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COMMUTATIVE CAPUTO FRACTIONAL KOROVKIN INEQUALITIES FOR STOCHASTIC PROCESSES

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Abstract. Here we consider and study expectation commutative stochastic positive linear operators acting on L^1-continuous stochastic processes which are Caputo fractional differentiable. Under some mild, general and natural assumptions on the stochastic processes we produce related Caputo fractional stochastic Shisha-Mond type inequalities pointwise and uniform. All convergences are produced with rates and are given by the fractional stochastic inequalities involving the first modulus of continuity of the expectation of the α-th right and left fractional derivatives of the engaged stochastic process, $\alpha > 0$, $\alpha \notin \mathbb{N}$. The amazing fact here is that the basic real Korovkin test functions assumptions impose the conclusions of our Caputo fractional stochastic Korovkin theory. We include also a detailed application to stochastic Bernstein operators.

1. Introduction

Our work is motivated by the following:

Korovkin's Theorem ([12], 1960) Let $(T_j)_{j\in\mathbb{N}}$ be a sequence of positive linear operators from $C([a,b])$ into itself, $[a,b] \subset \mathbb{R}$. In order to have $\lim_{j\to\infty} (T_j f)(t) = f(t)$ (in the sup-norm) for all $f \in C([a,b])$, it is enough to prove it for $f_0(t) = 1$, $f_1(t) = t$ and $f_2(t) = t^2$. The rate of the above convergence for abitrary $f \in C([a,b])$ can be determined exactly from the rates of convergence for f_0, f_1, f_2.

The above theorem was put in an inequality form:

Shisha-Mond inequality ([14]) We have

$$\|T_j(f) - f\| \le \|f\| \cdot \|T_j(1) - 1\| + \omega_1(f, \rho_j) \cdot (1 + \|T_j(1)\|),$$

where

$$\rho_j = \left(\left\|T_j\left((x-y)^2\right)(y)\right\|\right)^{\frac{1}{2}}.$$

In the last inequality $\|\cdot\|$ stands for the supremum norm and ω_1 for the first modulus of continuity. This inequality gives the rate of convergence of T_j to the unit operator I.

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Annastassiou in [2]-[4] established a series of sharp inequalities for various cases of the parameters of the problem. However, Weba in [15]-[18] was the first, among many workers in quantitative results of Shisha-Mond type, to produce inequalities for stochastic processes. He assumed that T_j are E-commutative (E means expectation) and stochastically simple. According to his work, if a stochastic process $X(t,\omega)$, $t \in Q$ - a compact convex subset of a real normed vector space, $\omega \in Q$ - probability space, is to be approximated by positive linear operators T_j, then the maximal error in the qth mean is $(q \geq 1)$

$$\|T_j X - X\| = \sup_{t \in Q} \left(E \left| (T_j X)(t,\omega) - X(t,\omega) \right|^q \right)^{\frac{1}{q}}.$$

So, Weba established upper bounds for $\|T_j X - X\|$ involving his own natural general first modulus of continuity of X with several interesting applications.

Anastassiou met ([5]) the pointwise case of $q = 1$. Without stochastic simplicity of T_j he found nearly best and best upper bounds for $|E(T_j X)(x_0) - (EX)(x_0)|$, $x_0 \in Q$.

The author here continues his above work on the approximation of stochastic processes, now at the Caputo stochastic fractional level. He derives pointwise and uniform Caputo fractional stochastic Shisha-Mond type inequalities, see tha main Theorems 4, 7 and the several related corollaries. He gives an extensive application to stochastic Bernstein operators. He finishes with a pointwise and a uniform fractional stochastic Korovkin theorem, derived by Theorems 4, 7. The stochastic convergences, about stochastic processes, of our fractional Korovkin Theorems 15, 16 are enforced only by the convergences of real basic non-stochastic functions.

2. Background

We need

Definition 1. ([10]) *Let non-integer $\alpha > 0$, $n = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $t \in [a,b] \subset \mathbb{R}$, $\omega \in \Omega$, where $(\Omega, \mathcal{F}, P)$ is a general probability space. Here $X(t,\omega)$ stands for a stochastic process. Assume that $X(\cdot,\omega) \in AC^n([a,b])$ (spaces of functions $X(\cdot,\omega)$ with $X^{(n-1)}(\cdot,\omega) \in AC([a,b])$ absolutely continuous functions), $\forall\ \omega \in \Omega$.*

We call stochastic left Caputo fractional derivative

$$D_{*a}^{\alpha} X(x,\omega) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} X^{(n)}(t,\omega)\,dt, \tag{1}$$

$\forall\ x \in [a,b]$, $\forall\ \omega \in \Omega$.

And, we call stochastic right Caputo fractional derivative

$$D_{b-}^{\alpha} X(x,\omega) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (z-x)^{n-\alpha-1} X^{(n)}(z,\omega)\,dz, \tag{2}$$

$\forall\ x \in [a,b]$, $\forall\ \omega \in \Omega$. Above Γ stands for the gamma function.

We make

Remark 2. *(to Definition 1) We further assume here that*

$$\left| X^{(n)}(t,\omega) \right| \leq M, \ \ \forall\ \ (t,\omega) \in [a,b] \times \Omega,$$

where $M > 0$.

