \frac{M(\psi(T))^m}{m\,|\Gamma(\theta)|} + nM^*.
\end{aligned}$$

Thus

$$\|(\mathscr{N}u)\|_\infty \leq |u_0| + \frac{nM(\psi(T))^m}{m\,|\Gamma(\theta)|} + \frac{M(\psi(T))^m}{m\,|\Gamma(\theta)|} + nM^* := \zeta.$$

Claim 3. $\mathscr{N}$ maps bounded set into equicontinuous set of $PC(J,\mathbb{R})$.

Let $t_1, t_2 \in J$, $t_1 < t_2$, B_{η^*} be a bounded set of $PC(J,\mathbb{R})$ as in Claim 2, and let $u \in B_{\eta^*}$. Then

$$\begin{aligned}
&|(\mathscr{N}u)(t_2) - (\mathscr{N}u)(t_1)| \\
&\leq \frac{1}{|\Gamma(\theta)|}\int_0^{t_1}\left|\psi\,'(s)\left(\psi(t_2)-\psi(s)\right)^{\theta-1} - \psi\,'(s)\left(\psi(t_1)-\psi(s)\right)^{\theta-1}\right||\mathscr{F}(s,u(s))|\,ds \\
&+ \frac{1}{|\Gamma(\theta)|}\int_{t_1}^{t_2}\left|\psi\,'(s)\left(\psi(t_2)-\psi(s)\right)^{\theta-1}\right||\mathscr{F}(s,u(s))|\,ds + \sum_{0<t_k<t_2-t_1}\left|I_k(u(t_k^-))\right| \\
&\leq \frac{M}{m\,|\Gamma(\theta)|}\left[2\left(\psi(t_2)-\psi(t_1)\right)^m + \left(\psi(t_2)\right)^m - \left(\psi(t_1)\right)^m\right] + \sum_{0<t_k<t_2-t_1}\left|I_k(u(t_k^-))\right|
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Claim 1 to Claim 3 together with Arzelá-Ascoli theorem, we can conclude

that $\mathscr{N}: PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is completely continuous.
Claim 4. A priori bounds.

Now it remains to show that the set

$$\chi = \{u \in PC(J,\mathbb{R}): \ u \in \lambda(\mathscr{N}u) \quad \text{for some } 0<\lambda<1\}$$

is bounded. Let $u \in \chi$, then $u = \lambda(\mathscr{N}u)$ for some $0<\lambda<1$. Thus, for each $t \in J$ we have

$$u(t) = \lambda u_0 + \frac{\lambda}{\Gamma(\theta)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \psi\,'(s)\left(\psi(t_k)-\psi(s)\right)^{\theta-1} \mathscr{F}(s,u(s))ds + \frac{\lambda}{\Gamma(\theta)} \int_{t_k}^{t} \psi\,'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1} \mathscr{F}(s,u(s))ds + \lambda \sum_{0<t_k<t} I_k(u(t_k^-)).$$

This implies by (A4) and (A5) (as in Claim 2) that for each $t \in J$ we have

$$|u(t)| \le |u_0| + \frac{nM(\psi(T))^m}{m\,|\Gamma(\theta)|} + \frac{M(\psi(T))^m}{m\,|\Gamma(\theta)|} + nM^*.$$

Thus for every $t \in J$, we have

$$\|u\|_\infty \le |u_0| + \frac{nM(\psi(T))^m}{m\,|\Gamma(\theta)|} + \frac{M(\psi(T))^m}{m\,|\Gamma(\theta)|} + nM^* := R.$$

This shows that the set χ is bounded. As a result of Schaefer's fixed point theorem, we deduce that $\mathscr{N}$ has a fixed point which is a solution of problem (1)-(3). $\square$

In the following theorem we present an existence result for problem (1)-(3) by using the nonlinear alternative of Leray-Schauder type [15] and which the assumptions (A4) and (A5) are weakened.

