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GLOBAL ASYMPTOTIC ATTRACTIVITY AND STABILITY THEOREMS FOR NONLINEAR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article we obtain some qualitative basic existence and uniqueness results concerning the global attractivity and asymptotic stability of mild solutions for a nonlinear fractional differential equation with Caputo fractional derivative involving the pulling function via the classical Schauder [14] and Dhage [5, 11] fixed point principles. A linear perturbation of first type is also considered for the discussion via a hybrid fixed point theorem due to Dhage [6]. Our abstract results are illustrated by indicating numerical examples.

1. Statement of the Problem

Let $t_0 \in \mathbb{R}_+ = [0,\infty)$ be a fixed real number and let $J_\infty = [t_0,\infty)$. A continuous function $a : J_\infty \to (0,\infty)$ is called a **pulling function** if $\lim\limits_{t\to\infty} a(t) = \infty$. The class of pulling functions on J_∞ is denoted by $\mathcal{CRB}(J_\infty)$ and it is is introduced in Dhage [9, 12], and Dhage *et al.* [13]. There do exist several pulling functions, however the most useful pulling functions are $a_1(t) = e^{at},\ a > 0$ and $a_2(t) = t^2 + 1$ on J_∞. Note that if $a \in \mathcal{CRB}(J_\infty)$, then the reciprocal function $\overline{a} : J_\infty \to \mathbb{R}_+$ defined by $\overline{a}(t) = \frac{1}{a(t)}$ is continuous and bounded on J_∞ with $\lim\limits_{t\to\infty} \overline{a}(t) = 0$. If $a, b \in \mathcal{CRB}(J_\infty)$ are two pulling functions, then (i) $a + b$, (ii) $\lambda\, a$, $\lambda \in \mathbb{R}_+$, (iii) $a \cdot b$, " $\cdot$ " denotes the product of two functions, and (iv) a^n are also pulling functions, where a^n is the n-times composition of a.

We need the following fundamental definitions from fractional calculus (see Podlubny [16], Kilbas *et al.* [15] and references therein) in what follows.

Definition 1.1. If $J_\infty = [t_0,\infty)$ be an interval of the real line $\mathbb{R}$ for some $t_0 \in \mathbb{R}$ with $t_0 \geq 0$, then for any $x \in L^1(J_\infty,\mathbb{R})$, the Riemann-Liouville fractional integral of fractional order $q > 0$ is defined as

$$I_{t_0}^q x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t \frac{x(s)}{(t-s)^{1-q}}\, ds,\ \ t \in J_\infty,$$

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provided the right hand side is pointwise defined on (t_0, ∞), where Γ is the Euler's gamma function defined by $\Gamma(q) = \int_0^\infty e^{-t} t^{q-1}\, dt$.

Definition 1.2. If $x \in C^n(J_\infty, \mathbb{R})$, then the Caputo fractional derivative ${}^C D^q_{t_0} x$ of x of fractional order q is defined as

$$ {}^C D^q_{t_0} x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^t (t-s)^{n-q-1} x^{(n)}(s)\, ds,\ n-1 < q < n, n = [q]+1, $$

where $[q]$ denotes the integer part of the real number q, and Γ is the Euler's gamma function. Here $C^n(J_\infty, \mathbb{R})$ denotes the space of real valued functions $x(t)$ which have continuous derivatives up to order n on J_∞.

Given a pulling function $a \in \mathcal{CRB}(J_\infty) \bigcap C^1(J_\infty, \mathbb{R})$, we consider the following nonlinear fractional differential equation (in short FRDE) involving the Caputo fractional derivative,

$$ {}^C D^q_{t_0} \big[a(t)x(t)\big] = f(t, x(t)),\ \ t \in J_\infty, \quad x(t_0) = x_0, \tag{1.1} $$

where ${}^C D^q$ is the Caputo fractional derivative of fractional order $0 < q \le 1$, Γ is a Euler's gamma function and $f\ :\ J_\infty \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Definition 1.3. By a *solution* for the fractional differential equation (1.1) we mean a function $x \in C^1(J_\infty, \mathbb{R})$ that satisfies the equations in (1.1) on J_∞, where $C^1(J_\infty, \mathbb{R})$ is the space of continuous real-valued functions defined on J_∞ whose first derivative x' exists and $x' \in C(J_\infty, \mathbb{R})$.

The FRDE (1.1) is a scalarly multiplicative perturbation of second type obtained by multiplying the unknown function under Caputo fractional derivative with a scalar function. The classification of the different types of perturbations of a differential equation is given in Dhage [10]. Now we state a couple of useful lemmas which are helpful in transforming the fractional Caputo differential equations into the Riemann-Liouville fractional integral equations.

Lemma 1.1. (Kilbas *et al.* [15, page 96]) *Let $x \in C^n(J, \mathbb{R})$ and $q > 0$. Then, we have*

$$ I^q_{t_0}\Big({}^C D^q_{t_0} x(t)\Big) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(t_0)}{k\,!}(t-t_0)^k = x(t) + \sum_{k=0}^{n-1} c_k (t-t_0)^k $$

for all $t \in J = [a, b]$, $n-1 < q \le n$, $n = [q]+1$ and $c_0, \ldots, c_{n-1}$ are constants.

The converse of the above lemma is not true. It is mentioned in Kilbas *et al.* [15, page 95] that if $q > 0$ and $x \in C(J, \mathbb{R})$, then ${}^C D^q_{t_0}\Big(I^q_{t_0} x(t)\Big) = x(t)$ for all $t \in J = [a, b]$. However, it has been proved recently in Cohen [3, 4] that it is not true for any continuous functions on J.

Remark 1.1. The conclusion of the above Lemma 1.1 also remains true if we replace the function spaces $C^n([a, b], \mathbb{R})$ and $C([a, b], \mathbb{R})$ with the function spaces $BC^n(J_\infty, \mathbb{R})$ and $BC(J_\infty, \mathbb{R})$ respectively. The proofs of these facts are similar to Lemmas 1.1 of Kilbas *et al.* [15, page 96]) with appropriate modifications.