Then, by (1), we have

$$\left|D_{*a}^{\alpha}X(x,\omega)\right| \leq \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}\left|X^{(n)}(t,\omega)\right|dt \leq$$

$$\frac{M}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}dt = \frac{M(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}.$$

That is

$$\left|D_{*a}^{\alpha}X(x,\omega)\right| \leq \frac{M(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \quad \forall\, x\in[a,b], \text{ any } \omega\in\Omega. \tag{3}$$

Also, from (2) we get

$$\left|D_{b-}^{\alpha}X(x,\omega)\right| \leq \frac{1}{\Gamma(n-\alpha)}\int_x^b (z-x)^{n-\alpha-1}\left|X^{(n)}(z,\omega)\right|dz \leq$$

$$\frac{M}{\Gamma(n-\alpha)}\int_x^b (z-x)^{n-\alpha-1}dz = \frac{M(b-x)^{n-\alpha}}{\Gamma(n-\alpha+1)}.$$

That is

$$\left|D_{b-}^{\alpha}X(x,\omega)\right| \leq \frac{M(b-x)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \quad \forall\, x\in[a,b], \text{ any } \omega\in\Omega. \tag{4}$$

It is not strange to assume that $D_{*a}^{\alpha}X$, $D_{b-}^{\alpha}X$ are stochastic processes.

By [7], p. 388, we get that $D_{*a}^{\alpha}X(\cdot,\omega)\in C([a,b])$, $\forall\,\omega\in\Omega$. And by [8], we get that $D_{b-}^{\alpha}X(\cdot,\omega)\in C([a,b])$, $\forall\,\omega\in\Omega$.

Similarly, we obtain

$$\left|D_{*t}^{\alpha}X(x,\omega)\right| \leq \frac{M(x-t)^{n-\alpha}}{\Gamma(n-\alpha+1)} \leq \frac{M(b-t)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \tag{5}$$

$\forall\, x\in[t,b]$, any $t\in[a,b]$, $\forall\,\omega\in\Omega$,

and

$$\left|D_{t-}^{\alpha}X(x,\omega)\right| \leq \frac{M(t-x)^{n-\alpha}}{\Gamma(n-\alpha+1)} \leq \frac{M(t-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \tag{6}$$

$\forall\, x\in[a,t]$, any $t\in[a,b]$, $\forall\,\omega\in\Omega$.

Above $D_{*t}^{\alpha}X$, $D_{t-}^{\alpha}X$ are assumed to be stochastic processes for any $t\in[a,b]$, and it holds $D_{*t}^{\alpha}X(\cdot,\omega)\in C([t,b])$, $D_{t-}^{\alpha}X(\cdot,\omega)\in C([a,t])$, $\forall\,\omega\in\Omega$.

Clearly, then

$$\left|E\left(D_{*t}^{\alpha}X\right)(x)\right| \leq \frac{M(b-t)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \tag{7}$$

$\forall\, x\in[t,b]$, any $t\in[a,b]$, where E is the expectation operator $(EX)(t) = \int_{\Omega} X(t,\omega)P(d\omega)$ and similarly,

$$\left|E\left(D_{t-}^{\alpha}X\right)(x)\right| \leq \frac{M(t-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \tag{8}$$

$\forall\, x\in[a,t]$, any $t\in[a,b]$.

We observe that the first modulus of continuity $(\delta>0)$

$$\omega_1\left(E\left(D_{*t}^{\alpha}X\right),\delta\right)_{[t,b]} := \sup_{\substack{x,y\in[t,b]:\\ |x-y|\leq\delta}} \left|E\left(D_{*t}^{\alpha}X\right)(x) - E\left(D_{*t}^{\alpha}X\right)(y)\right|$$

$$\overset{(7)}{\leq} \frac{2M(b-t)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \tag{9}$$

any $t \in [a,b]$.

Hence, it holds ($\delta > 0$)

$$\sup_{t\in[a,b]} \omega_1 \left(E\left(D_{*t}^{\alpha}X\right), \delta\right)_{[t,b]} \leq \frac{2M\left(b-a\right)^{n-\alpha}}{\Gamma\left(n-\alpha+1\right)}. \tag{10}$$

Similarly, it holds ($\delta > 0$)

$$\sup_{t\in[a,b]} \omega_1 \left(E\left(D_{t-}^{\alpha}X\right), \delta\right)_{[a,t]} \leq \frac{2M\left(b-a\right)^{n-\alpha}}{\Gamma\left(n-\alpha+1\right)}. \tag{11}$$

By [6], p. 209, we have that ($\delta_1 > 0$)

$$\left|E\left(D_{*t}^{\alpha}X\right)(z) - E\left(D_{*t}^{\alpha}X\right)(t)\right| \leq \omega_1\left(E\left(D_{*t}^{\alpha}X\right), \delta_1\right)_{[t,b]} \left\lceil \frac{\lceil z-t \rceil}{\delta_1} \right\rceil \leq$$

$$\omega_1\left(E\left(D_{*t}^{\alpha}X\right), \delta_1\right)_{[t,b]} \left(1 + \frac{(z-t)}{\delta_1}\right), \tag{12}$$

$\forall\ z \in [t,b]$,

and similarly ($\delta_2 > 0$),

$$\left|E\left(D_{t-}^{\alpha}X\right)(z) - E\left(D_{t-}^{\alpha}X\right)(t)\right| \leq \omega_1\left(E\left(D_{t-}^{\alpha}X\right), \delta_2\right)_{[a,t]} \left(1 + \frac{t-z}{\delta_2}\right), \tag{13}$$

$\forall\ z \in [a,t]$.

We also set

$$\omega_1\left(E\left(D_{t}^{\alpha}X\right), \delta\right) := \max\left\{\omega_1\left(E\left(D_{*t}^{\alpha}X\right), \delta\right)_{[t,b]}, \omega_1\left(E\left(D_{t-}^{\alpha}X\right), \delta\right)_{[a,t]}\right\}, \tag{14}$$

where $\delta > 0$.

