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NON-INSTANTANEOUS IMPULSIVE FRACTIONAL SEMILINEAR EVOLUTION EQUATION WITH FINITE DELAY

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ABSTRACT. This article deals with the existence of mild solution for a class of non-instantaneous impulsive fractional evolution equation with finite delay. With the aid of Burton-Kirk's fixed point theorem, fractional calculus and semigroup theory, a set of sufficient conditions for existence of the mild solution is established. The derived theory is illustrated by an example.

1. INTRODUCTION

Physical problems pertaining to memory and hereditary properties can be explained more realistically with the help of fractional calculus. Consequently, many partial differential equations and integro-differential equations representing physical phenomena have been recast in fractional set up for gaining more information about the system [20]. An extensive study of various types of abstract fractional differential equations with Caputo fractional derivative can be found in the book by Kostiĉ [15]. Using the concept of solution of integral equations developed in the book by Prüss [21], Bazhlekova [4] extended the classical theory of C_0-semigroup to discuss the solution operator of fractional Cauchy problem in abstract space. For recent development on qualitative study of fractional differential equations, readers are referred to [27].

Mathematical modeling of a dynamical system in which the trajectory of the state variable undergoes a short term abrupt jump after relatively long smooth evolution is known as instantaneous impulsive differential equation. This topic has been gaining more and more attention of late due to applications in various fields of engineering, science and economics. For details on theory on impulsive fractional differential equation, readers are referred to the monograph by Stamova and Stamov [23] and for current developments to the articles [10, 17, 22, 24, 26] and the references therein. Hernández and O'Regan [13] pointed out that some practical problems involving impulse cannot be properly represented by such models. For instance, in hemodynamical study, the introduction of drugs in the blood stream and subsequent absorption by human body is a continuous and gradual process.

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This type of phenomena can be appropriately explained by a non-instantaneous impulsive differential equation. In a Banach space $(X, \|.\|)$, Hernández and O'Regan [13] considered the following model problem:

$$x'(t) = Ax(t) + f(t, x(t)), t \in J_i = (s_i, t_{i+1}], i = 0, 1, \dots, N, \tag{1}$$

$$x(t) = g_i(t, x(t)), t \in (t_i, s_i], i = 1, 2, \dots, N, \tag{2}$$

$$x(0) = x_0, \tag{3}$$

where $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup of bounded linear operators $\{Q(t)\}_{t\geq 0}$ on X; $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < ... < t_N \leq s_N \leq t_{N+1} = T$ is a partition of the interval $[0, T]$; f and g_i are suitable functions. Recent developments on qualitative analysis of differential equations in classical as well as in fractional derivatives can be found in [2, 6, 8, 9, 11, 16, 18, 19].

Functional differential equations may be used to represent those phenomena in which the derivative of the state variable not only depends on its present state but also on the knowledge of past time. Abada et al. [1] established sufficient conditions for the existence of mild and extremal solutions for some impulsive functional differential equations in separable Banach spaces of the form

$$\begin{aligned} x'(t) - Ax(t) &= f(t, x_t), \text{ a.e. } t \in J = [0, T], t \neq t_k, k = 1, 2, \dots, N, \\ \Delta x|_{t=t_K} &= I_k(x(t_k^-)), k = 1, 2, \dots, N, \\ x(t) &= \phi(t), t \in [-r, 0], \end{aligned}$$

where $A : D(A) \subset X \to X$ generates a C_0-semigroup; f and $I_k, k = 1, 2, \dots, N$ are given functions, and $\mathscr{D} = \{\psi : [-r, 0] \to X, \psi$ is continuous everywhere except at a finite number of points s at which $\psi(s^-), \psi(s^+)$ exist and $\psi(s^-) = \psi(s^+)\}$.

Agarwal et al. [3] studied the existence of solution of a class of fractional neutral functional differential equations with bounded delay of the following form:

$$\begin{aligned} {}^C D^\alpha(x(t) - g(t, x_t)) &= f(t, x_t), t \in (t_0, \infty), t_0 \geq 0, \\ x_{t_0} &= \phi \in C([-r, 0], \mathbb{R}^n). \end{aligned}$$

The existence of solution was achieved by using Burton-Kirk's's fixed point theorem. Jiang [14] used analytic semigroup theory of linear operators and fixed point theory to prove the existence of mild solutions for a class of semilinear fractional differential equations with finite delay. Guo et al. [12] used Banach fixed point theorem and Schauder fixed point theorem to study the existence of solutions of impulsive fractional functional differential equations. Bellmekki et al. [5] established sufficient conditions for the existence and uniqueness of solution of semilinear functional differential equations with finite delay.

