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ON MILD SOLUTIONS OF VOLTERRA FRACTIONAL DIFFERENTIAL EQUATIONS OF SOBOLEV TYPE WITH FINITE DELAY

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Abstract. This paper studies the existence of mild solutions for a class of fractional differential equations of Sobolev type with finite delay. The problem is expressed in terms of Volterra integro-differential equation and nonlocal condition. Several results are established by means of measure of noncompactness and some familiar fixed point theorems.

1. Introduction

In the recent past, fractional differential equation has drawn reasonable attention due to its use in growing number of applications in many areas of science, economics and engineering [27]. It serves as an excellent and precise tool while describing memory and hereditary properties of materials with varying properties and various processes. It has been found that many physical phenomena can be modeled more accurately by utilizing fractional integrals or derivatives rather than integer order integrals or derivatives. Bonilla et al. [7] presented some fractional models involving Riemann-Liouville fractional derivative with solutions that were almost impossible to derive if the problems were modeled through classical differential equations.

There are several works that have generalized the notion of classical integer order derivatives and integrals to ones with non-integer order. Another definition of fractional derivative through Caputo has an important advantage over the Riemann-Liouville definition in the sense that it allows easy and simple interpretation of initial conditions such as $x(0) = x_0$, $x'(0) = x_1$, etc. For detailed discussion and description on the theory and applications in this area, the readers are referred to some relevant books [14, 17, 24, 21] along with the references in them.

Equations of Sobolev type arise in a number of physical problems, namely, flow of fluid through fissured rocks [5], propagation of long waves with small amplitudes [6] and many more. The abstract Cauchy problem of Sobolev type is to find a

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function x which satisfies the following initial value problem:

$$\frac{d}{dt}Bx(t) + Ax(t) = f(t, x(t)),$$
$$x(0) = x_0,$$

under different conditions on A and B, where A and B are linear operators with their domains and ranges contained, respectively, in a Banach space X and a Banach space Y.

Brill [8] and Showalter [26] considered semilinear evolution equations of Sobolev type in Banach spaces and established the existence of their solutions. Such fractional models are found to be more appropriate compared to those through integer order differential equations and hence has been considered by a good number of researchers. For more details we refer the reader to the monographs by Carroll and Showalter [9], Favini and Yagi [11], Kostić[18] and Sviridyuk and Fedorov [28].

Earlier works discussed problems of Sobolev type under the following conditions on the operators $A\colon D(A) \subset X \to Y$, $B\colon D(B) \subset X \to Y$:
(1) $D(B) \subset D(A)$, B is bijective, B^{-1} is compact, $B^{-1}A\colon X \to D(B)$ is continuous [3],
(2) $D(B) \subset D(A)$, B is bijective, B^{-1} is compact [13],
(3) $D(B) \subset D(A)$, B is bijective, B^{-1} is continuous [1].

In [12, 19, 23], the authors have considered different initial value problems with various fractional derivatives when the operator B satisfies $\ker B \neq \{0\}$. The work taken up here is different from some other similar works such as [20, 4] on two counts: (i) ODE of integer order replaced by ODE of fractional order in our case, (ii) Lipschitz condition was used in these works. Therefore, to fill the gap, we find it pertinent to consider the following fractional differential equation of Sobolev type:

$$\left.\begin{aligned} &{}^C D_{0^+}^{\alpha}(Bx(t)) + Ax(t) = f\Big(t, x_t, \int_0^t h(t,s,x_s)ds\Big),\ t \in J = [0,a],\ \alpha \in (0,1), \\ &x_0(\omega) + (g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(\omega) = \phi(\omega), \quad \omega \in [-d, 0], \end{aligned}\right\} \tag{1}$$

where X, Y are Banach spaces, $A\colon D(A) \subseteq X \to Y$ and $B\colon D(B) \subseteq X \to Y$ are linear operators, $f\colon J \times \mathcal{C} \times X \to Y$, $h\colon \Delta \times \mathcal{C} \to X$, where $\mathcal{C} := C([-d,0], X)$ and $\Delta = \{(t,s) \in \mathbb{R}^2 | 0 \le s \le t \le a\}$, are given functions. For $x \in C([-d,a], X)$ and each $t \in [0,a]$, $x_t \in \mathcal{C}$ is defined by

$$x_t(\omega) = x(t+\omega).$$

$\phi \in \mathcal{C}$, $g\colon \mathcal{C}^n \to \mathcal{C}$ with $\phi(0) \in D(B)$ and $(g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(0) \in D(B)$ are given functions.

In this paper, Section 2 describes some useful notations, definitions, lemmas and theorems which are considered important for the proofs here. Section 3 presents some sufficient conditions required for the existence of mild solution of problem (1).

2. Preliminaries

The following notations will be used throughout the present work:
X and Y are Banach spaces, respectively, with norms $\|.\|_X$ and $\|.\|_Y$. $B(Y)$ is a Banach space of all bounded linear operators on Y with the norm $\|.\|_{B(Y)}$. For a linear operator Q in Y, $\rho(Q)$ is the resolvent set of Q while $R(\lambda : Q)$, with $\lambda \in \rho(Q)$, denotes the resolvent of Q.

Let J be a closed interval and $C(J,X)$ be the Banach space of all continuous functions from J to X with respect to the supremum norm. Let $\mathcal{C}$ denote the space $C([-d,0],X)$ with the norm $\|x\|_{\mathcal{C}}$ and $\mathbb{D}$ denote the space $C([-d,a],X)$ with the norm $\|.\|_{\mathbb{D}}$.

Taking $\mathbb{R}^+ = [0,\infty)$ and $J=[0,a]$, let $L^p(J,X)$, $1 \le p \le \infty$, denote the Banach space of all measurable functions $f\colon J \to X$ endowed with the following norm:

$$\|f\|_{L^p} = \begin{cases} \Big(\int_J \|f\|_X^p dt \Big)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \inf_{\mu(\bar{J})=0} \big\{ \sup_{t \in J \setminus \bar{J}} \|f(t)\|_X \big\}, & p = \infty. \end{cases}$$

Theorem 1 [Hölder's inequality][29] Assume that $p, q \ge 1$ with $1/p + 1/q = 1$. If $f \in L^p(J,X)$ and $g \in L^q(J,X)$, then for $1 \le p \le \infty$, $fg \in L^1(J,X)$ and

$$\|fg\|_{L^1} \le \|f\|_{L^p} \|g\|_{L^q} \ .$$

Theorem 2 [Bochner's theorem][29] A measurable function $f\colon J \to X$ is Bochner integrable if $\|f\|$ is Lebesgue integrable.

Definition 1 [Riemann-Liouville integral][10] The Riemann-Liouville fractional integral of a function f of order $\alpha > 0$ is defined as

$$I_{0+}^{\alpha} f(t) \ = \ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \qquad t > 0,$$

provided that the right hand side above is point-wise defined on $[0,\infty)$ with $\Gamma(.)$ denoting the standard gamma function.

Definition 2 [Caputo derivative][10] The Caputo derivative of a function f of order α is defined as follows:

$$\begin{aligned} {}^C D_{0+}^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \qquad t > 0, \\ &= I_{0^+}^{n-\alpha} f^{(n)}(t), \end{aligned}$$

where n is the least integer greater than or equal to α.

Taking f to be an abstract function with values in X, the integrals appearing in the above two definitions are taken in Bochner's sense.

Throughout this article, it is assumed that the operators A and B satisfy the hypotheses as follows [1]:
(i) A and B are closed,
(ii) $D(B) \subset D(A)$ and B is bijective,
(iii) $B^{-1}\colon Y \to D(B)$ is continuous,
(iv) For each $t \in [0,a]$ and for some $\lambda \in \rho(-AB^{-1})$, $R(\lambda : -AB^{-1})$ is a compact operator.

Hypotheses (i)-(ii) and the closed graph theorem together give the boundedness of the linear operator $AB^{-1}\colon Y \to Y$.

Lemma 1[1] Let $Q(t)$ be a uniformly continuous semigroup. If $R(\lambda; Q)$ is compact for every $\lambda \in \rho(Q)$, then $Q(t)$ is a compact semigroup.

It follows that a compact semigroup $\{T(t)\}_{t\geq 0}$ in Y is generated by the operator $-AB^{-1}$. It is further assumed that there exists a constant $\mathcal{M} > 1$ such that $\sup_{t\in J} \|T(t)\|_{B(Y)} \leq \mathcal{M}$.