We make

Remark 3. *Let the positive linear operator L mapping $C([a,b])$ into $B([a,b])$ (the bounded functions). By the Riesz representation theorem ([13]) we have that there exists μ_t unique, completed Borel measure on $[a,b]$ with*

$$\mu_t([a,b]) = L(1)(t) > 0, \tag{15}$$

such that

$$L(f)(t) = \int_{[a,b]} f(s)\, d\mu_t(s), \quad \forall\ t \in [a,b], \forall\ f \in C([a,b]). \tag{16}$$

Let now $n = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $k = 1, ..., n-1$. Then by Hölder's inequality we obtain

$$\left|\int_{[a,b]} (s-t)^k\, d\mu_t(s)\right| \leq \int_{[a,b]} |s-t|^k\, d\mu_t(s) \leq$$

$$\left(\int_{[a,b]} |s-t|^{\alpha+1}\, d\mu_t(s)\right)^{\left(\frac{k}{\alpha+1}\right)} \left(\mu_t([a,b])\right)^{\left(\frac{\alpha+1-k}{\alpha+1}\right)}. \tag{17}$$

The last means

$$\left|L\left((s-t)^k\right)(t)\right| \leq L\left(|s-t|^k\right)(t) \leq \left(L\left(|s-t|^{\alpha+1}\right)(t)\right)^{\left(\frac{k}{\alpha+1}\right)} \left(L(1)(t)\right)^{\left(\frac{\alpha+1-k}{\alpha+1}\right)}, \tag{18}$$

all $k = 1, ..., n-1$. It is clear that

$$\left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_{\infty,[a,b]} < \infty.$$

Furthermore we derive

$$\left\| L\left((s-t)^{k}\right)(t)\right\|_{\infty,[a,b]} \leq \left\| L\left(|s-t|^{k}\right)(t)\right\|_{\infty,[a,b]} \leq$$
$$\left\| L\left(|s-t|^{\alpha+1}\right)(t)\right\|_{\infty,[a,b]}^{\left(\frac{k}{\alpha+1}\right)} \left\| L(1)\right\|_{\infty,[a,b]}^{\left(\frac{\alpha+1-k}{\alpha+1}\right)}, \tag{19}$$

all $k=1,...,n-1$.

From now on we will denote $\|\cdot\|_{\infty,[a,b]} = \|\cdot\|_{\infty}$ the supremum norm.

3. Preliminaries

Let $(\Omega,\mathcal{F},P)$ be a probabilistic space and $L^{1}(\Omega,\mathcal{F},P)$ be the space of all real-valued random variables $Y=Y(\omega)$ with

$$\int_{\Omega} |Y(\omega)| P(d\omega) < \infty.$$

Let $X = X(t,\omega)$ denote a stochastic process with index set $[a,b] \subset \mathbb{R}$ and real state space $(\mathbb{R},\mathcal{B})$, where $\mathcal{B}$ is the σ-field of Borel subsets of $\mathbb{R}$. Here $C([a,b])$ is the space of continuous real-valued functions on $[a,b]$ and $\mathcal{B}([a,b])$ is the space of bounded real-valued functions on $[a,b]$. Also $C_{\Omega}([a,b]) = C\left([a,b], L^{1}(\Omega,\mathcal{F},P)\right)$ is the space of L^{1}-continuous stochastic processes in t and
$B_{\Omega}([a,b]) = \left\{ X : \sup_{t\in[a,b]} \int_{\Omega} |X(t,\omega)| P(d\omega) < \infty \right\}$, obviously $C_{\Omega}([a,b]) \subset B_{\Omega}([a,b])$.

Let $\alpha>0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = n$, and consider the subspace of stochastic processes $C_{\Omega}^{\alpha,n}([a,b]) := \{X : X(\cdot,\omega) \in AC^{n}([a,b])$, $\forall\ \omega \in \Omega$ and $\left|X^{(n)}(t,\omega)\right| \leq M$, $\forall$ $(t,\omega) \in [a,b]\times\Omega$, where $M>0$; $X^{(k)}(t,\omega) \in C_{\Omega}([a,b])$, $k=0,1,...,n-1$; also $D_{*t}^{\alpha}X$, $D_{t-}^{\alpha}X$ are stochastic processes for any $t\in[a,b]\}$. That is, for every $\omega\in\Omega$ we have $X(t,\omega) \in C^{n-1}([a,b])$.

Consider the linear operator

$$L : C_{\Omega}([a,b]) \hookrightarrow B_{\Omega}([a,b]).$$

If $X \in C_{\Omega}([a,b])$ is nonnegative and LX, too, then L is called positive. If $EL=LE$, then L is called E-commutative.

4. Main Results

Following 3. Preliminaries we state

Theorem 4. *Consider the positive E-commutative linear operator $L : C_{\Omega}([a,b]) \hookrightarrow B_{\Omega}([a,b])$, and $\alpha>0$, $\alpha\notin\mathbb{N}$, $\lceil\alpha\rceil = n$, and let $X \in C_{\Omega}^{\alpha,n}([a,b])$, with $\delta>0$.*

Then

$$|(E(LX))(t) - (EX)(t)| \leq |(EX)(t)|\,|(L(1))(t)-1| + \tag{20}$$
$$\sum_{k=1}^{n-1} \frac{\left|\left(EX^{(k)}\right)(t)\right|}{k!} \left| L\left((s-t)^{k}\right)(t)\right| + \frac{\omega_{1}\left(E\left(D_{t}^{\alpha}X\right),\delta\right)}{\Gamma(\alpha+1)}$$
$$\left(L\left(|s-t|^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}} \left[(L(1)(t))^{\frac{1}{\alpha+1}} + \frac{\left(L\left(|s-t|^{\alpha+1}\right)(t)\right)^{\frac{1}{\alpha+1}}}{\delta(\alpha+1)} \right],$$

$\forall\ t\in[a,b]$.

Above $\omega_{1}\left(E\left(D_{t}^{\alpha}X\right),\delta\right)$ is as in (14).

Proof. We have that

$$D_{*t}^{\alpha}X(t,\omega)=D_{t-}^{\alpha}X(t,\omega)=0, \tag{21}$$

$\forall\ \omega\in\Omega$, see [9], pp. 358-359.