In this work, we establish sufficient conditions for the existence of mild solution for a class of impulsive fractional functional differential equations with finite delay of the following form:

$${}^C D_t^\alpha x(t) = Ax(t) + f(t, x_t), t \in J_i = J_i, i = 0, 1, ..., N, \tag{4}$$

$$x(t) = g_i(t, x_t), t \in (t_i, s_i], i = 1, 2, ..., N, \tag{5}$$

$$x(t) = \phi(t), t \in [-r, 0], \tag{6}$$

where A, X, the points s and t are same as in the description in [13]; $\mathscr{D}$ as defined in [1] is a Banach space with respect to the norm $\|\phi\|_{\mathscr{D}} = \sup_{-r\leq s\leq 0} \|\phi(s)\|$; $x_t(s) = x(t + s), -r \leq s \leq 0$, i.e., x_t represents the history of the state from $t - r$ up to the present time t. We further assume that the semigroup $\{Q(t)\}_{t>0}$ is uniformly bounded by $M > 1$.

We proceed as follows: in section 2 we recall some definitions and preliminaries which are required to develop our work; in section 3, we derive sufficient conditions for the existence of mild-solution of the system (4)–(6). At the end, an example is presented to support the obtained results.

2. Preliminaries

By $C(J, X)$ we denote the Banach space of bounded continuous functions from J into X with the norm

$$\|x\|_\infty = \sup_{t\in J}\|x(t)\|.$$

We consider the Banach space $\mathscr{D}_T = \{x : [-r, T] \to X$ such that $x|_{J_k} \in C(J_k, X)$, for $k = 0, 1, 2, ..., N$, and $x(t_k{}^+), x(t_k{}^-)$ exist ,$x(t_k{}^-) = x(t_k), k = 0, 1, 2, ..., N, x_0 = \phi \in \mathscr{D}$ and $\sup_{t\in[-r,T]} \|x(t)\| < \infty\}$, endowed with the norm

$$\|x\|_{\mathscr{D}_T} = \sup_{t\in[-r,T]} \|x(t)\|.$$

If $x \in \mathscr{D}_T$, then for each $t > 0$, x_t is an element of $\mathscr{D}$ and $x_t(\theta) = x(t+\theta), \theta \in [-r, 0]$. If $x \in \mathscr{P}_T$, then for any $i = 0, 1, 2, \dots, N$, the function $\tilde{x}_i \in C([t_i, t_{i+1}], X)$ is constructed as follows:

$$\tilde{x}_i(t) = \begin{cases} x(t), \text{ for } \ t \in (t_i, t_{i+1}], \\ x(t_i^+), \text{ for } \ t = t_i. \end{cases}$$

For $\mathscr{B} \subset \mathscr{D}_T$, we denote $\tilde{\mathscr{B}}_i = \{\tilde{x}_i : x \in \mathscr{D}_T\}$.

LEMMA 2.1. [13] *A set $\mathscr{B} \subset \mathscr{D}_T$ is relatively compact in $\mathscr{D}_T$ if and only if each set $\tilde{\mathscr{B}}_i$ is relatively compact in $C([t_i, t_{i+1}], X)$.*

Here we introduce some definitions and fundamental results of fractional calculus from the book by Zhou et al. [27].
Let $J = [a, b], -\infty < a < b < \infty$ be a finite interval on the real axis $\mathbb{R}$.

DEFINITION 2.2. *The Riemann-Liouville fractional integral ${}_aD_t^{-\alpha} f(t)$ of order $\alpha > 0$ is defined by*

$$I_a^\alpha f(t) ={}_a D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds,$$

provided the right-hand side is pointwise defined on $[a, b]$.

DEFINITION 2.3. *The Caputo fractional derivative of order $\alpha > 0$ for a function $f \in C_\alpha^n, n \in \mathbb{N}$ is defined as*

$${}_a^C D_t^\alpha f(t) ={}_a D_t^{-(n-\alpha)} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, t > a, n = [\alpha]+1,$$

where $[\alpha]$ denotes the integral part of α.

If there is no confusion about the base point of both the operators defined above, we simply remove it.

DEFINITION 2.4. [27] *Consider the fractional evolution equation*

$$\begin{aligned} {}^C D_t^\alpha x(t) &= Ax(t) + f(t, x_t), \ a.e.\ t \in J, 0 < \alpha < 1, \quad (7)\\ x(t) &= \phi \in \mathscr{D}. \quad (8) \end{aligned}$$

For continuous functions $f : J \times X \rightarrow X$ *with* A *generating the semigroup* $\{Q(t)\}_{t\geq 0}$, *a continuous function* $x : J \rightarrow X$ *satisfying the integral equation* $x(t) = S_\alpha(t)\phi(0) + \int_0^t P_\alpha(t-s)f(s, x_s)ds$ *is called a mild solution for the problem* (7)-(8). *Here*

$$S_\alpha(t) = \int_0^\infty M_\alpha(\theta)Q(t^\alpha\theta)d\theta, \ \ P_\alpha(t) = \alpha\int_0^\infty \theta M_\alpha(\theta)Q(t^\alpha\theta)d\theta, \ \ and$$

$$M_\alpha(\theta) = \sum_{n=1}^{\infty} \frac{(-\omega)^{n-1}}{(n-1)!\Gamma(1-\alpha n)}, 0 < \alpha < 1, \omega \in \mathbb{C},$$

where $M_\alpha(\omega)$ *satisfies the following equality*

$$\int_0^\infty M_\alpha(\theta)\theta^\delta d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\alpha\delta)}, \ for\ \delta \geq 0.$$