2. Characterizations of Solutions

We place the nonlinear problem (1.1) in the function space $BC(J_\infty, \mathbb{R})$ of continuous and bounded real-valued functions defined on J_∞ for better navigation. Define a standard supremum norm $\|\cdot\|$ in $BC(J_\infty, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J_\infty} |x(t)|.$$

Clearly, $BC(J_\infty, \mathbb{R})$ is a Banach space w.r.t. the above supremum norm. Let $\mathcal{T} : BC(J_\infty, \mathbb{R}) \to BC(J_\infty, \mathbb{R})$ be a continuous operator and consider the following operator equation in $BC(J_\infty, \mathbb{R})$,

$$\mathcal{T}x(t) = x(t) \tag{2.1}$$

for all $t \in J_\infty$. Below we give required characterizations of the solutions for the operator equation (2.1) in the space $BC(J_\infty, \mathbb{R})$. The details of these characterizations appear in Banas and Dhage [1], Dhage [8, 9], Dhage *et al.* [13] and references therein.

Definition 2.1. A solution $x = x(t)$ of the operator equation (2.1) is said to be **globally attractive** if

$$\lim_{t\to\infty} (x(t) - y(t)) = 0. \tag{2.2}$$

holds for each solution $y = y(t)$ of (2.1) in $BC(J_\infty, \mathbb{R})$. In other words, we may say that solutions of the equation (2.1) are globally attractive if for arbitrary solutions $x(t)$ and $y(t)$ of (2.1) in $BC(J_\infty, \mathbb{R})$, the condition (2.2) is satisfied. In the case when the condition (2.2) is satisfied uniformly with respect to the space $BC(J_\infty, \mathbb{R})$, i.e., if for every $\epsilon > 0$ there exists $T > 0$ such that the inequality

$$|x(t) - y(t)| \leq \epsilon \tag{2.3}$$

is satisfied for all solutions $x, y \in BC(J_\infty, \mathbb{R})$ of (2.1) and for $t \geq T$, we will say that solutions of the equation (2.1) are **uniformly globally attractive** on J_∞.

Now we introduce the concept of global asymptotic stability of the solutions for the operator equation (2.1) in the space $BC(J_\infty, \mathbb{R})$.

Definition 2.2. A solution $x \in BC(J_\infty, \mathbb{R})$ of the equation (2.1) is called **asymptotic to t-axis** or simply **asymptotic** if $\lim_{t\to\infty} x(t) = 0$. In the case when the limit is uniform with respect to the solution set of the operator equation (2.1) in $BC(J_\infty, \mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > t_0 \geq 0$ such that $|x(t)| < \varepsilon$ for all solutions x of (2.1) in $BC(J_\infty, \mathbb{R})$ and for all $t \geq T$, we will say that solutions of equation (2.1) are **uniformly asymptotic** to x-axis on J_∞.

Definition 2.3. If all the solutions of the operator equation (2.1) are asymptotic and globally uniformly attractive, we say that they are **uniformly asymptotically attractive** or **stable** on J_∞.

To state the required fixed point techniques that will be used in the proofs of main results, we need the following definitions from the Banach space $\mathfrak{X}$ in what follows.

Definition 2.4 (Dhage [5, 8])**.** An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function if $\psi(0) = 0$. The class of all $\mathcal{D}$-functions on $\mathbb{R}+$ is denoted by $\mathfrak{D}$.

Definition 2.5 (Dhage [5])**.** Let $\mathfrak{X}$ be a Banach space with norm $\|\cdot\|$. An operator $\mathcal{T}:\mathfrak{X}\to\mathfrak{X}$ is called $\mathcal{D}$-Lipschitz if there exists a $\mathcal{D}$-function $\psi_{\mathcal{T}}\in\mathfrak{D}$ such that

$$\|\mathcal{T}x-\mathcal{T}y\|\leq\psi_{\mathcal{T}}\big(\|x-y\|\big) \tag{2.4}$$

for all elements $x,y\in\mathfrak{X}$.

If $\psi_{\mathcal{T}}(r)=k\,r$, $k>0$, $\mathcal{T}$ is called a Lipschitz operator on $\mathfrak{X}$ with the Lipschitz constant k. Again, if $0\leq k<1$, then $\mathcal{T}$ is called a contraction on $\mathfrak{X}$ with contraction constant k. Furthermore, if $\psi_{\mathcal{T}}(r)<r$ for $r>0$, then $\mathcal{T}$ is called a nonlinear $\mathcal{D}$-contraction on $\mathfrak{X}$. The class of all $\mathcal{D}$-functions satisfying the condition of nonlinear $\mathcal{D}$-contraction is denoted by $\mathfrak{D}\mathfrak{N}$.

An operator $\mathcal{T}:\mathfrak{X}\to\mathfrak{X}$ is called compact if $\overline{\mathcal{T}(\mathfrak{X})}$ is a compact subset of $\mathfrak{X}$. $\mathcal{T}$ is called totally bounded if for any bounded subset S of $\mathfrak{X}$, $\mathcal{T}(S)$ is a totally bounded subset of $\mathfrak{X}$. T is called completely continuous if $\mathcal{T}$ is continuous and totally bounded on $\mathfrak{X}$. Every compact operator is totally bounded, but the converse may not be true, however, two notions are equivalent on bounded subsets of $\mathfrak{X}$. The details of different types of nonlinear contraction, compact and completely continuous operators appear in Granas and Dugundji [14].

To prove the main existence results of next section, we need the following classical topological and analytic fixed point principles.

Theorem 2.1 (Schauder [14])**.** *Let S be a closed convex and bounded subset of a Banach space $\mathfrak{X}$ and let $\mathcal{T}:S\to S$ be a completely continuous operator. Then the operator equation $\mathcal{T}x=x$ has a solution.*

Theorem 2.2 (Dhage [5])**.** *Let $\mathfrak{X}$ be a Banach space and let $\mathcal{T}:\mathfrak{X}\to\mathfrak{X}$ be a nonlinear $\mathcal{D}$-contraction. Then the operator equation $\mathcal{T}x=x$ has a unique solution x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^*.*

3. Attractivity and Stability Results

We need the following hypotheses in the sequel.