Definition 3 [Mild solution][30] A mild solution of problem (1) means a function $x \in \mathbb{D}$ which satisfies

$$\begin{cases} x(t) = B^{-1}S_\alpha(t)B[\phi(0) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0)] \\ \qquad + \displaystyle\int_{s=0}^{t} (t-s)^{\alpha-1} B^{-1} T_\alpha(t-s) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big) ds, \\ \qquad t \in J = [0,a], \\ x_0(\omega) + (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(\omega) = \phi(\omega), \quad \omega \in [-d, 0], \end{cases}$$

where

$$S_\alpha(t)x = \int_0^\infty \xi_\alpha(\omega) T(t^\alpha \omega) x d\omega, \quad T_\alpha(t)x = \alpha \int_0^\infty \omega \xi_\alpha(\omega) T(t^\alpha\omega) x d\omega,$$

$$\xi_\alpha(\omega) = \frac{1}{\alpha}\omega^{-1-\frac{1}{\alpha}} \bar{w}_\alpha(\omega^{-\frac{1}{\alpha}}), \ \bar{w}_\alpha(\omega) = \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^{n-1}\omega^{-(\alpha n+1)}\frac{\Gamma(\alpha n+1)}{n!}\sin(n\pi\alpha),$$

with $\xi_\alpha(\omega)$ being a probability density function on $(0, +\infty)$ satisfying

$$\xi_\alpha(\omega) \geq 0, \ \int_0^\infty \xi_\alpha(\omega) d\omega = 1, \quad \omega \in (0, +\infty).$$

Lemma 2 [30] The bounded linear operators $S_\alpha(t)$ and $T_\alpha(t)$ satisfy the following properties:
(i) for any fixed $t \geq 0$ and $y \in Y$,

$$\|S_\alpha(t)y\|_Y \leq \mathcal{M}\|y\|_Y \quad \text{and} \quad \|T_\alpha(t)y\|_Y \leq \frac{\mathcal{M}}{\Gamma(\alpha)}\|y\|_Y.$$

(ii) $\{S_\alpha(t)\}_{t\geq 0}$ and $\{T_\alpha(t)\}_{t\geq 0}$ are strongly continuous.
(iii) if $\{T(t)\}_{t>0}$ is compact, then $\{S_\alpha(t)\}_{t>0}$ and $\{T_\alpha(t)\}_{t>0}$ are compact operators.

Lemma 3 [22] Assume that $\{Q(t)\}_{t>0}$ is compact. Then $\{Q(t)\}_{t>0}$ is equicontinuous.

Theorem 3 [Krasnoselskii's fixed point theorem][25] Let S be a bounded, closed and convex subset of a Banach space X, and let P and Q map S into X such that
(i) for every pair $x, y \in S$, $Px + Qy \in S$,
(ii) P is a contraction,
(iii) Q is completely continuous.
Then there exists a fixed point of $P + Q$ in S.

Let $(Z, \|.\|)$ be a Banach space and Z_b be the set of all non-empty bounded subsets of Z. Then we state the following definition:
Definition 4 [Measure of Noncompactness][16] A map $\gamma\colon Z_b \to \mathbb{R}^+$ satisfying

$$\gamma(\overline{co}(O)) = \gamma(O), \text{ for every } O \in Z_b,$$

where $\overline{co}(O)$ is the closure of the convex hull of O, is called the measure of noncompactness in Z.

The measure of noncompactness γ is said to be [29]
(i) regular: if $\gamma(O) = 0 \Leftrightarrow O$ is relatively compact set,
(ii) monotone: if $O_1 \subset O_2 \implies \gamma(O_1) \leq \gamma(O_2)$,
(iii) algebraically semi-additive: if $\gamma(\{z\} \cup O) = \gamma(O)$, for every $z \in Z$,
(iv) nonsingular: if $\gamma(O_1 + O_2) \leq \gamma(O_1) + \gamma(O_1)$.

It may be noted that Hausdorff measure of noncompactness β is one of the important measures of noncompactness. It is defined for any $O \in Z_b$ as

$$\beta(O) = inf\{r > 0 | O \subseteq \cup_{i=1}^{n} O_r(z_i) \text{ where } z_i \in Z\},$$

with $O_r(z_i)$ as closed balls of radius $\leq r$ with center at $z_i, i = 1, 2, \ldots, n$.

Lemma 4 [29] Let $\Omega \subset Z$ be bounded. Then for every $\epsilon > 0$, there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset \Omega$ such that

$$\beta(\Omega) \leq 2\beta\big(\{z_n\}_{n=1}^{\infty}\big) + \epsilon.$$

Lemma 5 [29] Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of Bochner integrable functions from J into Z with

$$\|z_n(t)\| \leq u(t) \text{ for almost all t } \in \text{J and every n} \in \mathbb{N},$$

where $u \in L(J, \mathbb{R}^+)$. Then the function $\phi(t) = \beta\big(\{z_n\}_{n=1}^{\infty}\big) \in L(J, \mathbb{R}^+)$ satisfies

$$\beta\bigg(\bigg\{ \int_0^t z_n(s)ds \bigg| n \in \mathbb{N} \bigg\}\bigg) \leq 2 \int_0^t \phi(s)ds.$$

Lemma 6 [29] Let $\Omega \subset C(J, X)$ be equicontinuous and bounded. Then $\beta(\Omega(t))$ is continuous on J, and

$$\beta(D) = \sup_{t \in J} \beta(\Omega(t)).$$

Theorem 4 [Darbo-Sadovskii's fixed point theorem][29] Let S be a bounded, closed and convex subset of a Banach space X and the continuous mapping $Q\colon S \to S$ be a γ-contraction. Then the mapping Q has at least one fixed point in S.

Definition 5 [γ-condensing map][16] A continuous map $Q\colon \Omega \subseteq Z \to Z$ is called γ-condensing if for any bounded set $\Omega_0 \subseteq \Omega$ with $\gamma(\Omega) > 0$, we have $\gamma(Q(\Omega_0)) < \gamma(\Omega_0)$.

For γ, a monotone nonsingular measure of noncompactness in Z, the following fixed point theorem can be stated [16, 2, 15]:
Lemma 7 Assume Ω to be a closed convex bounded subset of Z and $Q\colon \Omega \to \Omega$ a γ-condensing map. Then the set of fixed points of Q forms a nonempty compact set.

Lemma 8 [30] For $\alpha \in (0, 1]$ and $0 < a \leq b$, we have $|a^\alpha - b^\alpha| \leq (b - a)^\alpha$.

3. Main Results

Let B_r, for each $r > 0$, denote the closed ball of radius r in $\mathbb{D}$.

Theorem 5 Assume that
[Hf1] for the function $f\colon J\times\mathcal{C}\times X\to Y$, the following conditions hold:
(i) for each $(\phi,x)\in\mathcal{C}\times X$, the function $t\to f(t,\phi,x)$ is strongly measurable.
(ii) $f\colon J\times\mathcal{C}\times X\to Y$ is continuous and there exist a constant $p_1\in(0,\alpha)$ and two functions $f_1,f_2\in L^{\frac{1}{p_1}}(J,\mathbb{R}^+)$ such that

$$\|f(t,\phi_1,x)-f(t,\phi_2,x_2)\|_Y\le f_1(t)\|\phi_1-\phi_2\|_{\mathcal{C}}+f_2(t)\|x_1-x_2\|_X,$$

for all $\phi_i\in\mathcal{C}$, $x_i\in X$ $(i=1,2)$, a.e. $t\in J=[0,a]$, and

$$I^{\alpha}_{0^+}k\in C(J,\mathbb{R}^+),\ \text{where } k(t):=tf_2(t),t\in J.$$

[Hh1] for a continuous function $h\colon\Delta\times\mathcal{C}\to X$ and a constant $H>0$, the following is satisfied:

$$\int_0^t\|h(t,s,x_s)-h(t,s,y_s)\|_X ds\le H\|x_s-y_s\|_{\mathcal{C}},$$

for all $x_s,y_s\in\mathcal{C}$ and $(t,s)\in\Delta$.
[Hg1] for a function $g\colon\mathcal{C}^n\to\mathcal{C}$, a constant $G>0$ exists such that

$$\|g(x_{t_1},\dots,x_{t_n})-g(y_{t_1},\dots,y_{t_n})\|_{\mathcal{C}}\le G\|x-y\|_{\mathbb{D}},$$

for $x,y\in\mathbb{D}$.

Then, problem (1) has a unique mild solution $x\in\mathbb{D}$ subject to

$$\Theta:=\mathcal{M}\|B^{-1}\|\left[\|B\|G+\frac{1}{\Gamma(\alpha)}\frac{b^{\alpha-p_1}}{\left(\frac{\alpha-p_1}{1-p_1}\right)^{1-p_1}}\left(\|f_1\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}+H\|f_2\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}\right)\right]<1.$$

Proof. Consider a map T defined on $\mathbb{D}$ by

$$(Tx)(t)=\begin{cases}B^{-1}S_\alpha(t)B[\phi(0)-(g(x_{t_1},x_{t_2},\dots,x_{t_n}))(0)]\\ +\int_{s=0}^t(t-s)^{\alpha-1}B^{-1}T_\alpha(t-s)f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)ds,\ t\in J=[0,a],\\ \phi(t)-(g(x_{t_1},x_{t_2},\dots,x_{t_n}))(t),\ t\in[-d,0].\end{cases}$$

<u>To show that T is well-defined on $B_r, r>0$:</u>
Define a function $v\in\mathbb{D}$ such that $\|v\|_X\equiv 0$ for each $t\in[-d,a]$. Then for any $r>0$ and $x\in B_r$, the following can be obtained for $t\in[0,a]$:

$$\begin{aligned}&\int_{s=0}^t\Big\|(t-s)^{\alpha-1}B^{-1}T_\alpha(t-s)f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_X ds\\ &\le\int_{s=0}^t\Big\|(t-s)^{\alpha-1}B^{-1}T_\alpha(t-s)f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_X ds\\ &\le\|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)}\int_{s=0}^t(t-s)^{\alpha-1}\Big\|f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds.\end{aligned}$$

Using [Hf1] and [Hh1], we have

$$\begin{aligned}\Big\|f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y&\le f_1(s)\|x_s-v_s\|_{\mathcal{C}}+f_2(s)\Big\|\int_0^s h(s,\tau,x_\tau)d\tau\Big\|_X\\ &\quad+\|f(s,v_s,0)\|_Y\\ &\le f_1(s)r+f_2(s)H\|x_\tau-v_\tau\|_{\mathcal{C}}+f_2(s)H_1\int_0^s d\tau+F\\ &\le f_1(s)r+Hrf_2(s)+H_1sf_2(s)+F,\end{aligned}$$

where $\|h(t,s,0)\|_X\le H_1\ \forall\ (t,s)\in\Delta$ and $\|f(t,0,0)\|_Y\le F\ \forall\ t\in J$.