LEMMA 2.5. [28] *For any* $t > 0$, $S_\alpha(t)$ *and* $P_\alpha(t)$ *are linear bounded operators, more precisely, for any* $x \in X$,

$$\|S_\alpha(t)x\| \leq M\|x\|, \|P_\alpha(t)x\| \leq \frac{M}{\Gamma(\alpha)}\|x\|.$$

LEMMA 2.6. [28] *For* $\{Q(t)\}_{t>0}$ *compact* ,$\{S_\alpha(t)\}_{t>0}$ *and* $\{P_\alpha(t)\}_{t>0}$ *are also compact.*

LEMMA 2.7. [28] *Operators* $\{S_\alpha(t)\}_{t>0}$ *and* $\{P_\alpha(t)\}_{t>0}$ *are strongly continuous.*

LEMMA 2.8. *(Burton-Kirk's fixed point theorem) Let* X *be a Banach space and* F_1, F_2 *be two operators satisfying*
(a) F_1 *is a contraction and*
(b) F_2 *is completely continuous.*
Then, either the operator equation $x = F_1(x) + F_2(x)$ *possesses a solution, or the set* $\mathscr{E} = \{x \in X : \lambda F_1(\frac{x}{\lambda}) + \lambda F_2(x) = x, \ for\ some\ \ 0 < \lambda < 1\}$ *is unbounded.*

3. Existence of PC-mild solution

In this section we first formulate the definition of PC-mild solution of our problem and then prove the existence of solutions with finite delay.
In view of definition 2.4 and the results by Hernández and O'Regan [13], we define the mild solution as follows:

DEFINITION 3.1. *A function* $x \in \mathscr{D}_T$ *satisfying the integral equation*

$$x(t) = \begin{cases} \phi(t), \ \ t \in [-r, 0], \\ S_\alpha(t)x_0 + \int_0^t P_\alpha(t-s)f(s, x_s)ds, t \in [0, t_1], \\ g_i(t, x_t), t \in (t_i, s_i], \\ S_\alpha(t-s_i)g_i(s_i, x_{s_i}) + \int_{s_i}^t P_\alpha(t-s)f(s, x_s)ds, t \in [s_i, t_{i+1}], \end{cases}$$

is to be called a PC-mild solution of the problem (4)-(6).

We introduce the following hypotheses:
(H1) The functions g_i are continuous and there are positive constants L_{g_i} such that $\|g_i(t,\psi_1) - g_i(t,\psi_2)\| \leq L_{g_i}\|\psi_1 - \psi_2\|_{\mathscr{D}}$, for all $\psi_1, \psi_2 \in \mathscr{D}, t \in (t_i, s_i]$ and each $i = 1, 2, ..., N$;
(H2) For each $\phi \in \mathscr{D}$, the function $f(.,\phi) : J \to X$ is strongly measurable and for each $t \in J$, the function $f(t,.) : \mathscr{D} \to X$ is continuous;
(H3) There exist a constant $\alpha_1 \in (0,\alpha)$ and a function $m \in L^{\frac{1}{\alpha_1}}(J, \mathbb{R}^+)$ such that

$$\|f(t,\psi)\| \leq m(t)W(\|\psi\|_{\mathscr{D}}), \text{a.e. } t \in J, \psi \in \mathscr{D},$$

where $W : [0,\infty) \to \mathbb{R}^+$ is a continuous nondecreasing function with

$$K_1 \int_{s_i}^{t_{i+1}} (t-s)^{\alpha-1} m(s) ds < \int_{K_0}^{\infty} \frac{ds}{W(s)},$$

where

$$0 = M\|\phi(0)\|, \quad K_1 = \frac{M}{\Gamma(\alpha)}, \text{ for } t \in [0, t_1], \text{ and}$$

$$\tilde{K}_0 = \max_{1\leq i\leq N} \frac{M\|g_i(s_i, 0)\|}{1 - ML_{g_i}}, \tilde{K}_1 = \max_{1\leq i\leq N} \frac{M}{(1 - ML_{g_i})\Gamma(\alpha)}, \qquad t \in [s_i, t_{i+1}], i = 1, 2, \ldots, N.$$

(H4) The operator A is the infinitesimal generator of a compact semigroup of uniformly bounded linear operators $\{Q(t)\}_{t\geq 0}$ such that there exists $M > 1$ satisfying

$$\|Q(t)\| \leq M.$$

It is to be noted that $a = \dfrac{\alpha - 1}{1 - \alpha_1} \in (-1, 0)$.