We also assume

$$D_{*t}^{\alpha}X(s,\omega)=0,\ \text{ for } s<t,$$

and

$$D_{t-}^{\alpha}X(s,\omega)=0,\ \text{ for } s>t,$$

$\forall\ \omega\in\Omega$.

We get by left Caputo fractional Taylor's formula that ([11], p. 54)

$$X(s,\omega)=\sum_{k=0}^{n-1}\frac{X^{(k)}(t,\omega)}{k!}(s-t)^{k}+ \tag{22}$$

$$\frac{1}{\Gamma(\alpha)}\int_{t}^{s}(s-z)^{\alpha-1}\left(D_{*t}^{\alpha}X(z,\omega)-D_{*t}^{\alpha}X(t,\omega)\right)dz,$$

for all $t\leq s\leq b$, $\forall\ \omega\in\Omega$.

Also, from [9], p. 341, using the right Caputo fractional Taylor formula we get

$$X(s,\omega)=\sum_{k=0}^{n-1}\frac{X^{(k)}(t,\omega)}{k!}(s-t)^{k}+ \tag{23}$$

$$\frac{1}{\Gamma(\alpha)}\int_{s}^{t}(z-s)^{\alpha-1}\left(D_{t-}^{\alpha}X(z,\omega)-D_{t-}^{\alpha}X(t,\omega)\right)dz,$$

for all $a\leq s\leq t$, $\forall\ \omega\in\Omega$.

Therefore we get

$$(EX)(s)=\sum_{k=0}^{n-1}\frac{\left(EX^{(k)}\right)(t)}{k!}(s-t)^{k}+$$

$$\frac{1}{\Gamma(\alpha)}E\left(\int_{t}^{s}(s-z)^{\alpha-1}\left(D_{*t}^{\alpha}X(z,\omega)-D_{*t}^{\alpha}X(t,\omega)\right)dz\right), \tag{24}$$

all $t\leq s\leq b$,

and

$$(EX)(s)=\sum_{k=0}^{n-1}\frac{\left(EX^{(k)}\right)(t)}{k!}(s-t)^{k}+$$

$$\frac{1}{\Gamma(\alpha)}E\left(\int_{s}^{t}(z-s)^{\alpha-1}\left(D_{t-}^{\alpha}X(z,\omega)-D_{t-}^{\alpha}X(t,\omega)\right)dz\right), \tag{25}$$

all $a\leq s\leq t$.

By [1], p. 156, Theorem 20.15, the functions $D_{*t}^{\alpha}X$ and $D_{t-}^{\alpha}X$ are jointly measurable.

By (5) and (6) we obtain that $E\left(\int_{t}^{s}(s-z)^{\alpha-1}\left|D_{*t}^{\alpha}X(z,\omega)-D_{*t}^{\alpha}X(t,\omega)\right|dz\right)$, $E\left(\int_{s}^{t}(z-s)^{\alpha-1}\left|D_{t-}^{\alpha}X(z,\omega)-D_{t-}^{\alpha}X(t,\omega)\right|dz\right)$ are finite.

Therefore, by Fubini-Tonelli's theorem ([13]), we get

$$(EX)(s)=\sum_{k=0}^{n-1}\frac{\left(EX^{(k)}\right)(t)}{k!}(s-t)^{k}+$$

$$\frac{1}{\Gamma(\alpha)}\int_t^s (s-z)^{\alpha-1}\left(E\left(D_{*t}^{\alpha}X\right)(z)-E\left(D_{*t}^{\alpha}X\right)(t)\right)dz, \tag{26}$$

all $t \leq s \leq b$,

and

$$(EX)(s)=\sum_{k=0}^{n-1}\frac{\left(EX^{(k)}\right)(t)}{k!}(s-t)^k+$$

$$\frac{1}{\Gamma(\alpha)}\int_s^t (z-s)^{\alpha-1}\left(E\left(D_{t-}^{\alpha}X\right)(z)-E\left(D_{t-}^{\alpha}X\right)(t)\right)dz, \tag{27}$$

all $a \leq s \leq t$.

Call the remainders

$$R_1(t,s):=\frac{1}{\Gamma(\alpha)}\int_t^s (s-z)^{\alpha-1}\left(E\left(D_{*t}^{\alpha}X\right)(z)-E\left(D_{*t}^{\alpha}X\right)(t)\right)dz, \tag{28}$$

all $t \leq s \leq b$,

and

$$R_2(s,t)=\frac{1}{\Gamma(\alpha)}\int_s^t (z-s)^{\alpha-1}\left(E\left(D_{t-}^{\alpha}X\right)(z)-E\left(D_{t-}^{\alpha}X\right)(t)\right)dz, \tag{29}$$

all $a \leq s \leq t$.

We observe that ($t \leq s \leq b$, $\delta_1 > 0$)

$$|R_1(t,s)| \leq \frac{1}{\Gamma(\alpha)}\int_t^s (s-z)^{\alpha-1}\left|E\left(D_{*t}^{\alpha}X\right)(z)-E\left(D_{*t}^{\alpha}X\right)(t)\right|dz \overset{(12)}{\leq}$$

$$\frac{\omega_1\left(E\left(D_{*t}^{\alpha}X\right),\delta_1\right)_{[t,b]}}{\Gamma(\alpha)}\int_t^s (s-z)^{\alpha-1}\left(1+\frac{(z-t)}{\delta_1}\right)dz=$$

$$\frac{\omega_1\left(E\left(D_{*t}^{\alpha}X\right),\delta_1\right)_{[t,b]}}{\Gamma(\alpha)}\left[\int_t^s (s-z)^{\alpha-1}dz+\frac{1}{\delta_1}\int_t^s (s-z)^{\alpha-1}(z-t)^{2-1}dz\right]= \tag{30}$$

$$\frac{\omega_1\left(E\left(D_{*t}^{\alpha}X\right),\delta_1\right)_{[t,b]}}{\Gamma(\alpha)}\left[\frac{(s-t)^{\alpha}}{\alpha}+\frac{1}{\delta_1}\frac{\Gamma(\alpha)\Gamma(2)}{\Gamma(\alpha+2)}(s-t)^{\alpha+1}\right]=$$

$$\frac{\omega_1\left(E\left(D_{*t}^{\alpha}X\right),\delta_1\right)_{[t,b]}}{\Gamma(\alpha+1)}\left[(s-t)^{\alpha}+\frac{(s-t)^{\alpha+1}}{\delta_1(\alpha+1)}\right].$$