(H$_1$) The function f is bounded on $J_\infty\times\mathbb{R}$ with bound M_f.

(H$_2$) There exists a $\mathcal{D}$-function $\psi_f\in\mathfrak{D}$ such that

$$|f(t,x)-f(t,y)|\leq\psi_f(|x-y|)$$

for all $(t,x),(t,y)\in J_\infty\times\mathbb{R}$.

(H$_3$) The pulling function a satisfies the condition $\lim\limits_{t\to\infty}\overline{a}(t)\,t^q=0$.

Remark 3.1. If $a\in\mathcal{CRB}(J_\infty)$, then $\overline{a}\in BC(J_\infty,\mathbb{R}_+)$ and so the number $\|\overline{a}\|=\sup_{t\in J_\infty}\overline{a}(t)$ exists. Again, since the hypothesis (H$_3$) holds, the function $w:\mathbb{R}_+\to\mathbb{R}_+$ defined by the expression $w(t)=\overline{a}(t)\,t^q$ is continuous on J_∞ and satisfies the relation $\lim\limits_{t\to\infty}w(t)=0$. So the number $W=\sup_{t\geq t_0}w(t)$ exists.

The following lemma useful in transforming the fractional differential equation into the fractional integral equation which follows by an applications of Lemmas 1.1.

Lemma 3.1. *For any function $h\in BC(J_\infty,\mathbb{R})$, if the function $x\in BC(J_\infty,\mathbb{R})$ is a solution of the FRDE*

$${}^CD^q_{t_0}\big[a(t)x(t)\big]=h(t),\;\; t\in J_\infty,\quad x(t_0)=x_0, \tag{3.1}$$

then, it satisfies the nonlinear fractional integral equation (FRIE)

$$x(t) = c_0\,\overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}h(s)\,ds \tag{3.2}$$

for all $t \in J_\infty$*, where* $c_0 = a(t_0)x_0$.

Definition 3.1. A solution $x \in BC(J_\infty, \mathbb{R})$ of the FRIE (3.2) is called a **mild solution** of the FRDE (3.1) defined on J_∞.

In the following we shall deal with the mild solution of the FRDE (3.1) on unbounded interval J_∞ of the real line $\mathbb{R}$. Our main existence and global attractivity result is as follows.

Theorem 3.1. *Assume that the hypotheses (*H_1*) and (*H_3*) hold. Then the FRDE (1.1) has a mild solution and mild solutions are uniformly globally asymptotically attractive defined on* J_∞.

Proof. Set $X = BC(J_\infty, \mathbb{R})$ and define a closed ball $\overline{B}_r(0)$ in X centered at origin 0 of radius r given by

$$r = |c_0|\,\|\overline{a}\| + \frac{M_f\,W}{\Gamma q}$$

where, c_0 is defined as in Lemma 3.1. Now, by an application of Lemma 3.1, the FRDE (1.1) is equivalent to the following fractional integral equation (in short FRDE)

$$x(t) = c_0\,\overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}f(s,x(s))\,ds \tag{3.3}$$

for all $t \in J_\infty$. Define the operator $\mathcal{T}$ on $\overline{B}_r(0)$ by

$$\mathcal{T}x(t) = c_0\,\overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}f(s,x(s))\,ds,\ \ t \in J_\infty. \tag{3.4}$$

Then the FRIE (3.3) is transformed into the operator equation as

$$\mathcal{T}x(t) = x(t),\ t \in J_\infty. \tag{3.5}$$

We show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.1 with $S = \overline{B}_r(0) \subset BC(J_\infty, \mathbb{R})$. Now, from continuity of the integral it follows that the function $t \to \mathcal{T}x(t)$ is continuous on J_∞ for each $x \in \overline{B}_r(0)$. Furthermore, by hypothesis (H_3),

$$\begin{aligned}
|\mathcal{T}x(t)| &\le |c_0|\,\overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}|f(s,x(s))|\,ds\\
&\le |c_0|\,\|\overline{a}\| + \frac{M_f}{\Gamma q}\overline{a}(t)\int_{t_0}^{t}(t-s)^{q-1}\,ds\\
&\le |c_0|\,\|\overline{a}\| + \frac{M_f}{\Gamma(q+1)}\overline{a}(t)\,t^q\\
&\le |c_0|\,\|\overline{a}\| + \frac{M_f W}{\Gamma q}
\end{aligned}$$

for all $t \in J_\infty$ and for all $x \in \overline{B}_r(0)$. Taking the supremum over t,

$$\|\mathcal{T}x\| \le |c_0|\,\|\overline{a}\| + \frac{M_f W}{\Gamma q} = r$$

for all $x \in \overline{B}_r(0)$. As a result, $\mathcal{T}$ defines the operator $\mathcal{T} : \overline{B}_r(0) \to \overline{B}_r(0)$.

Next we shows that $\mathcal{T}$ is a completely continuous operator on $\overline{B}_r(0)$. First, we show that $\mathcal{T}$ is continuous on $\overline{B}_r(0)$. To do this, let us fix arbitrarily $\epsilon > 0$ and let $\{x_n\}$ be a sequence of points in $\overline{B}_r(0)$ converging to a point $x \in \overline{B}_r(0)$. Then we get:

$$\begin{aligned} &|(\mathcal{T}x_n)(t) - (\mathcal{T}x)(t)| \\ &\qquad \le \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} \big|f(s,x_n(s)) - f(s,x(s))\big|\, ds \\ &\qquad \le \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} \Big[|f(s,x_n(s))| + |f(s,x(s))|\Big]\, ds \\ &\qquad \le \frac{2\,M_f \overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1}\, ds \\ &\qquad \le \frac{2\,M_f}{\Gamma q}\, w(t). \end{aligned} \tag{3.6}$$

Hence, by virtue of hypothesis (H$_3$), we infer that there exists a $T > 0$ such that $w(t) \le \dfrac{\epsilon}{M_f/\Gamma q}$ for $t \ge T$. Thus, for $t \ge T$, from the estimate (3.3) we derive that

$$|(\mathcal{T}x_n)(t) - (\mathcal{T}x)(t)| \le 2\epsilon \quad \text{as} \quad n \to \infty.$$

Furthermore, let us assume that $t \in [t_0, T]$. Then, by dominated convergence theorem, we obtain the estimate:

$$\begin{aligned} \lim_{n\to\infty} \mathcal{T}x_n(t) &= \lim_{n\to\infty} \left[c_0\, \overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} f(s,x_n(s))\, ds \right] \\ &= c_0\, \overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} \Big[\lim_{n\to\infty} f(s,x_n(s))\Big]\, ds \\ &= \mathcal{T}x(t) \end{aligned} \tag{3.7}$$

for all $t \in [t_0, T]$. This shows that $\mathcal{T}x_n \to \mathcal{T}x$ pointwise on J_∞. Moreover, it can be shown as below that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of functions in X. Now, following the arguments similar to that given in Granas *et al.* [14], it is proved that $\mathcal{T}$ is a continuous operator on $\overline{B}_r(0)$ into itself.