THEOREM 3.2. *Assume that the above hypotheses hold and $\|g_i(.,0)\|$ are bounded for each $i = 1, 2, \ldots, N$. Then, for every initial value $\phi \in \mathscr{D}$, the system of equations (4)-(6) has a unique PC-mild solution $x \in \mathscr{D}_T$, provided $(1 + M)L_{g_i} < 1$.*

Proof. Let $\mathscr{F} : \mathscr{D}_T \to \mathscr{D}_T$ be defined by

$$\mathscr{F}x(t) = \begin{cases} \phi(t), t \in [-r, 0], \\ g_i(t, x_t), t \in (t_i, s_i], \\ S_\alpha(t)\phi(0) + \int_0^t P_\alpha(t-s) f(s, x_s) ds, t \in [0, t_1], \\ S_\alpha(t - s_i) g_i(s_i, x_{s_i}) + \int_{s_i}^t P_\alpha(t-s) f(s, x_s) ds, t \in [s_i, t_{i+1}]. \end{cases}$$

By hypothesis $(H3)$ and the results by Chen et al. [9], it is easily observed that the operator is well-defined. To apply Burton-Kirk's fixed point theorem, we use the following decomposition of $\mathscr{F}$:

$$\mathscr{F} = \mathscr{F}^1 + \mathscr{F}^2 = \sum_{i=0}^{N} \mathscr{F}_i^1 + \sum_{i=0}^{N} \mathscr{F}_i^2,$$

where $\mathscr{F}_i^j : \mathscr{D}_T \to \mathscr{D}_T, i = 1, 2, \ldots, N,\ j = 1, 2$ are defined as

$$\mathscr{F}_i^1 x(t) = \begin{cases} g_i(t, x_t),\ t \in (t_i, s_i], i \geq 1, \\ S_\alpha(t-s) g_i(s, x_{s_i}),\ t \in J_i, i \geq 1, \\ S_\alpha(t) x_0,\ t \in [0, t_1], \\ 0,\ \text{otherwise}, \end{cases}$$

and

$$\mathscr{F}_i^2 x(t) = \begin{cases} \phi(t),\ t \in [-r, 0], \\ \int_{s_i}^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s, x_s) ds, t \in J_i, i \geq 0, \\ 0,\ \text{otherwise}. \end{cases}$$

Our proof consists of the following six steps.

Step I: To show that the function $\mathscr{F}^2$ is continuous.
Let $\{x^n\}_{n=1}^\infty$ be a sequence of functions in $\mathscr{D}_T$ such that x^n converges to $x \in \mathscr{D}_T$.
Then $\lim_{n\to\infty} x^n(s) = x(s)$, for $s \in [-r, T]$.
Since $\|x_s\| \leq \|x\|_\infty$, for $s \in J$, by condition $(H2)$, we have

$$\lim_{n\to\infty} f(s, x_s^n) = f(s, x_s) \ \text{ for each } s \in J_i.$$

Now, for each $s \in J_i$,

$$\|(\mathscr{F}_i^2 x^n)(s) - (\mathscr{F}_i^2 x)(s)\| \leq \frac{M}{\Gamma(\alpha)} \frac{t_{i+1}^\alpha}{\alpha} \sup_{s \in [s_i, t_{i+1}]} \|f(s, x_s^n) - f(s, x_s)\|.$$

Hence by Lebesgue's dominated convergence theorem, we have

$$\|(\mathscr{F}_i^2 x^n) - (\mathscr{F}_i^2 x)\| \to 0 \text{ as } n \to \infty.$$

Hence $\mathscr{F}^2$ is continuous in $\mathscr{D}_T$.

Step II: To show that $\mathscr{F}^2$ sends a bounded set to a bounded set in $\mathscr{D}_T$.
For $\eta > 0$, consider the ball $B_\eta = \{x \in \mathscr{D}_T : \|x\|_{\mathscr{D}_T} \leq \eta\}$.
Now, for any $x \in B_\eta$ and $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned} \|(\mathscr{F}_i^2 x)(t)\| &= \|\int_{s_i}^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s, x_s) ds\| \\ &\leq \frac{M}{\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} P_\alpha(t-s) m(s) W(\|x_s\|) ds \\ &\leq \frac{M}{\Gamma(\alpha)} W(\eta) \int_{s_i}^t (t-s)^{\alpha-1} m(s) ds \\ &\leq \frac{M}{\Gamma(\alpha)} W(\eta) \frac{(t_{i+1} - s_i)^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|m\|_{L^{\frac{1}{\alpha_1}}([s_i, t_{i+1}])} \\ &=: l_i,\ \text{a finite quantity}. \end{aligned}$$

Hence for each $x \in B_\eta$ and $i = 0, 1, \ldots, N$,

$$\|(\mathscr{F}_i^2 x)(t)\| \leq l_i.$$

Also the boundedness of $(\mathscr{F}_i^2 x)(t)$ is trivial for any $\widetilde{t \notin} J_i$.