That is

$$|R_1(t,s)| \leq \frac{\omega_1\left(E\left(D_{*t}^{\alpha}X\right),\delta_1\right)_{[t,b]}}{\Gamma(\alpha+1)}\left[(s-t)^{\alpha}+\frac{(s-t)^{\alpha+1}}{\delta_1(\alpha+1)}\right], \tag{31}$$

where $t \leq s \leq b$, any $t \in [a,b]$, $\delta_1 > 0$.

Similarly, we have that ($a \leq s \leq t$, $\delta_2 > 0$)

$$|R_2(s,t)| \leq \frac{1}{\Gamma(\alpha)}\int_s^t (z-s)^{\alpha-1}\left|E\left(D_{t-}^{\alpha}X\right)(z)-E\left(D_{t-}^{\alpha}X\right)(t)\right|dz \overset{(13)}{\leq}$$

$$\frac{\omega_1\left(E\left(D_{t-}^{\alpha}X\right),\delta_2\right)_{[a,t]}}{\Gamma(\alpha)}\int_s^t (z-s)^{\alpha-1}\left(1+\frac{t-z}{\delta_2}\right)dz=$$

$$\frac{\omega_1\left(E\left(D_{t-}^{\alpha}X\right),\delta_2\right)_{[a,t]}}{\Gamma(\alpha)}\left[\int_s^t (z-s)^{\alpha-1}dz+\frac{1}{\delta_2}\int_s^t (t-z)^{2-1}(z-s)^{\alpha-1}dz\right]= \tag{32}$$

$$\frac{\omega_1\left(E\left(D_{t-}^{\alpha}X\right),\delta_2\right)_{[a,t]}}{\Gamma(\alpha)}\left[\frac{(t-s)^{\alpha}}{\alpha}+\frac{1}{\delta_2}\frac{\Gamma(2)\Gamma(\alpha)}{\Gamma(\alpha+2)}(t-s)^{\alpha+1}\right]=$$

$$\frac{\omega_1\left(E\left(D_{t-}^{\alpha}X\right),\delta_2\right)_{[a,t]}}{\Gamma(\alpha+1)}\left[(t-s)^{\alpha}+\frac{(t-s)^{\alpha+1}}{\delta_2(\alpha+1)}\right].$$

That is

$$|R_2(s,t)|\leq\frac{\omega_1\left(E\left(D_{t-}^{\alpha}X\right),\delta_2\right)_{[a,t]}}{\Gamma(\alpha+1)}\left[(t-s)^{\alpha}+\frac{(t-s)^{\alpha+1}}{\delta_2(\alpha+1)}\right],\tag{33}$$

where $a\leq s\leq t$, any $t\in[a,b]$, $\delta_2>0$.

So, we have

$$(EX)(s)-(EX)(t)=\sum_{k=1}^{n-1}\frac{\left(EX^{(k)}\right)(t)}{k!}(s-t)^k+R_1(t,s),\tag{34}$$

all $t\leq s\leq b$,

and

$$(EX)(s)-(EX)(t)=\sum_{k=1}^{n-1}\frac{\left(EX^{(k)}\right)(t)}{k!}(s-t)^k+R_2(s,t),\tag{35}$$

all $a\leq s\leq t$.

From now on we take $\delta_1=\delta_2=:\delta>0$.

Therefore, it holds

$$|R_1(t,s)|,|R_2(s,t)|\leq\frac{\omega_1\left(E\left(D_{t}^{\alpha}X\right),\delta\right)}{\Gamma(\alpha+1)}\left[|t-s|^{\alpha}+\frac{|t-s|^{\alpha+1}}{\delta(\alpha+1)}\right],\tag{36}$$

for any $s,t\in[a,b]$.

We have that

$$L(EX)(t)-(EX)(t)L(1)(t)\overset{(16)}{=}$$

$$\int_{[a,b]}(EX)(s)\,d\mu_t(s)-(EX)(t)L(1)(t)\overset{(15)}{=}$$

$$\int_{[a,t)}(EX)(s)\,d\mu_t(s)+\int_{[t,b]}(EX)(s)\,d\mu_t(s)-$$

$$\int_{[a,t)}(EX)(t)\,d\mu_t(s)-\int_{[t,b]}(EX)(t)\,d\mu_t(s)=$$

$$\int_{[a,t)}((EX)(s)-(EX)(t))\,d\mu_t(s)+\int_{[t,b]}((EX)(s)-(EX)(t))\,d\mu_t(s)\overset{(\text{by }(34),\,(35))}{=}\tag{37}$$

$$\sum_{k=1}^{n-1}\frac{\left(EX^{(k)}\right)(t)}{k!}L\left((s-t)^k\right)(t)+\int_{[a,t)}R_2(s,t)\,d\mu_t(s)+\int_{[t,b]}R_1(t,s)\,d\mu_t(s).$$