Therefore, by using Hölder's inequality, the following can be obtained:

$$\begin{aligned}
&\int_{s=0}^{t} \Big\|(t-s)^{\alpha-1}B^{-1}T_\alpha(t-s)f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_X ds\\
&\leq \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)}\Big[r\int_{s=0}^{t}(t-s)^{\alpha-1}f_1(s)ds + Hr\int_{s=0}^{t}(t-s)^{\alpha-1}f_2(s)ds\\
&+H_1\int_{s=0}^{t} s(t-s)^{\alpha-1}f_2(s)ds + F\int_{s=0}^{t}(t-s)^{\alpha-1}ds\Big]\\
&\leq \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)}\Bigg[r\frac{b^{\alpha-p_1}}{\Big(\frac{\alpha-p_1}{1-p_1}\Big)^{1-p_1}}\Big(\|f_1\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} + H\|f_2\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}\Big)\\
&+H_1 b^{1+\alpha-p_1}\left\{\frac{\Gamma\Big(\frac{2-p_1}{1-p_1}\Big)\Gamma\Big(\frac{\alpha-p_1}{1-p_1}\Big)}{\Gamma\Big(\frac{2+\alpha-2p_1}{1-p_1}\Big)}\right\}^{1-p_1}\|f_2\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} + F\frac{b^\alpha}{\alpha}\Bigg].
\end{aligned}$$

It means that $\Big\|(t-s)^{\alpha-1}B^{-1}T_\alpha(t-s)f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)\Big)\Big\|_X$ is Lebesgue integrable with respect to $s\in[0,t]\ \forall\ t\in[0,a]$. Therefore, $(t-s)^{\alpha-1}B^{-1}T_\alpha(t-s)f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)\Big)$ is Bochner integrable with respect to $s\in[0,t]$ for all $t\in[0,a]$. Hence, $(Tx)(.)$ is well-defined on $[0,a]$ for any $x\in B_r$.

Also, $(Tx)(.)$ is well-defined on $[-d,0]$ for any $x\in B_r$. Thus, T is well-defined on $B_r\subset\mathbb{D}$.

<u>To show that $Tx\in\mathbb{D}$ for $x\in\mathbb{D}$:</u>

Let $x\in\mathbb{D}$ and $-d\leq s_1<s_2\leq 0$. Then

$$\begin{aligned}
\|(Tx)(s_2)-(Tx)(s_1)\|_X \leq\ &\|\phi(s_2)-\phi(s_1)\|_X\\
&+\|(g(x_{t_1},x_{t_2},\ldots,x_{t_n}))(s_2)-(g(x_{t_1},x_{t_2},\ldots,x_{t_n}))(s_1)\|_X\\
&\longrightarrow 0 \text{ as } s_2\to s_1.
\end{aligned}$$

Let $0<s_1<s_2\leq a$. Then

$$\begin{aligned}
&\|(Tx)(s_2)-(Tx)(s_1)\|_X\\
&\leq \big\|B^{-1}S_\alpha(s_2)B[\phi(0)-(g(x_{t_1},x_{t_2},\ldots,x_{t_n}))(0)]\\
&-B^{-1}S_\alpha(s_1)B[\phi(0)-(g(x_{t_1},x_{t_2},\ldots,x_{t_n}))(0)]\big\|_X\\
&+\Big\|\int_{s=0}^{s_2}(s_2-s)^{\alpha-1}B^{-1}T_\alpha(s_2-s)f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)ds\\
&-\int_{s=0}^{s_1}(s_1-s)^{\alpha-1}B^{-1}T_\alpha(s_1-s)f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)ds\Big\|_X.
\end{aligned}$$

Now, using Lemma 2(ii), we have

$$\begin{aligned}
&\big\|B^{-1}S_\alpha(s_2)B[\phi(0)-(g(x_{t_1},x_{t_2},\ldots,x_{t_n}))(0)]\\
&-B^{-1}S_\alpha(s_1)B[\phi(0)-(g(x_{t_1},x_{t_2},\ldots,x_{t_n}))(0)]\big\|_X\\
&\leq\|B^{-1}\|\|S_\alpha(s_2)B[\phi(0)-(g(x_{t_1},x_{t_2},\ldots,x_{t_n}))(0)]\\
&-S_\alpha(s_1)B[\phi(0)-(g(x_{t_1},x_{t_2},\ldots,x_{t_n}))(0)]\|_Y\\
&\longrightarrow 0 \text{ as } s_2\to s_1
\end{aligned}$$

and

$$\Big\| \int_{s=0}^{s_2} (s_2-s)^{\alpha-1} B^{-1} T_\alpha(s_2-s) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big) ds$$
$$- \int_{s=0}^{s_1} (s_1-s)^{\alpha-1} B^{-1} T_\alpha(s_1-s) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big) ds \Big\|_X$$
$$\le \|B^{-1}\| \frac{\mathcal{M}}{\Gamma(\alpha)} \int_{s_1}^{s_2} (s_2-s)^{\alpha-1} \Big\| f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds$$
$$+ \|B^{-1}\| \int_0^{s_1} (s_1-s)^{\alpha-1} \Big\| \big[T_\alpha(s_2-s) - T_\alpha(s_1-s)\big] f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds$$
$$+ \Big\| \int_0^{s_1} \big[(s_2-s)^{\alpha-1} - (s_1-s)^{\alpha-1}\big] B^{-1} T_\alpha(s_2-s) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big) ds \Big\|_X$$
$$=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

where

$$\mathcal{I}_1 = \|B^{-1}\| \frac{\mathcal{M}}{\Gamma(\alpha)} \int_{s_1}^{s_2} (s_2-s)^{\alpha-1} \Big\| f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds,$$
$$\mathcal{I}_2 = \|B^{-1}\| \int_0^{s_1} (s_1-s)^{\alpha-1} \Big\| \big[T_\alpha(s_2-s) - T_\alpha(s_1-s)\big] f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds,$$
$$\mathcal{I}_3 = \Big\| \int_0^{s_1} \big[(s_2-s)^{\alpha-1} - (s_1-s)^{\alpha-1}\big] B^{-1} T_\alpha(s_2-s) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big) ds \Big\|_X.$$

Now,

$$\mathcal{I}_1 \le \|B^{-1}\| \frac{\mathcal{M}}{\Gamma(\alpha)} \Big[\|x\|_{\mathbb{D}} \int_{s_1}^{s_2} (s_2-s)^{\alpha-1} f_1(s) ds + H \|x\|_{\mathbb{D}} \int_{s_1}^{s_2} (s_2-s)^{\alpha-1} f_2(s) ds$$
$$+ H_1 \int_{s_1}^{s_2} (s_2-s)^{\alpha-1} s f_2(s) ds + F \int_{s_1}^{s_2} (s_2-s)^{\alpha-1} ds \Big]$$
$$\le \|B^{-1}\| \frac{\mathcal{M}}{\Gamma(\alpha)} \Bigg[\|x\|_{\mathbb{D}} \frac{(s_2-s_1)^{\alpha-p_1}}{\Big(\frac{\alpha-p_1}{1-p_1}\Big)^{1-p_1}} \Big(\|f_1\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} + H \|f_2\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} \Big) + \mathcal{I}_1'$$
$$+ F \frac{(s_2-s_1)^\alpha}{\alpha} \Bigg],$$

where

$$\begin{aligned} \mathcal{I}_1' &= H_1 \int_{s_1}^{s_2} (s_2-s)^{\alpha-1} s f_2(s) ds \\ &\le H_1 \Big| \int_0^{s_2} (s_2-s)^{\alpha-1} s f_2(s) ds - \int_0^{s_1} (s_1-s)^{\alpha-1} s f_2(s) ds \Big| \\ &+ H_1 \Big| \int_0^{s_1} \big[(s_1-s)^{\alpha-1} - (s_2-s)^{\alpha-1}\big] s f_2(s) ds \Big| \\ &=: \mathcal{I}_{11} + \mathcal{I}_{12}, \end{aligned}$$

with

$$\begin{aligned} \mathcal{I}_{11} &:= H_1 \Big| \int_0^{s_2} (s_2-s)^{\alpha-1} s f_2(s) ds - \int_0^{s_1} (s_1-s)^{\alpha-1} s f_2(s) ds \Big| \\ &\longrightarrow 0 \text{ as } s_2 \to s_1 \text{ (using [Hf1])} \end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_{12} &:= H_1\Big|\int_0^{s_1}\big[(s_1-s)^{\alpha-1}-(s_2-s)^{\alpha-1}\big]sf_2(s)ds\Big|\\&\le H_1\int_0^{s_1}[(s_1-s)^{\alpha-1}-(s_2-s)^{\alpha-1}]sf_2(s)ds.\end{aligned}$$