Step III: To show that the set of functions $[\mathscr{F} \widetilde{x : x \in} B_r]_i, i = 0, 1, \ldots, N$, is an equicontinuous set in $C([t_i, t_{i+1}]; X)$.

Let $x \in B_\eta$ and $s_i < \tau_1 < \tau_2 \leq t_{i+1}$.
Now

$$\begin{aligned}
\|(\mathscr{F}_i^2 x)(\tau_2) - (\mathscr{F}_i^2 x)(\tau_1)\| &= \|\int_{s_i}^{\tau_2} (\tau_2 - s)^{\alpha-1} P_\alpha(\tau_2 - s) f(s, x_s) ds \\
&- \int_{s_i}^{\tau_1} (\tau_1 - s)^{\alpha-1} P_\alpha(\tau_1 - s) f(s, x_s) ds\| \\
&\leq \|\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} P_\alpha(\tau_2 - s) f(s, x_s) ds\| \\
&+ \|\int_{s_i}^{\tau_1} P_\alpha(\tau_2 - s)[(\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1}] f(s, x_s) ds\| \\
&+ \|\int_{s_i}^{\tau_1} (\tau_1 - s)^{\alpha-1} [P_\alpha(\tau_2 - s) f(s, x_s) - P_\alpha(\tau_1 - s) f(s, x_s)] ds\| \\
&\leq \frac{M}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} m(s) W(\|x_s\|_D) ds \\
&+ \frac{M}{\Gamma(\alpha)} \| \int_{s_i}^{\tau_1} [(\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1}] m(s) W(\|x_s\|_D) ds \\
&+ \|\int_{s_i}^{\tau_1} (\tau_1 - s)^{\alpha-1} [P_\alpha(\tau_2 - s) P_\alpha(\tau_1 - s)] m(s) W(\|x_s\|_D) dt \\
&\leq \frac{M}{\Gamma(\alpha)} W(\eta) \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} m(s) ds \\
&+ \frac{M}{\Gamma(\alpha)} W(\eta) \int_{s_i}^{\tau_1} [(\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1}] m(s) ds \\
&+ W(\eta) \int_{s_i}^{\tau_1} (\tau_1 - s)^{\alpha-1} \|P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\| m(s) ds \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

We have

$$\begin{aligned}
I_1 &= \frac{M}{\Gamma(\alpha)} W(\eta) \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} m(s) ds \\
&\leq \frac{M}{\Gamma(\alpha)} W(\eta) \frac{(\tau_2 - \tau_1)^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|m\|_{L^{\frac{1}{\alpha_1}}[s_i, t_{i+1}]} \\
&\to 0 \text{ as } \tau_2 \to \tau_1.
\end{aligned}$$

For $\tau_1 < \tau_2$,

$$
\begin{aligned}
I_2 &\leq \frac{M}{\Gamma(\alpha)} W(\eta) \Big[\int_{s_i}^{\tau_1} \big[(\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1} \big]^{\frac{1}{1-\alpha_1}} ds \Big]^{1-\alpha_1} \|m\|_{L^{\frac{1}{\alpha_1}}[s_i, t_{i+1}]} \\
&\leq \frac{M}{\Gamma(\alpha)} W(\eta) \Big[\int_{s_i}^{\tau_1} \big[(\tau_1 - s)^{a} - (\tau_2 - s)^{a} \big] ds \Big]^{1-\alpha_1} \|m\|_{L^{\frac{1}{\alpha_1}}[s_i, t_{i+1}]} \\
&\leq \frac{M}{\Gamma(\alpha)(1+a)^{(1-\alpha_1)}} W(\eta) \big[(\tau_2 - \tau_1)^{1+a} - ((\tau_2 - s_i)^{1+a} - (\tau_1 - s_i)^{1+a}) \big]^{1-\alpha_1} \|m\|_{L^{\frac{1}{\alpha_1}}[s_i, t_{i+1}]} \\
&\leq \frac{M}{\Gamma(\alpha)(1+a)^{(1-\alpha_1)}} W(\eta) (\tau_2 - \tau_1)^{(1+a)(1-\alpha_1)} \|m\|_{L^{\frac{1}{\alpha_1}}[s_i, t_{i+1}]} \\
&\to 0 \text{ as } \tau_2 \to \tau_1.
\end{aligned}
$$

For $\epsilon > 0$ small enough, we have

$$
\begin{aligned}
I_3 &\leq W(\eta) \int_{s_i}^{\tau_1 - \epsilon} (\tau_1 - s)^{\alpha-1} \|P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\| m(s) ds \\
&\quad + W(\eta) \int_{\tau_1 - \epsilon}^{\tau_1} (\tau_1 - s)^{\alpha-1} \|P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\| m(s) ds \\
&\leq W(\eta) \int_{s_i}^{\tau_1} (\tau_1 - s)^{\alpha-1} m(s) ds \sup_{s \in [s_i, t_1 - \epsilon]} \|P_\alpha(\tau_2 - s) - P_\alpha(\tau_1 - s)\| \\
&\quad + \frac{2M}{\Gamma(\alpha)} \int_{t_1 - \epsilon}^{t_1} (t_1 - s)^{(\alpha-1)} m(s) ds.
\end{aligned}
$$

First term on the right-hand side tends to zero as $\tau_2 \to \tau_1$ since $P_\alpha(t)$ is compact for $t > 0$ and hence continuous in the uniform operator topology. Second term tends to zero as $\epsilon \to 0$ by I_2.