That is

$$L(EX)(t)-(EX)(t)L(1)(t)=\sum_{k=1}^{n-1}\frac{\left(EX^{(k)}\right)(t)}{k!}L\left((s-t)^k\right)(t)$$

$$+\int_{[a,t)}R_2(s,t)\,d\mu_t(s)+\int_{[t,b]}R_1(t,s)\,d\mu_t(s).\tag{38}$$

From [17, pp. 3-5] we have the following results

(i) $C([a,b]) \subset C_{\Omega}([a,b])$,
(ii) if $X \in C_{\Omega}([a,b])$, then $EX \in C([a,b])$,
and
(iii) if L is E-commutative, then L maps the subspace $C([a,b])$ into $B([a,b])$.
One can rewrite

$$L(EX)(t)-(EX)(t)=(EX)(t)[(L(1))(t)-1]+$$
$$\sum_{k=1}^{n-1}\frac{\left(EX^{(k)}\right)(t)}{k!}L\left((s-t)^{k}\right)(t)+\int_{[a,t)}R_{2}(s,t)\,d\mu_{t}(s) \tag{39}$$
$$+\int_{[t,b]}R_{1}(t,s)\,d\mu_{t}(s).$$

Consequently, by E-commutativity of L we find

$$|E[(LX)(t,\omega)-X(t,\omega)]|\leq|(EX)(t)|\,|(L(1))(t)-1|+$$
$$\sum_{k=1}^{n-1}\frac{\left|\left(EX^{(k)}\right)(t)\right|}{k!}\left|L\left((s-t)^{k}\right)(t)\right|+\int_{[a,t)}|R_{2}(s,t)|\,d\mu_{t}(s)$$
$$+\int_{[t,b]}|R_{1}(t,s)|\,d\mu_{t}(s)\overset{(36)}{\leq}$$
$$|(EX)(t)|\,|(L(1))(t)-1|+\sum_{k=1}^{n-1}\frac{\left|\left(EX^{(k)}\right)(t)\right|}{k!}\left|L\left((s-t)^{k}\right)(t)\right|+$$
$$\frac{\omega_{1}\left(E\left(D_{t}^{\alpha}X\right),\delta\right)}{\Gamma(\alpha+1)}\left[\int_{[a,b]}|t-s|^{\alpha}\,d\mu_{t}(s)+\frac{1}{\delta(\alpha+1)}\int_{[a,b]}|t-s|^{\alpha+1}\,d\mu_{t}(s)\right]\leq$$

(by Hölder's inequality)

$$|(EX)(t)|\,|(L(1))(t)-1|+\sum_{k=1}^{n-1}\frac{\left|\left(EX^{(k)}\right)(t)\right|}{k!}\left|L\left((s-t)^{k}\right)(t)\right|+$$
$$\frac{\omega_{1}\left(E\left(D_{t}^{\alpha}X\right),\delta\right)}{\Gamma(\alpha+1)}\left[\left(\int_{[a,b]}|t-s|^{\alpha+1}\,d\mu_{t}(s)\right)^{\frac{\alpha}{\alpha+1}}(L(1)(t))^{\frac{1}{\alpha+1}} \right. \tag{40}$$
$$\left. +\frac{1}{\delta(\alpha+1)}\int_{[a,b]}|t-s|^{\alpha+1}\,d\mu_{t}(s)\right].$$

We have proved that

$$|(E(LX))(t)-(EX)(t)|\leq|(EX)(t)|\,|(L(1))(t)-1|+$$
$$\sum_{k=1}^{n-1}\frac{\left|\left(EX^{(k)}\right)(t)\right|}{k!}\left|L\left((s-t)^{k}\right)(t)\right|+\frac{\omega_{1}\left(E\left(D_{t}^{\alpha}X\right),\delta\right)}{\Gamma(\alpha+1)}$$
$$\left[\left(L\left(|s-t|^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}(L(1)(t))^{\frac{1}{\alpha+1}}+\frac{1}{\delta(\alpha+1)}\left(L\left(|s-t|^{\alpha+1}\right)(t)\right)\right]= \tag{41}$$
$$|(EX)(t)|\,|(L(1))(t)-1|+\sum_{k=1}^{n-1}\frac{\left|\left(EX^{(k)}\right)(t)\right|}{k!}\left|L\left((s-t)^{k}\right)(t)\right|+$$
$$\frac{\omega_{1}\left(E\left(D_{t}^{\alpha}X\right),\delta\right)}{\Gamma(\alpha+1)}\left(L\left(|s-t|^{\alpha+1}\right)(t)\right)^{\frac{\alpha}{\alpha+1}}$$

$$\left[(L(1)(t))^{\frac{1}{\alpha+1}} + \frac{\left(L\left(|s-t|^{\alpha+1} \right)(t) \right)^{\frac{1}{\alpha+1}}}{\delta(\alpha+1)} \right].$$

The theorem now is valid. □

We need

Definition 5. *If $0 < \alpha < 1$, then $n = 1$, and $C_{\Omega}^{\alpha,1}([a,b]) := \{X : X(\cdot,\omega) \in AC([a,b]),\ \forall\ \omega \in \Omega$ and $\left|X^{(1)}(t,\omega)\right| \leq M,\ \forall\ (t,\omega) \in [a,b] \times \Omega$, where $M > 0$; $X(t,\omega) \in C_{\Omega}([a,b])$; also $D_{*t}^{\alpha}X$, $D_{t-}^{\alpha}X$ are stochastic processes for any $t \in [a,b]\}$.*

We give

Corollary 6. *Consider the positive E-commutative linear operator $L : C_{\Omega}([a,b]) \hookrightarrow B_{\Omega}([a,b])$, and $0 < \alpha < 1$ and let $X \in C_{\Omega}^{\alpha,1}([a,b])$, with $\delta > 0$.*

Then

$$|(E(LX))(t) - (EX)(t)| \leq |(EX)(t)| \, |(L(1))(t) - 1| + \frac{\omega_1(E(D_t^{\alpha}X),\delta)}{\Gamma(\alpha+1)} \left(L\left(|s-t|^{\alpha+1} \right)(t) \right)^{\frac{\alpha}{\alpha+1}} \left[(L(1)(t))^{\frac{1}{\alpha+1}} + \frac{\left(L\left(|s-t|^{\alpha+1} \right)(t) \right)^{\frac{1}{\alpha+1}}}{\delta(\alpha+1)} \right], \quad (42)$$

$\forall\ t \in [a,b]$.