Next, we show that $\mathcal{T}$ is a compact operator on $\overline{B}_r(0)$. To finish, it is enough to show that every sequence $\{\mathcal{T}x_n\}$ in $\mathcal{T}(\overline{B}_r(0))$ has a Cauchy subsequence. Now, proceeding with the earlier arguments it is proved that

$$\|\mathcal{T}x_n\| \le |c_0|\, \|\overline{a}\| + \frac{M_f W}{\Gamma q} = r$$

for all $n \in \mathbb{N}$. This shows that $\{\mathcal{T}x_n\}$ is a uniformly bounded sequence in $\mathcal{T}(\overline{B}_r(0))$.

Next, we show that $\{\mathcal{T}x_n\}$ is also a equicontinuous sequence in $\mathcal{T}(\overline{B}_r(0))$. Let $\epsilon > 0$ be given. Since $\lim_{t\to\infty} w(t) = 0$, there is a real number $T_1 > t_0 \ge 0$ such that $|w(t)| < \dfrac{\epsilon}{8M_f/\Gamma(q+1)}$ for all $t \ge T_1$. Similarly, since $\lim\limits_{t\to\infty} \overline{a}(t) = 0$, for above $\epsilon > 0$, there is a real number $T_2 > t_0 \ge 0$ such that $|\overline{a}(t)| < \dfrac{\epsilon}{8|c_0|}$ for all $t \ge T_2$.

Thus, if $T = \max\{T_1, T_2\}$, then

$$|w(t)| < \frac{\epsilon}{8M_f/\Gamma(q+1)} \quad \text{and} \quad |\overline{a}(t)| < \frac{\epsilon}{8|c_0|} \tag{3.8}$$

for all $t \geq T$. Let $t, \tau \in J_\infty$ be arbitrary. If $t, \tau \in [t_0, T]$, then we have

$$\begin{aligned}
&|\mathcal{T}x_n(t) - \mathcal{T}x_n(\tau)| \\
&\quad \leq |c_0|\,|\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + \left|\frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}f(s,x(s))\,ds - \frac{\overline{a}(\tau)}{\Gamma q}\int_{t_0}^{\tau}(\tau-s)^{q-1}f(s,x(s))\,ds\right| \\
&\quad \leq |c_0|\,|\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + \left|\frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}f(s,x(s))\,ds - \frac{\overline{a}(\tau)}{\Gamma q}\int_{t_0}^{t}(\tau-s)^{q-1}f(s,x(s))\,ds\right| \\
&\quad + \left|\frac{\overline{a}(\tau)}{\Gamma q}\int_{t_0}^{t}(\tau-s)^{q-1}f(s,x(s))\,ds - \frac{\overline{a}(\tau)}{\Gamma q}\int_{t_0}^{\tau}(\tau-s)^{q-1}f(s,x(s))\,ds\right| \\
&\quad \leq |c_0|\,|\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + \frac{M_f}{\Gamma q}\int_{t_0}^{t}\left|\overline{a}(t)(t-s)^{q-1} - \overline{a}(\tau)(\tau-s)^{q-1}\right|\,ds \\
&\quad + \frac{M_f}{\Gamma q}\left|\int_{\tau}^{t}\left|\overline{a}(\tau)(\tau-s)^{q-1}\right|ds\right| \\
&\quad \leq |c_0|\,|\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + \frac{M_f}{\Gamma q}\int_{t_0}^{T}\left|\overline{a}(t)(t-s)^{q-1} - \overline{a}(\tau)(\tau-s)^{q-1}\right|\,ds \\
&\quad + \frac{M_f\,\|\overline{a}\|}{\Gamma(q+1)}\,|(\tau-t)^q|\,.
\end{aligned}$$

Since the functions $t \mapsto \overline{a}(t)$ and $t \mapsto \overline{a}(t)(t-s)^{q-1}$ are continuous on compact $[t_0, T]$, they are uniformly continuous there. Therefore, by the uniform continuity, for above ϵ we have the real numbers $\delta_1 > 0$ and $\delta_2 > 0$ depending only on ϵ such that

$$|t-\tau| < \delta_1 \Longrightarrow |\overline{a}(t) - \overline{a}(\tau)| < \frac{\epsilon}{9|c_0|}$$

and

$$|t-\tau| < \delta_2 \Longrightarrow \left|\overline{a}(t)(t-s)^{q-1} - \overline{a}(\tau)(\tau-s)^{q-1}\right| < \frac{\epsilon}{9M_f T/\Gamma q}.$$

Similarly, choose the real number $\delta_3 = \left(\frac{\epsilon}{9M_f\|\overline{a}\|\big/\Gamma(q+1)}\right)^{1/q} > 0$ so that

$$|t-\tau| < \delta_3 \Longrightarrow |(t-\tau)^q| < \frac{\epsilon}{9M_f\|\overline{a}\|/\Gamma(q+1)}.$$

Let $\delta_4 = \min\{\delta_1, \delta_2, \delta_3\}$. Then

$$|t-\tau| < \delta_4 \Longrightarrow |\mathcal{T}x_n(t) - \mathcal{T}x_n(\tau)| < \frac{\epsilon}{3}$$

for all $n \in \mathbb{N}$.