Now we have

$$\big[(s_1-s)^{\alpha-1}-(s_2-s)^{\alpha-1}\big]sf_2(s)\le(s_1-s)^{\alpha-1}sf_2(s)$$

and since $\int_0^{s_1}(s_1-s)^{\alpha-1}sf_2(s)ds$ exists, therefore Lebesgue's dominated convergence theorem gives $\mathcal{I}_{12}\to 0$ as $s_2\to s_1$. Thus, $\mathcal{I}_1\to 0$ as $s_2\to s_1$.

$$\begin{aligned}\mathcal{I}_2 &\le \|B^{-1}\|\int_0^{s_1}(s_1-s)^{\alpha-1}\|T_\alpha(s_2-s)-T_\alpha(s_1-s)\|_{B(Y)}\big[f_1(s)\|x\|_{\mathbb{D}}+Hf_2(s)\|x\|_{\mathbb{D}}\\&\quad+H_1f_2(s)s+F\big]ds\\&=:\mathcal{I}_{21}+\mathcal{I}_{22}+\mathcal{I}_{23}+\mathcal{I}_{24},\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_{21}&=\|B^{-1}\|\|x\|_{\mathbb{D}}\int_0^{s_1}(s_1-s)^{\alpha-1}\|T_\alpha(s_2-s)-T_\alpha(s_1-s)\|_{B(Y)}f_1(s)ds,\\\mathcal{I}_{22}&=H\|B^{-1}\|\|x\|_{\mathbb{D}}\int_0^{s_1}(s_1-s)^{\alpha-1}\|T_\alpha(s_2-s)-T_\alpha(s_1-s)\|_{B(Y)}f_2(s)ds,\\\mathcal{I}_{23}&=H_1\|B^{-1}\|\int_0^{s_1}(s_1-s)^{\alpha-1}\|T_\alpha(s_2-s)-T_\alpha(s_1-s)\|_{B(Y)}sf_2(s)ds,\\\mathcal{I}_{24}&=F\|B^{-1}\|\int_0^{s_1}(s_1-s)^{\alpha-1}\|T_\alpha(s_2-s)-T_\alpha(s_1-s)\|_{B(Y)}ds.\end{aligned}$$

Let us consider $\epsilon>0$ to be sufficiently small. Consequently,

$$\begin{aligned}\mathcal{I}_{21}&=\|B^{-1}\|\|x\|_{\mathbb{D}}\int_0^{s_1-\epsilon}(s_1-s)^{\alpha-1}f_1(s)\|T_\alpha(s_2-s)-T_\alpha(s_1-s)\|_{B(Y)}ds\\&\quad+\big\|B^{-1}\big\|\|x\|_{\mathbb{D}}\int_{s_1-\epsilon}^{s_1}(s_1-s)^{\alpha-1}f_1(s)\|T_\alpha(s_2-s)-T_\alpha(s_1-s)\|_{B(Y)}ds\\&\le\|B^{-1}\|\|x\|_{\mathbb{D}}\int_0^{s_1}(s_1-s)^{\alpha-1}f_1(s)ds\sup_{s\in[0,s_1-\epsilon]}\|T_\alpha(s_2-s)-T_\alpha(s_1-s)\|_{B(Y)}ds\\&\quad+\|B^{-1}\|\|x\|_{\mathbb{D}}\frac{2M}{\Gamma(\alpha)}\int_{s_1-\epsilon}^{s_1}(s_1-s)^{\alpha-1}f_1(s)ds\\&\le\|B^{-1}\|\|x\|_{\mathbb{D}}\int_0^{s_1}(s_1-s)^{\alpha-1}f_1(s)ds\sup_{s\in[0,s_1-\epsilon]}\|T_\alpha(s_2-s)-T_\alpha(s_1-s)\|_{B(Y)}ds\\&\quad+\frac{\epsilon^{\alpha-p_1}}{\left(\frac{\alpha-p_1}{1-p_1}\right)^{1-p_1}}\|f_1\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}.\end{aligned}$$

As $T_\alpha(t)$ is known to be continuous in the uniform operator topology for $t>0$, we have $\mathcal{I}_{21}\to 0$ as $s_2\to s_1$, $\epsilon\to 0$. Similarly, it can be shown that $\mathcal{I}_{22},\mathcal{I}_{23}$ and $\mathcal{I}_{24}$ also tend to zero as $s_2\to s_1$, $\epsilon\to 0$. Therefore, $\mathcal{I}_2\to 0$ as $s_2\to s_1$.

Now

$$\begin{aligned}
\mathcal{I}_3 &\leq \|B^{-1}\| \frac{\mathcal{M}}{\Gamma(\alpha)} \|x\|_{\mathbb{D}} \int_0^{s_1} \left[(s_1-s)^{\alpha-1} - (s_2-s)^{\alpha-1}\right] f_1(s)ds \\
&+ H\|B^{-1}\| \frac{\mathcal{M}}{\Gamma(\alpha)} \|x\|_{\mathbb{D}} \int_0^{s_1} [(s_1-s)^{\alpha-1} - (s_2-s)^{\alpha-1}] f_2(s)ds \\
&+ H_1\|B^{-1}\| \frac{\mathcal{M}}{\Gamma(\alpha)} \int_0^{s_1} \left[(s_1-s)^{\alpha-1} - (s_2-s)^{\alpha-1}\right] f(s)sds \\
&+ F\|B^{-1}\| \frac{\mathcal{M}}{\Gamma(\alpha)} \int_0^{s_1} \left[(s_1-s)^{\alpha-1} - (s_2-s)^{\alpha-1}\right] ds \\
&=: \mathcal{I}_{31} + \mathcal{I}_{32} + \mathcal{I}_{33} + \mathcal{I}_{34}.
\end{aligned}$$

We have

$$\begin{aligned}
\mathcal{I}_{31} &= \|B^{-1}\| \frac{\mathcal{M}}{\Gamma(\alpha)} \|x\|_{\mathbb{D}} \int_0^{s_1} \left[(s_1-s)^{\alpha-1} - (s_2-s)^{\alpha-1}\right] f_1(s)ds \\
&\leq \|B^{-1}\| \frac{\mathcal{M}}{\Gamma(\alpha)} \|x\|_{\mathbb{D}} \frac{\|f_1\|_{L^{\frac{1}{\alpha_1}}(J,\mathbb{R}^+)}}{\left(\frac{\alpha-p_1}{1-p_1}\right)^{1-p_1}} (s_2-s_1)^{\alpha-p_1} \\
&\longrightarrow 0 \text{ as } s_2 \to s_1.
\end{aligned}$$

Similarly, it can be shown that $\mathcal{I}_{32}, \mathcal{I}_{33}$ and $\mathcal{I}_{34}$ also tend to zero as $s_2 \to s_1$. Therefore, $\mathcal{I}_3 \to 0$ as $s_2 \to s_1$.

Thus, for $0 < s_1 < s_2 \leq a$,

$$\|(Tx)(s_2) - (Tx)(s_1)\|_X \longrightarrow 0 \text{ as } s_2 \to s_1.$$

Therefore, $Tx \in \mathbb{D}$ for any $x \in \mathbb{D}$.