Proof. By Theorem 4. □

We further present

Theorem 7. *All as in Theorem 4. Then*

$$\|E(LX) - EX\|_{\infty} \leq \|EX\|_{\infty} \|L(1) - 1\|_{\infty} + \sum_{k=1}^{n-1} \frac{\left\|EX^{(k)}\right\|_{\infty}}{k!} \left\| L\left((s-t)^k \right)(t) \right\|_{\infty} + \sup_{t \in [a,b]} \frac{\omega_1(E(D_t^{\alpha}X),\delta)}{\Gamma(\alpha+1)} \left\| L\left(|s-t|^{\alpha+1} \right)(t) \right\|_{\infty}^{\frac{\alpha}{\alpha+1}} \left[\|L(1)\|_{\infty}^{\frac{1}{\alpha+1}} + \frac{\left\| L\left(|s-t|^{\alpha+1} \right)(t) \right\|_{\infty}^{\frac{1}{\alpha+1}}}{\delta(\alpha+1)} \right] < \infty. \quad (43)$$

Proof. By Theorem 4, see (10), (11) and (14). □

Corollary 8. *All as in Corollary 6. Then*

$$\|E(LX) - EX\|_{\infty} \leq \|EX\|_{\infty} \|L(1) - 1\|_{\infty} + \sup_{t \in [a,b]} \frac{\omega_1(E(D_t^{\alpha}X),\delta)}{\Gamma(\alpha+1)} \left\| L\left(|s-t|^{\alpha+1} \right)(t) \right\|_{\infty}^{\frac{\alpha}{\alpha+1}} \left[\|L(1)\|_{\infty}^{\frac{1}{\alpha+1}} + \frac{\left\| L\left(|s-t|^{\alpha+1} \right)(t) \right\|_{\infty}^{\frac{1}{\alpha+1}}}{\delta(\alpha+1)} \right]. \quad (44)$$

Proof. By Corollary 6. □

Corollary 9. *All as in Corollary 6, and $L(1) = 1$. Then*

$$\|E(LX) - EX\|_\infty \leq \sup_{t\in[a,b]} \frac{\omega_1\left(E\left(D_t^\alpha X\right), \delta\right)}{\Gamma(\alpha+1)}$$

$$\left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_\infty^{\frac{\alpha}{\alpha+1}} \left[1 + \frac{\left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_\infty^{\frac{1}{\alpha+1}}}{\delta(\alpha+1)}\right]. \tag{45}$$

Proof. By (44). □

We give

Corollary 10. *All as in Corollary 6, and $L(1) = 1$. Then*

$$\|E(LX) - EX\|_\infty \leq \frac{2\sup_{t\in[a,b]} \omega_1\left(E\left(D_t^\alpha X\right), \frac{1}{(\alpha+1)}\left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_\infty^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)}$$

$$\left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_\infty^{\frac{\alpha}{\alpha+1}}. \tag{46}$$

Proof. By (45): we take there

$$\delta = \frac{1}{(\alpha+1)}\left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_\infty^{\frac{1}{\alpha+1}} > 0.$$

In case of $\left\|L\left(|s-t|^{\alpha+1}\right)(t)\right\|_\infty = 0$, we have that $L\left(|s-t|^{\alpha+1}\right)(t) = 0$, $\forall$ $t \in [a,b]$. That is, by (16) $\int_{[a,b]} |s-t|^{\alpha+1} d\mu_t(s) = 0$, $\forall$ $t \in [a,b]$, where μ_t is a probability measure, any $t \in [a,b]$.

The last implies $|s-t|^{\alpha+1} = 0$, a.e., hence $|s-t| = 0$, a.e., thus $s = t$, a.e., which means $\mu_t\{s \in [a,b] : s \neq t\} = 0$, i.e. $\mu_t = \delta_t$, $\forall$ $t \in [a,b]$, where δ_t is the unit Dirac measure.

Consequently we have $E(LX)(t) = L(EX)(t) = \int_{[a,b]} (EX)(s)\, d\delta_t(s) = (EX)(t)$, $\forall$ $t \in [a,b]$.

That is $E(LX) = EX$ over $[a,b]$. Therefore both sides of inequality (46) equal to zero.

Hence (46) is always true. □

5. Application

Let $f \in C([0,1])$ and the Bernstein polynomials

$$B_N(f)(t) := \sum_{k=0}^{N} f\left(\frac{k}{N}\right)\binom{N}{k} t^k (1-t)^{N-k}, \tag{47}$$

$\forall$ $t \in [0,1]$, $\forall$ $N \in \mathbb{N}$.

We have that $B_N 1 = 1$ and B_N is a positive linear operator.

We have that

$$B_N\left((\cdot - t)^2\right)(t) = \frac{t(1-t)}{N}, \ \forall\, t \in [0,1], \tag{48}$$

and

$$\left\| B_N \left((\cdot - t)^2 \right)(t) \right\|_\infty^{\frac{1}{2}} \leq \frac{1}{2\sqrt{N}}, \quad \forall\, N \in \mathbb{N}. \tag{49}$$

Define the corresponding stochastic application of B_N by

$$B_N(X)(t,\omega) := B_N(X(\cdot,\omega))(t) = \sum_{k=0}^{N} X\left(\frac{k}{N},\omega\right) \binom{N}{k} t^k (1-t)^{N-k}, \tag{50}$$

$\forall\, t \in [0,1]$, $\forall\, \omega \in \Omega$, $N \in \mathbb{N}$, where X is a stochastic process. Clearly $B_N(X)$ is a stochastic process and $B_N : C_\Omega([0,1]) \hookrightarrow C_\Omega([0,1])$. Notice that

$$(EB_N(X))(t) = \sum_{k=0}^{N} (EX)\left(\frac{k}{N}\right) \binom{N}{k} t^k (1-t)^{N-k} = (B_N(EX))(t), \tag{51}$$

$\forall\, t \in [0,1]$.