Again, if $t, \tau > T$, then we have a $\delta_5 > 0$ depending only on ϵ such that

$$
\begin{aligned}
&|\mathcal{T}x_n(t) - \mathcal{T}x_n(\tau)| \\
&\quad \le |c_0|\ |a(t) - a(\tau)| + \frac{\overline{a}(t)}{\Gamma q}\left|\int_{t_0}^{t}(t-s)^{q-1}f(s,x_n(s))\,ds\right| \\
&\qquad + \frac{\overline{a}(\tau)}{\Gamma q}\left|\int_{t_0}^{\tau}(\tau-s)^{q-1}f(s,x_n(s))\,ds\right| \\
&\quad \le \big|c_0\big[\,|\overline{a}(t)| + |\overline{a}(\tau)|\big] + \frac{M_f}{\Gamma(q+1)}\big[w(t)+w(\tau)\big] \\
&\quad < \frac{\epsilon}{2} < \epsilon
\end{aligned}
$$

for all $n \in \mathbb{N}$ whenever $|t-\tau| < \delta_5$. Similarly, if $t, \tau \in \mathbb{R}_+$ with $t < T < \tau$, then we have

$$|\mathcal{T}x_n(t) - \mathcal{T}x_n(\tau)| \le |\mathcal{T}x_n(t) - \mathcal{T}x_n(T)| + |\mathcal{T}x_n(T) - \mathcal{T}x_n(\tau)|.$$

Take $\delta = \min\{\delta_4, \delta_5\} > 0$ depending only on ϵ. Therefore, from the above obtained estimates, it follows that

$$|\mathcal{T}x_n(t) - \mathcal{T}x_n(T)| < \frac{\epsilon}{2} \quad \text{and} \quad |\mathcal{T}x_n(T) - \mathcal{T}x_n(\tau)| < \frac{\epsilon}{2}$$

for all $n \in \mathbb{N}$ whenever $|t-\tau| < \delta$. As a result, $|\mathcal{T}x_n(t) - \mathcal{T}x_n(\tau)| < \epsilon$ for all $t, \tau \in J_\infty$ and for all $n \in \mathbb{N}$ whenever $|t-\tau| < \delta$. This shows that $\{\mathcal{T}x_n\}$ is a equicontinuous sequence in X. Now an application of Arzelà-Ascoli theorem yields that $\{\mathcal{T}x_n\}$ has a uniformly convergent subsequence on the compact subset $[t_0, T]$ of J_∞. Without loss of generality, call the subsequence to be the sequence itself. We show that $\{\mathcal{T}x_n\}$ is Cauchy in X. Now $|\mathcal{T}x_n(t) - \mathcal{T}x(t)| \to 0$ as $n \to \infty$ for all $t \in [t_0, T]$. Then for given $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\sup_{t_0 \le t \le T} \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}\big|f(s,x_m(s)) - f(s,x_n(s))\big|\,ds < \frac{\epsilon}{2}$$

for all $m, n \ge n_0$. Therefore, if $m, n \ge n_0$, then we have

$$
\begin{aligned}
&\|\mathcal{T}x_m - \mathcal{T}x_n\| \\
&\quad = \sup_{t_0 \le t < \infty}\left|\frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}\big|f(s,x_m(s)) - f(s,x_n(s))\big|\,ds\right| \\
&\quad \le \sup_{t_0 \le t \le T}\left|\frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}\big|f(s,x_m(s)) - f(s,x_n(s))\big|\,ds\right| \\
&\qquad + \sup_{t \ge T}\left|\frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}\Big[\big|f(s,x_m(s))\big| + \big|f(s,x_n(s))\big|\Big]\,ds\right| \\
&\quad < \epsilon.
\end{aligned}
$$

This shows that $\{\mathcal{T}x_n\} \subset \mathcal{T}(\overline{B}_r(0)) \subset X$ is Cauchy. Since X is complete, $\{\mathcal{T}x_n\}$ converges to a point in X. As $\mathcal{T}(\overline{B}_r(0))$ is closed, we have that $\{\mathcal{T}x_n\}$ converges to a point in $\mathcal{T}(\overline{B}_r(0))$. Hence $\mathcal{T}(\overline{B}_r(0))$ is relatively compact and consequently $\mathcal{T}$ is a continuous and compact operator on $\overline{B}_r(0)$ into itself. Now an application of Theorem 2.1 to the operator equation $\mathcal{T}x = x$ shows that the FRDE (1.1) has a mild solution on J_∞. Moreover, the mild solutions of the FRDE (1.1) are in $\overline{B}_r(0)$. Hence, mild solutions are global in nature.

To prove the attractivity of mild solutions, let $x, y \in \overline{B}_r(0)$ are any two mild solutions of the FRDE (1.1) on J_∞. Then,

$$\begin{aligned}|x(t) - y(t)| &\leq \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}\big|f(s,x(s)) - f(s,y(s))\big|\,ds \\ &\leq \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}\big[|f(s,x(s))| + |f(s,y(s))|\big]\,ds \\ &\leq \frac{2\,M_f}{\Gamma(q+1)}w(t)\end{aligned}$$

for all $t \in J_\infty$. From (3.8), for $\epsilon > 0$, there exists a real number $T > 0$ such that

$$|x(t) - y(t)| \leq \frac{\epsilon}{4} < \epsilon$$

for all $t \geq T$. Hence the mild solutions of the FRDE (1.1) are globally uniformly attractive on J_∞. Finally, for any mild solution x of the FRDE (1.1) defined on J_∞, one has a real number $T > t_0 \geq 0$ such that

$$\begin{aligned}|x(t)| &\leq |c_0|\overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}|f(s,x(s))|\,ds \\ &\leq |c_0|\,\overline{a}(t) + \frac{M_f}{\Gamma(q+1)}w(t) \\ &< \epsilon\end{aligned}$$

for all $t \geq T$. Hence, the mild solutions of the FRDE (1.1) are globally asymptotically uniformly attractive and stable on J_∞. This completes the proof. $\square$

The uniqueness of the uniformly stable mild solution is embodied in the following theorem.