In order to show that T has a unique fixed point on $\mathbb{D}$, it needs to be established that T has a unique fixed point on $B_{r_0} \subset \mathbb{D}$, where r_0 satisfies

$$\begin{aligned}
r_0 &\geq \mathcal{M}\|B\|\|B^{-1}\|\big(\|\phi\|_{\mathcal{C}} + \|g(v_{t_1}, v_{t_2}, \ldots, v_{t_n})\|_{\mathcal{C}}\big) + \Theta r_0 \\
&+ H_1 b^{1+\alpha-p_1} \left\{ \frac{\Gamma\left(\frac{2-p_1}{1-p_1}\right)\Gamma\left(\frac{\alpha-p_1}{1-p_1}\right)}{\Gamma\left(\frac{2+\alpha-2p_1}{1-p_1}\right)} \right\}^{1-p_1} \|f_2\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} + F\frac{b^\alpha}{\alpha}.
\end{aligned}$$

<u>To show that $T(B_{r_0}) \subset B_{r_0}$.</u>

For $x \in B_{r_0}$ and $t \in [0,a]$, the following is obtained:

$$\begin{aligned}
\|(Tx)(t)\|_X &\leq \left\|B^{-1}S_\alpha(t)B[\phi(0) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0)]\right\|_X \\
&+ \left\| \int_{s=0}^t (t-s)^{\alpha-1} B^{-1} T_\alpha(t-s) f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds \right\|_X \\
&\leq \mathcal{M}\|B\|\|B^{-1}\|\big(\|\phi\|_{\mathcal{C}} + Gr_0 + \|g(v_{t_1}, v_{t_2}, \ldots, v_{t_n})\|_{\mathcal{C}}\big) \\
&+ \|B^{-1}\| \frac{\mathcal{M}}{\Gamma(\alpha)} \Bigg[r_0 \frac{b^{\alpha-p_1}}{\left(\frac{\alpha-p_1}{1-p_1}\right)^{1-p_1}} \Big(\|f_1\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} + H\|f_2\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} \Big) \\
&+ H_1 b^{1+\alpha-p_1} \left\{ \frac{\Gamma\left(\frac{2-p_1}{1-p_1}\right)\Gamma\left(\frac{\alpha-p_1}{1-p_1}\right)}{\Gamma\left(\frac{2+\alpha-2p_1}{1-p_1}\right)} \right\}^{1-\alpha_1} \|f_2\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} + F\frac{b^\alpha}{\alpha} \Bigg],
\end{aligned}$$

and for $t \in [-d, 0]$,

$$\begin{aligned}\|(Tx)(t)\|_X &\leq \|\phi(t)\|_X + \|(g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(t)\|_X \\ &\leq \|\phi\|_{\mathcal{C}} + Gr_0 + \|g(v_{t_1}, v_{t_2}, \dots, v_{t_n})\|_{\mathcal{C}}.\end{aligned}$$

Thus, $Tx \in B_{r_0}$ for any $x \in B_{r_0}$.
To show that T is a contraction on B_{r_0}.
Let $x, y \in B_{r_0}$. For $t \in [0, a]$, using [Hf1], [Hh1] and [Hg1], we have

$$\begin{aligned}&\|(Tx)(t) - (Ty)(t)\|_X \\ &\leq \big\|B^{-1}S_\alpha(t)B\big[(g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(0) - (g(y_{t_1}, y_{t_2}, \dots, y_{t_n}))(0)\big]\big\|_X \\ &+ \Big\| \int_{s=0}^{t} (t-s)^{\alpha-1} B^{-1} T_\alpha(t-s) \Big[f\Big(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau\Big) \\ &- f\Big(s, y_s, \int_0^s h(s, \tau, y_\tau) d\tau\Big)\Big] ds \Big\|_X \\ &\leq \mathcal{M}\|B\|\big\|B^{-1}\big\|\|(g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(0) - (g(y_{t_1}, y_{t_2}, \dots, y_{t_n}))(0)\|_X \\ &+ \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)} \int_{s=0}^{t} (t-s)^{\alpha-1} \Big\| f\Big(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau\Big) \\ &- f\Big(s, y_s, \int_0^s h(s, \tau, y_\tau) d\tau\Big)\Big\|_Y ds \\ &\leq \mathcal{M}G\|B\|\|B^{-1}\|\|x - y\|_D + \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\alpha-1} f_1(s) ds\Big)\|x - y\|_D \\ &+ \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)} \int_{s=0}^{t} (t-s)^{\alpha-1} \Big(\int_0^s \|h(s, \tau, x_\tau) - h(s, \tau, y_\tau)\|_X d\tau\Big) f_2(s) ds \\ &\leq \Theta\|x - y\|_D.\end{aligned}$$

Also, for $t \in [-d, 0]$,

$$\|(Tx)(t) - (Ty)(t)\|_X \leq G\|x - y\|_D.$$

Therefore,

$$\|Tx - Ty\|_D \leq \Theta\|x - y\|_D.$$

Therefore, by means of Banach fixed point theorem, it is established that T has a unique fixed point in $\mathbb{D}$.

Theorem 6 Assume that
[Hf2] a function $f\colon J \times \mathcal{C} \times X \to Y$ is such that
(i) for a.e. $t \in J$, the function $(\phi, x) \to f(t, \phi, x)$ is continuous, and for each $(\phi, x) \in \mathcal{C} \times X$, the function $t \to f(t, \phi, x)$ is strongly measurable.
(ii) there exist a constant $p_1 \in (0, \alpha)$ and positive functions $f_1, f_2, f_3 \in L^{\frac{1}{p_1}}(J, \mathbb{R}^+)$ such that

$$\|f(t, \phi, x)\|_Y \leq f_1(t) + f_2(t)\|\phi\|_{\mathcal{C}} + f_3(t)\|x\|_X,$$

for any $\phi \in \mathcal{C}$, $x \in X$ and $t \in J$.
[Hh2] a function $h\colon \Delta \times \mathcal{C} \to X$ is such that
(i) for each $(t, s) \in \Delta$, the function $h(t, s, .)\colon \mathcal{C} \to X$ is continuous, and for each $x \in X$, the function $h(., ., x)\colon \Delta \to X$ is strongly measurable.
(ii) there exists a function $H(t, s) \in C(\Delta, \mathbb{R}^+)$ such that

$$\|h(t, s, \phi)\|_X \leq H(t, s)\|\phi\|_{\mathcal{C}}, \text{ for } (t, s) \in \Delta, \ \phi \in \mathcal{C}$$

and $H^* = \sup_{t\in J}\int_0^t H(t,s)ds < \infty$.

and **[Hg1]** holds. Then problem (1) admits a mild solution in $\mathbb{D}$ subject to

$$\Sigma := \mathcal{M}\|B^{-1}\|\left[\|B\|G+\frac{1}{\Gamma(\alpha)}\frac{b^{\alpha-p_1}}{\left(\frac{\alpha-p_1}{1-p_1}\right)^{1-p_1}}\left(\|f_2\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}+H^*\|f_3\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}\right)\right] < 1.$$

Proof. Consider a map T defined on $\mathbb{D}$ by

$$(Tx)(t) = \begin{cases} B^{-1}S_\alpha(t)B[\phi(0) - (g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(0)] \\ + \int_{s=0}^t (t-s)^{\alpha-1}B^{-1}T_\alpha(t-s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds, \ t \in J = [0,a], \\ \phi(t) - (g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(t), \ t \in [-d, 0]. \end{cases}$$

Then under the given assumptions, it is clearly evident that the map T is well-defined on B_r for each $r > 0$.

Choose

$$\begin{aligned} r_0 \geq\ & \mathcal{M}\|B\|\big\|B^{-1}\big\|\big(\|\phi\|_{\mathcal{C}} + \|g(v_{t_1}, v_{t_2}, \dots, v_{t_n})\|_{\mathcal{C}}\big) + \Sigma r_0 \\ & + \big\|B^{-1}\big\|\frac{\mathcal{M}}{\Gamma(\alpha)}\frac{b^{\alpha-p_1}}{\left(\frac{\alpha-p_1}{1-p_1}\right)^{1-p_1}}\|f_1\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} \end{aligned}$$

and define two operators T_1 and T_2 on B_{r_0} given by

$$(T_1x)(t) = \begin{cases} B^{-1}S_\alpha(t)B[\phi(0) - (g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(0)], \ t \in J = [0,a], \\ \phi(t) - (g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(t), \ t \in [-d, 0], \end{cases}$$

and

$$(T_2x)(t) = \begin{cases} \int_{s=0}^t (t-s)^{\alpha-1}B^{-1}T_\alpha(t-s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds, \ t \in J = [0,a], \\ 0, \ t \in [-d, 0]. \end{cases}$$

***Step 1*:** To show that $T_1x + T_2y \in B_r$ whenever $x, y \in B_r$.

Proceeding in a similar manner as was followed in Theorem 3, it can be shown that $T_1x, T_2x \in \mathbb{D}$ for any $x \in B_{r_0}$. For $x, y \in B_{r_0}$ and $t \in [-d, 0]$, the following can be obtained:

$$\|(T_1x)(t) + (T_2x)(t)\|_X \leq \|\phi\|_{\mathcal{C}} + Gr_0 + \|g(v_{t_1}, v_{t_2}, \dots, v_{t_n})\|_{\mathcal{C}}.$$

Now, for $t \in [0,a]$,

$$\begin{aligned} &\|(T_1x)(t) + (T_2x)(t)\|_X \\ &\leq \|B^{-1}S_\alpha(t)B[\phi(0) - (g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(0)]\|_{\mathcal{C}} + \int_{s=0}^t \Big\|(t-s)^{\alpha-1}B^{-1}T_\alpha(t-s) \\ &\times f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_X ds \\ &\leq \mathcal{M}\|B\|\|B^{-1}\|\big(\|\phi\|_{\mathcal{C}} + Gr_0 + \|g(v_{t_1}, v_{t_2}, \dots, v_{t_n})\|_{\mathcal{C}}\big) + \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)}\frac{b^{\alpha-p_1}}{\left(\frac{\alpha-p_1}{1-p_1}\right)^{1-p_1}} \\ &\times \left(\|f_1\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} + \|f_2\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}r_0 + H^*\|f_3\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}r_0\right). \end{aligned}$$

Therefore, $T_1x + T_2y \in B_{r_0}$ for any $x, y \in B_{r_0}$.