Step IV: To show that for $i = 0, 1, \ldots, N$ and $s_i < s < t \leq t_{i+1}$, the set $V(\tau) = \bigcup_{\tau \in [s,t]} \{(\mathscr{F}_i^2 x)(\tau) : x \in B_\eta\}$ is a pre-compact set in X.

For $0 < \epsilon < t - s_i$ and any $\delta > 0$, define an operator $(\mathscr{F}_i^2)_\epsilon^\delta$ on B_η by the formula

$$
\begin{aligned}
(\mathscr{F}_i^2)_\epsilon^\delta x(\tau) &= \alpha \int_{s_i}^{\tau - \epsilon} \int_\delta^\infty \theta (\tau - s)^{\alpha-1} M_\alpha(\theta) Q((\tau - s)^\alpha \theta) f(s, x_s) d\theta ds \\
&= \alpha Q(\epsilon^\alpha \delta) \int_{s_i}^{\tau - \epsilon} \int_\delta^\infty \theta (\tau - s)^{\alpha-1} M_\alpha(\theta) Q((\tau - s)^\alpha \theta - \epsilon^\alpha \delta) f(s, x_s) d\theta ds.
\end{aligned}
$$

From the compactness of the operator $Q(\epsilon^\alpha\delta)$, we see that the set $V_\epsilon^\delta(\tau) = \{(\mathscr{F}_i^2)_\epsilon^\delta x(\tau) : x \in B_\eta\}$ is relatively compact in X. Moreover for any $x \in B_\eta$, we have

$$\begin{aligned}\|(\mathscr{F}_i^2)x(\tau) - (\mathscr{F}_i^2)_\epsilon^\delta x(\tau)\| &\leq \alpha\|\int_{s_i}^{\tau}\int_0^{\delta}\theta(\tau-s)^{\alpha-1}M_\alpha(\theta)Q((\tau-s)^\alpha\theta-\epsilon^\alpha\delta)f(s,x_s)d\theta ds\| \\ &+ \alpha\|\int_{\tau-\epsilon}^{\tau}\int_{\delta}^{\infty}\theta(\tau-s)^{\alpha-1}M_\alpha(\theta)Q((\tau-s)^\alpha\theta-\epsilon^\alpha\delta)f(s,x_s)d\theta ds\| \\ &\leq W(\eta)\alpha M\int_{s_i}^{\tau}(\tau-s)^{\alpha-1}m(s)ds\int_0^{\delta}M_\alpha(\theta)d\theta \\ &+ W(\eta)\frac{M}{\Gamma(\alpha)}\int_{\tau-\epsilon}^{\tau}(\tau-s)^{\alpha-1}m(s)ds \\ &\to 0 \text{ as } \epsilon\to 0, \delta\to 0.\end{aligned}$$

Therefore, there are pre-compact sets arbitrarily close to the set $V_\epsilon^\delta(\tau) = \{(\mathscr{F}_i^2)_\epsilon^\delta x(\tau) : x \in B_\eta\}$. Hence the set $V(\tau)$ is pre-compact in X. Consequently, the operator $\mathscr{T}^2 : \mathscr{D}_T \to \mathscr{D}_T$ is completely continuous.

Step V: To show that $\mathscr{F}^1$ is a contraction on B_η.
Let $x, y \in B_\eta$ and $t \in (t_i, t_{i+1}], i = 1, 2, \ldots, N$. Therefore

$$\begin{aligned}\|\mathscr{T}_i^1 x(t) - \mathscr{T}_i^1 y(t)\| &\leq (1+M)L_{g_i}\|x_t - y_t\|_{\mathscr{D}} \\ &\leq (1+M)L_{g_i}\|x-y\|_\infty.\end{aligned}$$

This implies that

$$\|\mathscr{T}^1 x - \mathscr{T}^1 y\|_{\mathscr{D}_T} \leq \Theta\|x-y\|_{\mathscr{D}_T},$$

which is a contraction since $\Theta < 1$.