Theorem 3.2. *Assume that the hypotheses (H_2) and (H_3) hold. Furthermore, if*

$$\frac{\sup_{t\geq t_0}\overline{a}(t)\,t^q}{\Gamma q}\psi_f(r) < r, \quad r > 0, \tag{3.9}$$

then the FRDE (1.1) has a unique uniformly asymptotically stable mild solution x^* defined on J_∞ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$\left.\begin{aligned} x_0(t) &= a(t_0)x_0\,\overline{a}(t), \\ x_{n+1}(t) &= c_0\,\overline{a}(t) + \tfrac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}f(s,x_n(s))\,ds, \end{aligned}\right\} \tag{3.10}$$

for all $t \in J_\infty$, converges to x^*.

Proof. Set $X = BC(J_\infty, \mathbb{R})$ and define the operator $\mathcal{T} : X \to X$ by (3.4). We show that $\mathcal{T}$ is a nonlinear $\mathcal{D}$-contraction on X. Let $x, y \in X$ be any two elements. Then by hypothesis (H_2), we obtain

$$\begin{aligned}|\mathcal{T}x(t) - \mathcal{T}x(t)| &\leq \frac{\overline{a}(t)}{\Gamma q}\left|\int_{t_0}^{t}(t-s)^{q-1}\big|f(s,x(s)) - f(s,y(s))\big|\,ds\right| \\ &\leq \frac{\overline{a}(t)}{\Gamma q}\left|\int_{t_0}^{t}(t-s)^{q-1}\psi_f\big(|x(s) - y(s)|\big)\,ds\right|\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\overline{a}(t)}{\Gamma q}\int_{t_0}^{t}(t-s)^{q-1}\psi_f\big(\|x-y\|\big)\,ds\\
&\le \frac{w(t)}{\Gamma(q+1)}\psi_f\big(\|x-y\|\big)\\
&\le \frac{W}{\Gamma(q)}\psi_f\big(\|x-y\|\big)
\end{aligned}
$$

for all $t \in t_\infty$. Taking the supremum over t in the above inequality yields

$$\|\mathcal{T}x-\mathcal{T}y\| \le \frac{W}{\Gamma(q)}\psi_f\big(\|x-y\|\big)$$

for all $x, y \in X$, where $\frac{W}{\Gamma(q)}\psi_f(r) < r$ for $r > 0$ in view of condition (3.9). This shows that $\mathcal{T}$ is a nonlinear $\mathcal{D}$-contraction on X. Now, by an application of Theorem 2.2 we obtain that the FRDE (1.1) has a unique mild solutions x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.10) converges geometrically to x^*. Moreover, it can be shown as in Theorem 3.1 that the mild solutions x^* is uniformly asymptotically stable on t_∞. This completes the proof. □

Example 3.1. Given a closed but unbounded interval $J_\infty = \mathbb{R}_+$, consider the IVP of FRDE with Caputo fractional derivative,

$$
\left.
\begin{aligned}
{}^C D_0^q\big[(t^2+1)x(t)\big] &= \frac{\log(|x(t)|+1)}{(t+1)\big(x^2(t)+2\big)},\ \ t\in\mathbb{R}_+,\\
x(0) &= 1,
\end{aligned}
\right\}
\tag{3.11}
$$

where ${}^C D_0^q$ is the Caputo fractional derivative of fractional order $0 < q \le 1$.

Here, $a(t) = t^2 + 1$, $t \in \mathbb{R}_+$ and $f(t, x) = \dfrac{\ln(|x| + 1)}{(t + 1)(x^2 + 2)}$ for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Clearly, f is a continuous and bounded function on $\mathbb{R}_+ \times \mathbb{R}$ in $\mathbb{R}$. Therefore, hypotheses (H$_1$) is satisfied with $M_f = 1$. Also, $a \in \mathcal{CRB}(\mathbb{R}_+) \bigcap C^1(J_\infty, \mathbb{R})$ with the estimate:

$$\lim_{t\to\infty}\overline{a}(t)\,t^q = \lim_{t\to\infty}\frac{t^q}{t^2+1} = 0$$

and so, hypothesis (H$_3$) is satisfied. Hence, by an application of Theorem 3.1, the FRDE (3.11) has a mild solution and the mild solutions are globally uniformly asymptotically attractive and stable on $\mathbb{R}_+$. Particularly, the FRDE

$${}^C D_0^{2/3}\big[(t^2+1)x(t)\big] = \frac{\log(|x(t)|+1)}{(t+1)\big(x^2(t)+2\big)},\ \ t\in\mathbb{R}_+,\quad x(0)=1,$$

has a mild solution and the mild solutions are globally uniformly asymptotically attractive and stable on $\mathbb{R}_+$.

Example 3.2. Given a closed but unbounded interval $J_\infty = \mathbb{R}_+$, consider the IVP of FRDE with Caputo fractional derivative,

$$
\left.
\begin{aligned}
{}^C D_0^q\big[(t^2+1)x(t)\big] &= e^{-t}\log(1+|x(t)|),\ \ t\in\mathbb{R}_+,\\
x(0) &= 1,
\end{aligned}
\right\}
\tag{3.12}
$$

where ${}^C D_0^q$ is the Caputo fractional derivative of fractional order $0 < q \le 1$.

Here, $a(t) = t^2 + 1$, $t \in \mathbb{R}_+$ and $f(t, x) = e^{-t}\log(1 + |x|)$ for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Clearly, f is continuous and satisfies the hypothesis (H$_2$) on $\mathbb{R}_+ \times \mathbb{R}$ with $\mathcal{D}$-function

$\psi_f(r) = \log(1+r)$ (see Dhage [6]). Also,$a \in \mathcal{CRB}(\mathbb{R}_+)$ and satisfies the hypothesis (H$_3$). Moreover, $\frac{1}{\Gamma q} \cdot \sup_{t\geq 0} \frac{t^q}{t^2+1} \leq 1$ and therefore, we have

$$\frac{\sup_{t\geq 0} \overline{a}(t)\, t^q}{\Gamma q}\psi_f(r) = \frac{\sup_{t\geq 0} \frac{t^q}{t^2+1}}{\Gamma q} \cdot \log(1+r) < r$$

for each $r > 0$. Hence, by Theorem 3.1, the FRDE (3.12) has a unique mild solution x^* defined on J_∞ and the sequence $\{x_n\}$ of successive approximations defined by

$$x_0(t) = \frac{1}{t^2+1},\ x_{n+1}(t) = \frac{1}{t^2+1} + \frac{1}{t^2+1} \cdot \frac{1}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} f(s, x_n(s))\, ds,$$

for $n \geq 0$, converges to x^*. Moreover, the mild solution x^* is uniformly asymptotically stable on t_∞. Particularly, the FRDE

$${}^C D_0^{1/2}\big[(t^2+1)x(t)\big] = e^{-t}\log(1+|x(t)|),\ \ t \in \mathbb{R}_+, \quad x(0) = 1,$$

has a unique mild solution which is globally uniformly asymptotically attractive and stable on $\mathbb{R}_+$.