***Step 2*:** To show that T_1 is a contraction.

For $x, y \in B_{r_0}$ and $t \in [-d, a]$, we have $\|(T_1x)(t) - (T_1y)(t)\|_X \le \mathcal{M}\|B\|\|B^{-1}\|G\|x - y\|_{\mathbb{D}}$, which shows that T_1 is a contraction.
***Step 3*:** To show that T_2 is completely continuous.
$\{T_2x | x \in B_{r_0}\}$ **is uniformly bounded**: It follows easily from ***Step 1***.
$\{T_2x | x \in B_{r_0}\}$ **is equicontinuous**: Let $x \in B_{r_0}$ and $0 \le s_1 < s_2 \le a$. Then

$$
\begin{aligned}
&\|(T_2x)(s_2) - (T_2x)(s_1)\|_X \\
&\le \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)}\int_{s_1}^{s_2}(s_2-s)^{\alpha-1}\Big\|f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds \\
&+ \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)}\int_0^{s_1}[(s_1-s)^{\alpha-1} - (s_2-s)^{\alpha-1}]\Big\|f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds \\
&+ \|B^{-1}\|\int_0^{s_1}(s_1-s)^{\alpha-1}\|[T_\alpha(s_2-s) - T_\alpha(s_1-s)]\|_{B(Y)}\Big\|f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds \\
&=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}_1 &= \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)}\int_{s_1}^{s_2}(s_2-s)^{\alpha-1}\Big\|f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds, \\
\mathcal{I}_2 &= \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)}\int_0^{s_1}[(s_1-s)^{\alpha-1} - (s_2-s)^{\alpha-1}]\Big\|f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds, \\
\mathcal{I}_3 &= \|B^{-1}\|\int_0^{s_1}(s_1-s)^{\alpha-1}\|[T_\alpha(s_2-s) - T_\alpha(s_1-s)]\|_{B(Y)}\Big\|f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds.
\end{aligned}
$$

Now, using [Hf2],

$$
\begin{aligned}
\mathcal{I}_1 &\le \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)}\int_{s_1}^{s_2}(s_2-s)^{\alpha-1}\Big\{f_1(s) + f_2(s)\|x_s\|_{\mathcal{C}} + f_3(s)\Big\|\int_0^s h(s,\tau,x_\tau)d\tau\Big\|_X\Big\}ds \\
&\le \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)}\frac{(s_2-s_1)^{\alpha-p_1}}{\Big(\frac{\alpha-p_1}{1-p_1}\Big)^{1-p_1}}\Big[\|f_1\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} + \|x\|_{\mathbb{D}}\|f_2\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} + H^*\|x\|_{\mathbb{D}}\|f_3\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}\Big] \\
&\longrightarrow 0 \text{ as } s_2 \to s_1.
\end{aligned}
$$

Again, using [Hf2],

$$
\begin{aligned}
\mathcal{I}_2 &\le \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)}\int_0^{s_1}[(s_1-s)^{\alpha-1} - (s_2-s)^{\alpha-1}]\Big(f_1(s) + f_2(s)\|x_s\|_{\mathcal{C}} \\
&\quad + f_3(s)\Big\|\int_0^s h(s,\tau,x_\tau)d\tau\Big\|_X\Big)ds \\
&\le \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)}\Big(\|f_1\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} + \|x\|_{\mathbb{D}}\|f_2\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} + H^*\|x\|_{\mathbb{D}}\|f_3\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}\Big) \\
&\quad \times \Big(\int_0^{s_1}\Big(\Big[(s_1-s)^{\frac{\alpha-1}{1-p_1}} - (s_1-s)^{\frac{\alpha-1}{1-p_1}}\Big]\Big)ds\Big)^{1-p_1} \\
&\le \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)}\frac{(s_2-s_1)^{\alpha-p_1}}{\Big(\frac{\alpha-p_1}{1-p_1}\Big)^{1-p_1}}\Big[\|f_1\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} + \|x\|_{\mathbb{D}}\|f_2\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)} \\
&\quad + H^*\|x\|_{\mathbb{D}}\|f_3\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}\Big] \\
&\longrightarrow 0 \text{ as } s_2 \to s_1.
\end{aligned}
$$

We further have

$$
\begin{aligned}
\mathcal{I}_3 &\leq \|B^{-1}\| \int_0^{s_1} (s_1-s)^{\alpha-1} \|T_\alpha(s_2-s) - T_\alpha(s_1-s)\|_{B(Y)} f_1(s) ds \\
&+ \|x\|_{\mathbb{D}} \|B^{-1}\| \int_0^{s_1} (s_1-s)^{\alpha-1} \|T_\alpha(s_2-s) - T_\alpha(s_1-s)\|_{B(Y)} f_2(s) ds \\
&+ H^* \|x\|_{\mathbb{D}} \|B^{-1}\| \int_0^{s_1} (s_1-s)^{\alpha-1} \|T_\alpha(s_2-s) - T_\alpha(s_1-s)\|_{B(Y)} f_3(s) ds \\
&=: \mathcal{I}_{31} + \mathcal{I}_{32} + \mathcal{I}_{33},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}_{31} &= \|B^{-1}\| \int_0^{s_1} (s_1-s)^{\alpha-1} \|T_\alpha(s_2-s) - T_\alpha(s_1-s)\|_{B(Y)} f_1(s) ds, \\
\mathcal{I}_{32} &= \|x\|_{\mathbb{D}} \|B^{-1}\| \int_0^{s_1} (s_1-s)^{\alpha-1} \|T_\alpha(s_2-s) - T_\alpha(s_1-s)\|_{B(Y)} f_2(s) ds, \\
\mathcal{I}_{33} &= H^* \|x\|_{\mathbb{D}} \|B^{-1}\| \int_0^{s_1} (s_1-s)^{\alpha-1} \|T_\alpha(s_2-s) - T_\alpha(s_1-s)\|_{B(Y)} f_3(s) ds.
\end{aligned}
$$

For $s_1 = 0$ and $0 < s_2 \leq a$, $\mathcal{I}_3 = 0$. Therefore, for $s_1 > 0$ and $\epsilon > 0$ small enough,

$$
\begin{aligned}
\mathcal{I}_{31} = \|B^{-1}\| &\int_0^{s_1-\epsilon} (s_1-s)^{\alpha-1} \|T_\alpha(s_2-s) - T_\alpha(s_1-s)\|_{B(Y)} f_1(s) ds \\
&+ \|B^{-1}\| \int_{s_1-\epsilon}^{s_1} (s_1-s)^{\alpha-1} \|T_\alpha(s_2-s) - T_\alpha(s_1-s)\|_{B(Y)} f_1(s) ds.
\end{aligned}
$$

Now, following similar arguments as in Theorem 3, it can be shown that $\mathcal{I}_{31}, \mathcal{I}_{32}$ and $\mathcal{I}_{33}$ tend to zero as $s_2 \to s_1$, $\epsilon \to 0$. Thus, $\|(T_2x)(s_2) - (T_2x)(s_1)\|_X \to 0$ as $s_2 \to s_1$ implying that $\{T_2 x | x \in B_{r_0}\}$ is equicontinuous.

***Step 4*: To show that T_2 is continuous on B_{r_0}:**

Let $(x^{(k)}) \subset B_{r_0}$ and $x \in B_{r_0}$ such that $x^{(k)} \to x$ as $k \to \infty$. Then, for $t \in [0, a]$, we have

$$
\begin{aligned}
\|(T_2x^{(k)})(t) - (T_2x)(t)\|_X \leq \|B^{-1}\| \frac{\mathcal{M}}{\Gamma(\alpha)} \int_{s=0}^t (t-s)^{\alpha-1} \Big\| f\Big(s, x_s^{(k)}, \int_0^s h(s,\tau,x_\tau^{(k)}) d\tau\Big) \\
- f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau) d\tau\Big) \Big\|_Y ds.
\end{aligned}
$$

Using [Hh2] and Lebesgue's dominated convergence theorem, we get

$$
\int_0^s h(s,\tau,x_\tau^{(k)}) d\tau \longrightarrow \int_0^s h(s,\tau,x_\tau) d\tau \quad \text{as } k \to \infty.
$$

Consequently

$$
f\Big(s, x_s^{(k)}, \int_0^s h(s,\tau,x_\tau^{(k)}) d\tau\Big) \longrightarrow f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau) d\tau\Big) \quad \text{as } k \to \infty.
$$

Also, for each $t \in J$,

$$
\begin{aligned}
&(t-s)^{\alpha-1} \Big\| f\Big(s, x_s^{(k)}, \int_0^s h(s,\tau,x_\tau^{(k)}) d\tau\Big) - f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau) d\tau\Big) \Big\|_Y \\
&\leq 2(t-s)^{\alpha-1} \big[f_1(s) + r_0 f_2(s) + r_0 H^* f_3(s) \big],
\end{aligned}
$$

which is integrable for $s \in [0,t)$ and $t \in [0,a]$. Hence, application of Lebesgue's dominated convergence theorem gives

$$\int_{s=0}^{t}(t-s)^{\alpha-1}\Big\|f\Big(s,x_s^{(k)},\int_0^s h(s,\tau,x_\tau^{(k)})d\tau\Big)-f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)\Big\|_Y ds \longrightarrow 0 \quad \text{as } k\to\infty.$$

Therefore, T_2 is continuous on B_{r_0}.