Step VI: To find the a priori bounds.
Consider the set

$$\mathscr{E} = \{x \in \mathscr{D}_T : x = \lambda\mathscr{T}^2(x) + \lambda\mathscr{T}^1\left(\frac{x}{\lambda}\right) \text{ for some } 0<\lambda<1\}.$$

For each $t \in [0, t_1]$, we have

$$x(t) = \lambda S_\alpha(t)\phi(0) + \lambda\int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s,x_s)ds.$$

Hence for each $t \in [0, t_1]$, we have

$$\|x(t)\| \leq M\|\phi(0)\| + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)W(\|x_s\|_{\mathscr{D}})ds. \tag{9}$$

But $\|x_t\|_{\mathscr{D}\leq\{\sup\|x(t+s)\|::-r\leq s\leq 0\},0\leq t\leq t_1}$.
If we define $\mu(t) = \{\sup\|x(s)\| :: -r \leq s \leq t\}, 0 \leq t \leq t_1$, then (9) becomes

$$\|x(t)\| \leq M\|\phi(0)\| + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)W(\mu(s))ds. \tag{10}$$

Hence from the definition of μ, we have

$$\mu(t) \leq M\|\phi(0)\| + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}m(s)W(\mu(s))ds.$$

Thus we have

$$\mu(t) \leq K_0 + K_1\int_0^t (t-s)^{\alpha-1}m(s)W(\mu(s))ds,$$

where $K_0 = M\|\phi(0)\|, K_1 = \frac{M}{\Gamma(\alpha)}$.
If we denote the right-hand side inequality by $v(t)$, then

$$\mu(t) \le v(t) \ \forall \ t \in [0, t_1], \ v(0) = K_0,$$

and

$$v'(t) = (s-t)^{\alpha-1} m(t) W(\mu(t)).$$

This gives

$$v'(t) \le (s-t)^{\alpha-1} m(t) W(v(t)).$$

Therefore,

$$\int_{v(0)}^{v(t)} \frac{du}{W(u)} \le K_1 \int_0^t (t-s)^{\alpha-1} m(s) ds < \int_{K_0}^{\infty} \frac{du}{W(u)}.$$

Hence there exists a constant C such that

$$\mu(t) \le v(t) \le C, \ \forall \ t \in [0, t_1].$$

Now from the definition of μ, it follows that

$$\|x\|_{\mathscr{D}_T} \le \mu(t_1) \le C, \ \forall \ x \in \mathscr{E}.$$

For each $t \in (t_i, s_i], i = 1, 2, \dots, N$,

$$x(t) = \lambda g_i \left(t, \frac{x_t}{\lambda}\right).$$

This implies that for each $t \in (t_i, s_i]$,

$$\|x(t)\| \le L_{g_i} \|x_t\| + \lambda \|g_i(t, 0)\|,$$

$$\|x(t)\| \le L_{g_i} \|x_t\|_{\mathscr{D}_T} + \|g_i(t, 0)\|. \tag{11}$$

If $\|x_t\|_{\mathscr{D}_T} \le \mu(t)$, then (11) becomes

$$\|x(t)\| \le L_{g_i} \mu(t) + \|g_i(t, 0)\|. \tag{12}$$

Using the definition of μ in (12), we have

$$\mu(t) \le L_{g_i} \mu(t) + \|g_i(t, 0)\|. \tag{13}$$

Thus

$$\mu(t) \le \frac{\|g_i(t, 0)\|}{1 - L_{g_i}} = M_{t_i}.$$

This gives $\mu(t) \le M_{t_i}, t \in (t_i, s_i]$.
Hence from (12), we have

$$\|x(t)\| \le L_{g_i} M_{t_i} + \|g_i(t, 0)\| =: L_i.$$

Thus

$$\|x\|_{\mathscr{D}_T} \le L_i.$$

Finally for $t \in [s_i, t_{i+1}], i = 1, 2, \dots, N$, we have

$$x(t) = \lambda S_\alpha(t - s_i) g_i(s_i, \frac{x_{s_i}}{\lambda}) + \lambda \int_{s_i}^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s, x_s) ds.$$

Hence for each $t \in [s_i, t_{i+1}]$, we have

$$\|x(t)\| \le M L_{g_i} \|x_{s_i}\| + M\lambda \|g_i(s_i, 0)\| + \lambda \frac{M}{\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} m(s) W(\|x_s\|_{\mathscr{D}}) ds.$$

Therefore,

$$\|x(t)\| \leq ML_{g_i}\|x_{s_i}\| + M\|g_i(s_i,0)\| + \frac{M}{\Gamma(\alpha)}\int_{s_i}^{t}(t-s)^{\alpha-1}m(s)W(\|x_s\|_{\mathscr{D}})ds. \quad (14)$$

If $\|x_t\|_{\mathscr{D}_T} \leq \mu(t)$, then (14) becomes

$$\|x(t)\| \leq ML_{g_i}\mu(t) + M\|g_i(s_i,0)\| + \frac{M}{\Gamma(\alpha)}\int_{s_i}^{t}(t-s)^{\alpha-1}m(s)W(\mu(s))ds. \quad (15)$$

Using the definition of μ in (15), we have

$$\mu(t) \leq \tilde{K}_0 + \tilde{K}_1\int_{s_i}^{t}(t-s)^{\alpha-1}m(s)W(\mu(s))ds, \quad (16)$$

where

$$\tilde{K}_0 = \max_{1\leq i\leq N}\frac{M\|g_i(s_i,0)\|}{1-ML_{g_i}}, \tilde{K}_1 = \max_{1\leq i\leq N}\frac{M}{(1-ML_{g_i})\Gamma(\alpha)}.$$