Theorem 2.9. *Assume that assumptions (A3), (A6)-(A8) are satisfied. Then problem* (1)-(3) *has at least one solution on J.*

Proof. Consider the operator $\mathscr{N}$ defined in Theorem 2.7. It can be easily shown, as in Theorem 2.8, that $\mathscr{N}$ is continuous and completely continuous (see, Theorem 6.1, [17]). For $0<\lambda<1$, let u be such that for each $t \in J$ we have $u(t) = \lambda(\mathscr{N}u)(t)$. Then from (A6)-(A7) we have for each $t \in J$

$$\begin{aligned} |u(t)| &\le |u_0| + \frac{1}{|\Gamma(\theta)|} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \left|\psi\,'(s)\left(\psi(t_k)-\psi(s)\right)^{\theta-1}\right| p(s)\varphi\left(|u(s)|\right)ds \\ &\quad + \frac{1}{|\Gamma(\theta)|} \sum_{0<t_k<t} \int_{t_k}^{t} \left|\psi\,'(s)\left(\psi(t)-\psi(s)\right)^{\theta-1}\right| p(s)\varphi\left(|u(s)|\right)ds + \sum_{0<t_k<t} \varphi^*\left(|u(s)|\right) \\ &\le |u_0| + \varphi(\|u\|_\infty)\frac{n(\psi(T))^m\|p\|_\infty}{m\,|\Gamma(\theta)|} + \varphi(\|u\|_\infty)\frac{(\psi(T))^m\|p\|_\infty}{m\,|\Gamma(\theta)|} + n\varphi^*(\|u\|_\infty). \end{aligned}$$

Thus

$$\frac{\|u\|_\infty}{|u_0| + \varphi(\|u\|_\infty)\frac{n(\psi(T))^m\|p\|_\infty}{m|\Gamma(\theta)|} + \varphi(\|u\|_\infty)\frac{(\psi(T))^m\|p\|_\infty}{m|\Gamma(\theta)|} + n\varphi^*(\|u\|_\infty)} \le 1.$$

Then by (A8), there exists $\overline{M}$ such that $\|u\|_\infty \ne \overline{M}$. Let

$$X = \left\{u \in PC(J,\mathbb{R}): \ \|u\|_\infty < \overline{M}\right\}.$$

The operator $\mathscr{N}:\overline{X}\to PC(J,\mathbb{R})$ is continuous and completely. From the choice of X, there is no $u\in\partial U$ such that $u=\lambda(\mathscr{N}u)$ for some $\lambda\in(0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that $\mathscr{N}$ has a fixed point u in $\overline{U}$ which is a solution of problem (1)-(3). This completes the proof. □

3. An example

Consider the following ψ-type complex-order IDEs of the form

$$\left({}^{c}\mathscr{D}^{\theta;\psi}u\right)(t)=\frac{|u(t)|}{10\left(1+|u(t)|\right)},\ t\in J:=[0,1],\ t\neq\frac{1}{2},\tag{9}$$

$$\Delta u|_{t=\frac{1}{2}}=\frac{\left|u(\frac{1}{2}^{-})\right|}{3+\left|u(\frac{1}{2}^{-})\right|},\tag{10}$$

$$u(0)=u_0,\tag{11}$$

where $\theta=m+i\alpha$, $\alpha=\frac{1}{2}$ and $m=1$.
Set

$$\mathscr{F}(t,x)=\frac{x}{10(1+x)},\quad (t,x)\in J\times[0,\infty),$$

and

$$I_k(x)=\frac{x}{3+x},\quad x\in[0,\infty).$$

Let $x,y\in[0,\infty)$ and $t\in J$. Then we have

$$|\mathscr{F}(t,x)-\mathscr{F}(t,y)|\leq\frac{1}{10}|x-y|.$$

Hence the condition (A2) is satisfied with $l=\frac{1}{10}$. Let $x,y\in[0,\infty)$. Then we have

$$|I_k(x)-I_k(y)|\leq\frac{1}{3}|x-y|.$$

Therefore, the condition (A3) is satisfied with $l^*=\frac{1}{3}$. We shall check that condition (8) is fulfilled with $\psi(T)=1$ and $n=1$. Indeed, the conditon (8) holds. Then by Theorem 2.7 problem (9)-(11) has a unique solution on J.

Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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