***Step 5*: To show that, for any $t \in [-d,a]$, $\{(T_2x)(t)|x \in B_{r_0}\}$ is relatively compact in X:**

Let $V(t) = \{(T_2x)(t)|x \in B_{r_0}\}$, $t \in [-d,a]$. For $t \in [-d,0]$, it is obvious that $V(t)$ is relatively compact in X. Now, let $0 < t \le a$ be fixed and $\forall\ \epsilon \in (0,t)$ and $\forall\ \theta > 0$, the following operator $T_2^{\epsilon,\theta}$ is defined:

$$\begin{aligned}(T_2^{\epsilon,\theta}x)(t) &= B^{-1}\int_0^{t-\epsilon}\int_\theta^\infty \alpha\omega(t-s)^{\alpha-1}\xi_\alpha(\omega)T((t-s)^\alpha\omega)f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)d\omega ds\\ &= B^{-1}T(\epsilon^\alpha\theta)\int_0^{t-\epsilon}\int_\theta^\infty \alpha\omega(t-s)^{\alpha-1}\xi_\alpha(\omega)T((t-s)^\alpha\omega-\epsilon^\alpha\theta)\\ &\quad\times f\Big(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau\Big)d\omega ds.\end{aligned}$$

From the compactness of $T(\epsilon^\alpha\theta)$, $(\epsilon^\alpha\theta > 0)$, it is established that the set $V^{\epsilon,\theta}(t) = \{(T_2^{\epsilon,\theta}x)(t)|x \in B_{r_0}\}$ is relatively compact in X $\forall\ \epsilon \in (0,t)$ and $\forall\ \theta > 0$. Also, for any $x \in B_{r_0}$, the following holds:

$$\begin{aligned}\|(T_2x)(t)-(T_2^{\epsilon,\theta}x)(t)\|_X &\le \alpha\|B^{-1}\|\frac{\mathcal{M}}{\left(\frac{\alpha-p_1}{1-p_1}\right)^{1-p_1}}\Big(\|f_1\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}+r_0\|f_2\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}\\ &\quad+r_0H^*\|f_3\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}\Big)\left[b^{\alpha-p_1}\int_0^\theta\omega\xi_\alpha(\omega)d\omega+\frac{\epsilon^{\alpha-p_1}}{\Gamma(\alpha+1)}\right]\\ &\longrightarrow 0 \text{ as } \epsilon\to 0,\ \theta\to 0.\end{aligned}$$

Therefore, application of Arzelá-Ascoli theorem tells that $\{T_2x|x \in B_{r_0}\}$ is relatively compact which in turn implies that T_2 is completely continuous. Consequently, Krasnoselskii's fixed point theorem guarantees that $T_1 + T_2$ has a fixed point on $B_{r_0} \subset \mathbb{D}$.

Theorem 7 Assume that earlier hypotheses [Hf2], [Hh2] and the following condition hold:

[Hg2] $g\colon \mathcal{C}^n \to \mathcal{C}$ is completely continuous and there exist constants $G_1, G_2 > 0$ such that

$$\|g(x_{t_1},\ldots,x_{t_n})\|_{\mathcal{C}} \le G_1\|x\|_{\mathbb{D}} + G_2,$$

for all $x \in \mathbb{D}$.

Then problem (1) admits a mild solution in $\mathbb{D}$ provided

$$\chi := \mathcal{M}\|B^{-1}\|\left[\|B\|G_1+\frac{1}{\Gamma(\alpha)}\frac{b^{\alpha-p_1}}{\left(\frac{\alpha-p_1}{1-p_1}\right)^{1-p_1}}\Big(\|f_2\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}+H^*\|f_3\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}\Big)\right] < 1.$$

Proof. The proof can be accomplished in a similar manner like the one for Theorem 3. Therefore, only the new steps in this proof are presented.

Consider a map T defined on $\mathbb{D}$ by

$$(Tx)(t) = \begin{cases} B^{-1}S_\alpha(t)B[\phi(0) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0)] \\ + \int_{s=0}^{t}(t-s)^{\alpha-1}B^{-1}T_\alpha(t-s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds,\ t \in J = [0,a], \\ \phi(t) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(t),\ t \in [-d, 0]. \end{cases}$$

Then T is well-defined on B_r for each $r > 0$ and $T(B_{r_0}) \subset B_{r_0}$ where r_0 is chosen such that

$$r_0 \geq \mathcal{M}\|B\|\|B^{-1}\|\big(\|\phi\|_{\mathcal{C}} + G_2\big) + \chi r_0 + \|B^{-1}\|\frac{\mathcal{M}}{\Gamma(\alpha)}\frac{b^{\alpha-p_1}}{\Big(\frac{\alpha-p_1}{1-p_1}\Big)^{1-p_1}}\|f_1\|_{L^{\frac{1}{p_1}}(J,\mathbb{R}^+)}.$$

The operator T is split into the following two operators T_1 and T_2 on B_{r_0}:

$$(T_1x)(t) = \begin{cases} B^{-1}S_\alpha(t)B[\phi(0) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0)],\ t \in J = [0,a], \\ 0,\ t \in [-d, 0], \end{cases}$$

and

$$(T_2x)(t) = \begin{cases} \int_{s=0}^{t}(t-s)^{\alpha-1}B^{-1}T_\alpha(t-s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds,\ t \in J = [0,a], \\ \phi(t) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(t),\ t \in [-d, 0]. \end{cases}$$

In order to show that T has a unique fixed point on $\mathbb{D}$, it is required to establish that T is completely continuous on B_{r_0}.

Obviously $\{Tx|x \in B_{r_0}\}$ is uniformly bounded and that $\{Tx|x \in B_{r_0}\}$ is equicontinuous follows from Theorem 3 and Lemma 2. Further, [Hg2] gives that T is continuous on B_{r_0}. In order to establish that for any $t \in [-d, a]$, $\{(Tx)(t)|x \in B_{r_0}\}$ are relatively compact in X, it is sufficient to show that for $t \in [-d, 0]$, $\{(T_1x)(t)|x \in B_{r_0}\}$ and $\{(T_2x)(t)|x \in B_{r_0}\}$ is relatively compact in X. The fact that $\{(T_2x)(t)|x \in B_{r_0}\}$ for $t \in [-d, a]$ are relatively compact in X easily follows from hypothesis [Hg2] and Theorem 3.

Let $V(t) = \{(T_1x)(t)|x \in B_{r_0}\}$, $t \in [-d, a]$. For $t \in [-d, 0]$, it is obvious that $V(t) = \{0\}$ which is relatively compact in X.

Now, for $0 < t \leq a$ fixed and $\forall\ \theta > 0$, an operator T_1^θ is defined by

$$\begin{aligned}(T_1^\theta x)(t) &= B^{-1}\int_\theta^\infty \xi_\alpha(\omega)T(t^\alpha\omega)B[\phi(0) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0)]d\omega \\ &= B^{-1}T(\epsilon^\alpha\theta)\int_\theta^\infty \xi_\alpha(\omega)T(t^\alpha\omega - \epsilon^\alpha\theta)B[\phi(0) - (g(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))(0)]d\omega.\end{aligned}$$

From the compactness of $T(\epsilon^\alpha\theta)$, $(\epsilon^\alpha\theta > 0)$, it is obtained that the set $V^\theta(t) = \{(T_1^\theta x)(t)|x \in B_{r_0}\}$ is relatively compact in $X\ \forall\ \theta > 0$. Now, for any $x \in B_{r_0}$, the following holds:

$$\begin{aligned}\|(T_1x)(t) - (T_1^\theta x)(t)\|_X &\leq \mathcal{M}\|B\|\|B^{-1}\|\big[\|\phi\|_{\mathcal{C}} + G_1r_0 + G_2\big]\int_0^\theta \xi_\alpha(\omega)d\omega \\ &\longrightarrow 0 \text{ as } \epsilon \to 0,\ \theta \to 0,\end{aligned}$$

which gives $\{(T_1x)(t)|x \in B_{r_0}\}$ for $t \in [-d, a]$ to be relatively compact in X. By Arzelá-Ascoli theorem, is can be concluded that $\{Tx|x \in B_{r_0}\}$ is relatively compact. Therefore $\gamma(T(B_{r_0})) = 0$ and subsequently, by Darbo-Sadovskii's fixed point theorem, it is established that T has a fixed point in $B_{r_0} \subset \mathbb{D}$ which is the mild solution of problem (1).

For establishing the results of the next theorem, we consider β_X, $\beta_{\mathcal{C}}$ and $\beta_{\mathbb{D}}$ to be the Hausdorff measure of noncompactness in X, $\mathcal{C}$ and $\mathbb{D}$, respectively.