That is $EB_N = B_N E$, i.e. B_N is an E-commutative positive linear operator.

We give

Proposition 11. *Let $0 < \alpha < 1$ and $X \in C_\Omega^{\alpha,1}([0,1])$. Then*

$$\|E(B_N X) - EX\|_\infty \leq \frac{2 \sup_{t\in[0,1]} \omega_1 \left(E(D_t^\alpha X), \frac{1}{(\alpha+1)} \left\| B_N \left(|s-t|^{\alpha+1} \right)(t) \right\|_\infty^{\frac{1}{\alpha+1}} \right)}{\Gamma(\alpha+1)}$$
$$\left\| B_N \left(|s-t|^{\alpha+1} \right)(t) \right\|_\infty^{\frac{\alpha}{\alpha+1}}. \tag{52}$$

Proof. By Corollary 10. ∎

In particular we get:

Corollary 12. *Let $X \in C_\Omega^{\frac{1}{2},1}([0,1])$. Then*

$$\|E(B_N X) - EX\|_\infty \leq \frac{4}{\sqrt{\pi}} \sup_{t\in[0,1]} \omega_1 \left(E\left(D_t^{\frac{1}{2}} X \right), \frac{2}{3} \left\| B_N \left(|s-t|^{\frac{3}{2}} \right)(t) \right\|_\infty^{\frac{2}{3}} \right)$$
$$\left\| B_N \left(|s-t|^{\frac{3}{2}} \right)(t) \right\|_\infty^{\frac{1}{3}}, \quad \forall\, N \in \mathbb{N}. \tag{53}$$

Proof. Apply (52) for $\alpha = \frac{1}{2}$. ∎

We make

Remark 13. *We notice that*

$$B_N\left(|s-t|^{\frac{3}{2}} \right)(t) = \sum_{k=0}^{N} \left| t - \frac{k}{N} \right|^{\frac{3}{2}} \binom{N}{k} t^k (1-t)^{N-k}$$

(by discrete Hölder's inequality)

$$\leq \left(\sum_{k=0}^{N} \left| t - \frac{k}{N} \right|^2 \binom{N}{k} t^k (1-t)^{N-k} \right)^{\frac{3}{4}} \tag{54}$$

$$\overset{(48)}{=} \left(\frac{1}{N} t(1-t) \right)^{\frac{3}{4}} \leq \frac{1}{(4N)^{\frac{3}{4}}}, \quad \forall\, t \in [0,1].$$

That is

$$\left\| B_N \left(|s-t|^{\frac{3}{2}} \right)(t) \right\|_{\infty} \leq \frac{1}{(4N)^{\frac{3}{4}}}, \tag{55}$$

and

$$\left\| B_N \left(|s-t|^{\frac{3}{2}} \right)(t) \right\|_{\infty}^{\frac{1}{3}} \leq \frac{1}{(4N)^{\frac{1}{4}}}, \tag{56}$$

and

$$\left\| B_N \left(|s-t|^{\frac{3}{2}} \right)(t) \right\|_{\infty}^{\frac{2}{3}} \leq \frac{1}{2\sqrt{N}}, \tag{57}$$

$\forall\ N \in \mathbb{N}$.

We derive

Proposition 14. *Let* $X \in C_{\Omega}^{\frac{1}{2},1}([0,1])$. *Then*

$$\|E(B_N X) - EX\|_{\infty} \leq \frac{\left(\sqrt{2}\right)^3}{\sqrt{\pi}\sqrt[4]{N}} \sup_{t\in[0,1]} \omega_1 \left(E\left(D_t^{\frac{1}{2}} X \right), \frac{1}{3\sqrt{N}} \right), \tag{58}$$

$\forall\ N \in \mathbb{N}$.

Hence $\lim_{N\to\infty} E(B_N X) = EX$, *uniformly.*

Proof. By (53) and Remark 13. □

6. Caputo Fractional Stochastic Korovkin theory

Here L is meant as a sequence of positive E-commutative linear operators and all assumptions are as in Theorem 4.

We give

Theorem 15. *We further assume that* $L(1)(t) \to 1$ *and* $L\left(|s-t|^{\alpha+1}\right)(t) \to 0$, *then* $(E(LX))(t) \to (EX)(t)$, *for any* $X \in C_{\Omega}^{\alpha,n}([a,b])$, $\forall\ t \in [a,b]$, *a pointwise convergence; where* $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = n$.

Proof. Based on (20) and (18), and that $L(1)(t)$ is bounded as a sequence of functions. Also $\omega_1(E(D_t^{\alpha}X), \delta)$ is bounded, see (10), (11) and (14). □

We continue with

Theorem 16. *We further assume that* $L(1)(t) \to 1$, *uniformly and* $\left\| L\left(|s-t|^{\alpha+1}\right)(t) \right\|_{\infty} \to 0$, *then* $E(LX) \to EX$, *uniformly over* $[a,b]$, *for any* $X \in C_{\Omega}^{\alpha,n}([a,b])$; *where* $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = n$.

Proof. Based on (43) and (19), and that $\|L(1)\|_{\infty}$ is bounded.
Also it is $\sup_{t\in[a,b]} \omega_1(E(D_t^{\alpha}X), \delta) < \infty$, by (10), (11) and (14). □

We finish with

Remark 17. *The stochastic convergences of Theorems 15, 16 are derived by the convergences of the basic and simple real non-stochastic functions* $\left\{1, |s-t|^{\alpha+1}\right\}$, *an amazing fact!*

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