4. Linear Perturbation of First Type

Next, given a pulling function $a \in \mathcal{CRB}(J_\infty) \bigcap C^1(J_\infty, \mathbb{R})$, we consider the perturbed nonlinear fractional differential equation of the type

$${}^C D_{t_0}^q\big[a(t)x(t)\big] = f(t, x(t)) + g(t, x(t)),\ \ t \in J_\infty, \tag{4.1}$$

satisfying the initial condition (in short IC)

$$x(t_0) = x_0, \tag{4.2}$$

where ${}^C D_{t_0}^q$ is the Caputo fractional derivative of fractional order $0 < q \leq 1$ on $[t_0, \infty)$ and $f, g\ :\ J_\infty \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Definition 4.1. By a *mild solution* for the fractional differential equation (4.1)-(4.2) we mean a function $x \in C^1(J_\infty, \mathbb{R})$ that satisfies the equations in (4.1)-(4.2) on J_∞, where $C^1(J_\infty, \mathbb{R})$ is the space of continuous real-valued functions defined on J_∞ whose first derivative x' exist and $x' \in C(J_\infty, \mathbb{R})$.

The FRDE (4.1)–(4.2) is a linear perturbation of first type obtained by adding a nnlinearity $g(t, x)$ containing unknown function to the right side of the equation (see Dhage [10, 11]). We employ the following hybrid fixed point theorem of Dhage [5, 8] involving the sum of two nonlinear operators in a Banach space while proving the existence and attractivity result for the FRDE (4.1) and (4.2) on J_∞.

Theorem 4.1 (Dhage [5, 6, 11])**.** *Let S be a closed convex and bounded subset of a Banach space $\mathfrak{X}$ and let $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ and $\mathcal{B} : S \to \mathfrak{X}$ be two operators satisfying the following conditions.*

(a) *$\mathcal{A}$ is nonlinear $\mathcal{D}$-contraction,*
(b) *$\mathcal{B}$ is completely continuous, and*
(c) *$\mathcal{A}x + \mathcal{B}y = x \implies x \in S$ for all $y \in S$.*

Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution in S.

We need the following hypotheses in the sequel.

(H$_4$) The function g is continuous and bounded on $J_\infty \times \mathbb{R}$ with bound M_g.

Theorem 4.2. *Assume that the hypotheses (H_1) through (H_4) hold. Furthermore, if the condition (3.9) is satisfied, then the FRDE (4.1)–(4.2) has a mild solution and mild solutions are global uniformly asymptotically attractive or stable defined on J_∞.*

Proof. Set $X = BC(J_\infty, \mathbb{R})$ and define a closed ball $\overline{B}_r(0)$ in X centered at origin 0 of radius r given by

$$r = |c_0|\, \|\overline{a}\| + \frac{M_f\, W}{\Gamma q} + \frac{M_g\, W}{\Gamma q} \tag{4.3}$$

where, c_0 is defined as in Lemma 3.1. Now, by an application of Lemma 3.1, the mild solution of the FRDE (1.1) is given by the following hybrid functional integral equation (in short FRIE)

$$\begin{aligned} x(t) &= c_0\, \overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))\, ds \\ &\quad + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} g(s, x(s))\, ds \end{aligned} \tag{4.4}$$

for all $t \in J_\infty$. Define two operators $\mathcal{A} : X \to X$ and $\mathcal{B} : \overline{B}_r(0) \to X$ by

$$\mathcal{A}x(t) = \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))\, ds, \;\; t \in J_\infty, \tag{4.5}$$

and

$$\mathcal{B}x(t) = c_0\, \overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} g(s, x(s))\, ds, \;\; t \in J_\infty. \tag{4.6}$$

Then the FRIE (4.4) is transformed into the operator equation as

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \;\; t \in J_\infty. \tag{4.7}$$

We show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.1. Now proceeding with the arguments as in Theorem 3.1 it can be shown that $\mathcal{A}$ is a nonlinear $\mathcal{D}$-contraction on X and $\mathcal{B}$ is completely continuous on $\overline{B}_r(0)$. We show that condition (c) of Theorem 4.1 is satisfied. Let $y \in \overline{B}_r(0)$ be arbitrary and consider the operator equation $x = \mathcal{A}x + \mathcal{B}y$. Then, we obtain

$$\begin{aligned} |x(t)| &\le |\mathcal{A}x(t)| + |\mathcal{B}y(t)| \\ &\le |c_0|\, \overline{a}(t) + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} |f(s, x(s))|\, ds \\ &\quad + \frac{\overline{a}(t)}{\Gamma q} \int_{t_0}^{t} (t-s)^{q-1} |g(s, x(s))|\, ds \\ &\le |c_0|\, \|\overline{a}\| + \frac{M_f W}{\Gamma q} + \frac{M_f W}{\Gamma q} \\ &= r \end{aligned}$$

for all $t \in J_\infty$. Taking the supremum over t, we get

$$\|x\| \le |c_0|\, \|\overline{a}\| + + \frac{M_f W}{\Gamma q} + \frac{M_f W}{\Gamma q} = r$$

for all $y \in \overline{B}_r(0)$. As result, $x \in \overline{B}_r(0)$ for all $y \in \overline{B}_r(0)$ and that the condition (c) of Theorem 4.1 holds. Hence, the operator equation (4.7) and consequently the FRDE (4.1)-(4.2) has a mild solution defined on J_∞.