Theorem 8 Assume that
[Hf3] $f\colon J \times \mathcal{C} \times X \to Y$ satisfies the following:
(i) for a.e. $t \in J$, the function $(\phi, x) \to f(t, \phi, x)$ is continuous, and for each $(\phi, x) \in \mathcal{C} \times X$, the function $t \to f(t, \phi, x)$ is strongly measurable.
(ii) there exist a function $F \in L^1(J, \mathbb{R}^+)$ and a monotone decreasing function $\overline{F}\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t, \phi, x)\|_Y \le F(t)\overline{F}\big(\|\phi\|_{\mathcal{C}} + \|x\|_X\big),$$

for a.e. $t \in J$, $\phi \in \mathcal{C}$ and $x \in X$, and

$$I^{\alpha}_{0+}F \in C(J, \mathbb{R}^+).$$

(iii) there exists a function $N(t, s) \in C(\Delta, \mathbb{R}^+)$ such that

$$\beta_X\Big(B^{-1}T_\alpha(t-s)f(s, \tilde{C}, \mathcal{D})\Big) \le N(t, s)\Bigg[\sup_{\theta \in [-d, 0]} \beta_X(\tilde{C}(\theta)) + \beta_X(\mathcal{D})\Bigg],$$

for every bounded subsets $\tilde{C} \subset \mathcal{C}$ and $\mathcal{D} \subset X$.
[Hh3] $h\colon \Delta \times \mathcal{C} \to X$ satisfies the following:
(i) for each $(t, s) \in \Delta$, the function $h(t, s, .)\colon \mathcal{C} \to X$ is continuous, and for each $x \in X$, the function $h(., ., x)\colon \Delta \to X$ is strongly measurable.
(ii) there exist a function $L(t, s) \in C(\Delta, \mathbb{R}^+)$ and a monotone nondecreasing continuous function $\overline{L}\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|h(t, s, \phi)\|_X \le L(t, s)\overline{L}(\|\phi\|_{\mathcal{C}}), \text{ for } (t, s) \in \Delta, \ \phi \in \mathcal{C}$$

and $L^* = \sup_{t \in J} \int_0^t L(t, s)ds < \infty$.
(iii) there exists a function $H(t, s) \in C(\Delta, \mathbb{R}^+)$ such that for any bounded set $\tilde{C} \subset \mathcal{C}$,

$$\beta_X(h(t, s, \tilde{C})) \le H(t, s) \sup_{\theta \in [-d, 0]} \beta_X(\tilde{C}(\theta))$$

and $H^* = \sup_{t \in J} \int_0^t H(t, s)ds < \infty$.
[Hg3] $g\colon \mathcal{C}^n \to \mathcal{C}$ is continuous and
(i) there exists a monotone nondecreasing continuous function $G\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|g(x_{t_1}, \dots, x_{t_n})\|_{\mathcal{C}} \le G(\|x\|_{\mathbb{D}}), \text{ for all } x \in \mathbb{D}.$$

(ii) there exists a constant $\overline{G} > 0$ such that for any bounded subset $\Omega \subset \mathbb{D}$,

$$\beta_{\mathcal{C}}\Big(g(\Omega_{t_1}, \dots, \Omega_{t_n})\Big) \le \overline{G}\beta_{\mathbb{D}}(\Omega).$$

[Hr] there exists a constant $r > 0$ such that

$$\mathcal{M}\|B^{-1}\|\|B\|(\|\phi\|_{\mathcal{C}} + G(r)) + \overline{F}(r + L^*\overline{L}(r))\|B^{-1}\|\mathcal{M}F^* \le r,$$

where $F^* = \sup_{t \in J} I^{\alpha}_{0+}F(t)$.
Then problem (1) has a mild solution in $\mathbb{D}$ provided

$$\mathcal{M}\|B^{-1}\|\|B\|\overline{G} + 4(1 + 2H^*)\sup_{t \in J}\int_0^t (t-s)^{\alpha-1}N(t, s)ds < 1.$$

Proof. Defining an operator $T\colon \mathbb{D} \to \mathbb{D}$ as in the previous theorem and then proceeding in a similar manner and by using the given assumptions, it can be shown that T is well-defined and continuous on B_r, for every $r > 0$. Also, we have $T(B_r) \subseteq B_r$, for $r > 0$, satisfying assumption [Hr].

Now, T is split into two parts T_1 and T_2 as follows:

$$(T_1x)(t) = \begin{cases} B^{-1}S_\alpha(t)B[\phi(0) - (g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(0)], \ t \in J = [0,a], \\ \phi(t) - (g(x_{t_1}, x_{t_2}, \dots, x_{t_n}))(t), \ t \in [-d, 0], \end{cases}$$

and

$$(T_2x)(t) = \begin{cases} \int_{s=0}^t (t-s)^{\alpha-1}B^{-1}T_\alpha(t-s)f\Big(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\Big)ds, \ t \in J = [0,a], \\ 0, \ t \in [-d, 0]. \end{cases}$$

Let $\Omega \subseteq B_r$ be a bounded set. Then using the algebraically semi-additive property of $\beta_{\mathbb{D}}$, we have

$$\beta_{\mathbb{D}}(T(\Omega)) \leq \beta_{\mathbb{D}}(T_1(\Omega)) + \beta_{\mathbb{D}}(T_2(\Omega)),$$

where

$$\beta_{\mathbb{D}}(T_1(\Omega)) \leq \mathcal{M}\|B^{-1}\|\|B\|\overline{G}\beta_{\mathbb{D}}(\Omega).$$

Now, using Lemma 2, for $\epsilon > 0$, we can choose $\{x_n\}_{n=1}^\infty \subset \Omega$ such that

$$\beta_{\mathbb{D}}(T_2(\Omega)) \leq 2\beta_{\mathbb{D}}\big(T_2(\{x_n\}\big) + \epsilon.$$

Because $T_2(B_r)$ is equicontinuous, by using Lemma 2, we obtain

$$\begin{aligned} \beta_{\mathbb{D}}\big(T_2(\{x_n\})\big) &= \sup_{t\in[-d,a]} \beta_X(T_2\{x_n\}(t)) \\ &= \sup_{t\in[0,a]} \beta_X\bigg(\bigg\{\int_{s=0}^t (t-s)^{\alpha-1}B^{-1}T_\alpha(t-s) \\ &\quad \times f\Big(s, (x_n)_s, \int_0^s h(s,\tau,(x_n)_\tau)d\tau\Big)ds\bigg\}\bigg). \end{aligned}$$

Now, using Lemma 2, [Hf3](iii), [Hh3](iii), we get

$$\begin{aligned} &\beta_X\bigg(\bigg\{\int_{s=0}^t (t-s)^{\alpha-1}B^{-1}T_\alpha(t-s)f\Big(s,(x_n)_s, \int_0^s h(s,\tau,(x_n)_\tau)d\tau\Big)ds\bigg\}\bigg) \\ &\leq 2\int_{s=0}^t \beta_X\bigg(\bigg\{(t-s)^{\alpha-1}B^{-1}T_\alpha(t-s)f\Big(s,(x_n)_s, \int_0^s h(s,\tau,(x_n)_\tau)d\tau\Big)\bigg\}\bigg)ds \\ &\leq 2\int_{s=0}^t (t-s)^{\alpha-1}N(t,s)\bigg[\beta_{\mathcal{C}}\big\{(x_n)_s\big\} + \beta_X\bigg(\bigg\{\int_0^s h(s,\tau,(x_n)_\tau)d\tau\bigg\}\bigg)\bigg]ds \\ &\leq 2(1+2H^*)\beta_{\mathbb{D}}(\Omega)\sup_{t\in[0,b]}\int_0^t (t-s)^{\alpha-1}N(t,s)ds. \end{aligned}$$

Therefore,

$$\beta_{\mathbb{D}}(T_2(\Omega)) \leq \bigg[4(1+2H^*)\sup_{t\in J}\int_0^t (t-s)^{\alpha-1}N(t,s)ds\bigg]\beta_{\mathbb{D}}(\Omega),$$

since $\epsilon > 0$ is arbitrary.

Thus

$$\beta_{\mathbb{D}}(T(\Omega)) \leq \bigg[\mathcal{M}\|B^{-1}\|\|B\|\overline{G} + 4(1+2H^*)\sup_{t\in J}\int_0^t (t-s)^{\alpha-1}K(t,s)ds\bigg]\beta_{\mathbb{D}}(\Omega).$$

By using Lemma 2, it can be concluded that T has a fixed point in $\mathbb{D}$, which is the required mild solution of our problem.

Conclusion

In this work, the existence and uniqueness of mild solutions of a class of fractional differential equations of Sobolev type with finite delay is discussed. The problem is expressed in terms of Volterra integro-differential equation and nonlocal condition. The first three results are established by applying Banach fixed point theorem, Krasnoselskii's fixed point theorem and Darbo-Sadovskii's fixed point theorem, respectively. In the last result, we drop the compactness assumption on the nonlocal function g and instead use measure of noncompactness to obtain some sufficient conditions which ensure the existence of mild solutions.

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