If we denote the right hand side inequality by $v(t)$, then

$$\mu(t) \leq v(t),\ \forall\ t \in [s_i,t_{i+1}],\ v(s_i) = \tilde{K}_0,$$

and

$$v'(t) = \tilde{K}_1(s-t)^{\alpha-1}m(t)W(\mu(t)),\ t \in [s_i,t_{i+1}].$$

By the increasing property of W, we obtain

$$v'(t) \leq \tilde{K}_1(s-t)^{\alpha-1}m(t)W(v(t)),\ t \in [s_i,t_{i+1}].$$

Hence upon integration, we get

$$\int_{v(s_i)}^{v(t)}\frac{ds}{W(s)} \leq \tilde{K}_1\int_{s_i}^{t}(t-s)^{\alpha-1}m(s)ds < \int_{\tilde{K}_0}^{\infty}\frac{ds}{W(s)}.$$

Therefore there exists a constant $\tilde{C}_i$ such that $\mu(t) \leq v(t) \leq \tilde{C}_i,\ \forall t \in [s_i,t_{i+1}]$.
Consequently,

$$\|x\|_{\mathscr{D}_T} \leq ML_{g_i}\tilde{C}_i + M\|g_i(t,0)\| + \frac{M}{\Gamma(\alpha)}\int_{s_i}^{t}(t-s)^{\alpha-1}m(s)W(\tilde{C}_i)ds.$$

This implies that the set $\mathscr{E}$ is bounded.
Thus by Burton-Kirk's fixed point theorem, the operator $\mathscr{F}$ has a fixed point in $\mathscr{D}_T$ which is a mild solution of the system (4)-(6). □

4. Example

We consider the fractional reaction-diffusion equation with delay described by

$$\begin{aligned} {}^{C}D_t^{\alpha}x(t,z) &= \frac{\partial^2}{\partial z^2}x(t,z) + F(t,x(t-r,z)), t \in J_{i+1}, i = 0,1,...,N, z \in [0,\pi], && (17)\\ x(t,z) &= G_i(t,x(t-r,z)), z \in [0,\pi], t \in (t_i,s_i], i = 1,2,...,N, && (18)\\ x(t,0) &= x(t,\pi) = 0, t \in [0,T], && (19)\\ x(t,z) &= \phi(t,z), t \in [-r,0], z \in [0,\pi], && (20)\end{aligned}$$

where $r > 0, \phi \in \mathscr{D} = \{\psi : [-r,0]\times[0,\pi] \to \mathbb{R}, \psi$ is continuous everywhere except at a finite number of points s at which $\psi(s^-), \psi(s^+)$ exist and $\psi(s^-) = \psi(s^+)\}$, the impulse time t_i satisfies $t_0 = s_0 < t_1 \leq s_1 < t_2 < ... < t_N \leq s_N < t_{N+1} = T$ and F, G_i are given functions.

Let us take $X = L^2([0,\pi])$ and define $A : D(A) \subset X \to X$ by $Ax = x''$ with domain

$$D(A) = \{x \in X : x, x' \text{ are absolutely continuous }, x'' \in X, x(0) = x(\pi) = 0\}.$$

Then

$$Aw = \sum_{n=1}^{\infty} e^{-n^2 t}(w, w_n)w_n, w \in X,$$

where $(.,.)$ is an inner product in L^2 and $w_n(s) = \sqrt{\frac{2}{\pi}}\sin ns, n = 1, 2, \dots$ is the orthogonal set of eigenvectors in A. Then A generates a C_0-semigroup $\{Q(t)\}_{t\geq 0}$ on X. There exists $M \geq 1$ such that

$$\|Q(t)\|_{B(X)} \leq M.$$

Let $x(t)z = x(t, z), t \in J, z \in [0, \pi]$.
For the case $(t, \phi) \in [-r, b] \times \mathscr{D}$:
Assume that (i) For all $i = 0, 1, \dots, N$, the function $f : [s_i, t_{i+1}] \times \mathscr{D} \to X$ defined by $f(t, x_t)z = F(t, x(t - r, z)), t \in J_i, z \in [0, \pi]$ is continuous and satisfies hypotheses $(H2)$ and $(H3)$.
(ii) For all $i = 1, \dots, N$, the functions $g_i : (t_i, s_i] \times \mathscr{D} \to X$ defined by $g_i(t, x_t)z = G_i(t, x(t - r, z)), t \in (t_i, s_i], z \in [0, \pi]$ are continuous and satisfy hypothesis $(H1)$.
With the above setting, the system of equations (17)-(19) gets transformed to the abstract form (4)-(6). Since all the conditions of Theorem 3.2 are satisfied, the problem (17)-(19) has a mild solution x on $[-r, T] \times [0, \pi]$.

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