To prove the attractivity of mild solutions, let $x, y \in \overline{B}_r(0)$ are any two mild solutions of the FRDE (1.1) on J_∞. Then,

$$|x(t) - y(t)| \leq \frac{2\,M_f}{\Gamma(q+1)} w(t) + \frac{2\,M_g}{\Gamma(q+1)} w(t) \tag{4.8}$$

for all $t \in J_\infty$. Since $\lim_{t\to\infty} w(t) = 0$, for $\epsilon > 0$ there exists a real numbers $T_1 > 0$ and $T_2 > 0$ such that $w(t) < \frac{\epsilon}{6\,M_f \big/ \Gamma q}$ for all $t \geq T_1$ and $w(t) < \frac{\epsilon}{6\,M_g \big/ \Gamma q}$ for all $t \geq T_2$. If $T = \max\{T_1\,,\,T_2\}$, then from the above inequality (4.8) we obtain

$$|x(t) - y(t)| < \epsilon$$

for all $t \geq T$. Hence, the mild solutions of the FRDE (4.1)–(4.2) are globally uniformly attractive or stable on J_∞. Moreover, sinnce $\lim_{t\to\infty} \overline{a}(t) = 0$, for $\epsilon > 0$ there exists a $T_3 > 0$ such that $\overline{a}(t) < \dfrac{\epsilon}{3\,|c_0|}$ for all $t \geq T_3$. Therefore, if we choose $T = \{T_1, T_2, T_3\}$, then for any mild solution x of the FRDE (4.1)–(4.2), one has

$$|x(t)| \leq |c_0|\,\overline{a}(t) + \frac{M_f}{\Gamma q} w(t) + \frac{M_g}{\Gamma q} w(t) < \epsilon$$

for all $t \geq T$. This shows that every mild solution x of FRDE (4.1)–(4.2) is asymptotic to the line $x(t) = 0$ as $t \to \infty$. Hence, the mild solutions of the FRDE (4.1)–(4.2) are globally uniformly asymptotically attractive or stable on J_∞. This completes the proof. □

Example 4.1. Given a closed but unbounded interval $J_\infty = \mathbb{R}_+$, consider the IVP of FRDE with Caputo fractional derivative,

$$\left.\begin{aligned} {}^{C}D_0^q\big[(t^2+1)x(t)\big] &= \frac{e^{-t}|x(t)|}{1+|x(t)|} + \frac{\log(|x(t)|+1)}{(t+1)\big(x^2(t)+2\big)},\ \ t \in \mathbb{R}_+, \\ x(0) &= 1, \end{aligned}\right\} \tag{4.9}$$

where ${}^{C}D_0^q$ is the Caputo fractional derivative of fractional order $0 < q \leq 1$.

Here, $f(t,x) = \dfrac{e^{-t}|x|}{1+|x|}$ and $g(t,x) = \dfrac{\log(|x|+1)}{(t+1)(x^2+2)}$. Obviously f is continuous and bounded on $\mathbb{R}_+ \times \mathbb{R}$ with bound $M_f = 1$. So the hypotheses (H_1) is satisfied. Next, we have

$$|f(t,x) - f(t,y)| \leq \frac{|x-y|}{1+|x-y|} = \psi_f(|x-y|)$$

for all $(t,x), (t,y) \in \mathbb{R}_+ \times \mathbb{R}$, where $\psi_f(r) = \dfrac{r}{1+r}$ is a $\mathcal{D}$-function. Hence the hypothesis (H_2) holds. Moreover, since $\frac{1}{\Gamma q} \cdot \sup_{t\geq 0} \frac{t^q}{t^2+1} \leq 1$, the $\mathcal{D}$-function ψ_f satisfies the condition (3.9) of Theorem 4.2. Because

$$\frac{\sup_{t\geq 0}\,\overline{a}(t)\,t^q}{\Gamma q} \psi_f(r) = \frac{\sup_{t\geq 0}\,\frac{t^q}{t^2+1}}{\Gamma q} \cdot \frac{r}{1+r} < r$$

for each $r > 0$. Similarly, it can be shown as in Example 3.1 that the nonlinearity g satisfies the hypothesis (H_4) on $\mathbb{R}_+ \times \mathbb{R}$. Here, the pulling function is $a(t) = t^2 + 1$, $t \in \mathbb{R}_+$, and so it satisfies the hypothesis (H_3). Now we apply Theorem 4.2 that the FRDE (4.9) has a mild solution and the mild solutions are globally uniformly asymptotically attractive or stable on $\mathbb{R}_+$. In particular, the FRDE

$$ {}^C D_0^{4/5}\big[(t^2+1)x(t)\big] = \frac{e^{-t}|x(t)|}{1+|x(t)|} + \frac{\log(|x(t)|+1)}{(t+1)\big(x^2(t)+2\big)}, \ \ t \in \mathbb{R}_+, \ \ x(0)=1, $$

has a mild solution and the mild solutions are globally uniformly asymptotically attractive or stable defined on $\mathbb{R}_+$.

Remark 4.1. The existence and attractiity results for mild solution of the FRDE (1.1) may be proved via another approach of using the technique of measure of noncompactness in the Banach space $BC(J_\infty, \mathbb{R})$. In that case we need to construct a handy tool for the measure of noncompactness which is not the case with the present approach in the qualitative study of such nonlinear fractional equations. See the details of this procedure that appears in Banas and Dhage [1], Dhage [8] and the references therein.

Remark 4.2. Finally, while concluding this paper, we conjecture that the asymptotic stability theorems for the FRDE (1.1) may not be possible or are very difficult without the use of pulling function as presented in the equation of (1.1) on unbounded interval J_∞ provided $x(t_0) = x_0 \neq 0$. Again, the choice of pulling functions depends upon the nature of the given nonlinear fractional differential equation. A cleverer mathematician makes a cleverer and intelligent selection of the pulling function suitable for the given nonlinear fractional differential equation to yield desired characterizations of the mild solution on unbounded intervals of the real line. Some of the results in this direction for different perturbations of the nonlinear FRDE (1.1) with Caputo fractional derivative will be reported elsewhere.

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