Journal of Fractional Calculus and Applications
Vol. 12(2) July 2021, pp. 114-124
ISSN: 2090-5858.
http://math-frac.oreg/Journals/JFCA/

BSDES DRIVEN BY BOTH STANDARD AND FRACTIONAL BROWNIAN MOTIONS WITH NON-LIPSCHITZ CONDITIONS

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Abstract. In this work, we deal with a backward stochastic differential equation driven by both standard and fractional Brownian motion with Hurst parameter $H \geq \frac{1}{2}$ whose generator satisfies the Mao's condition in y and the Lipschitz condition in z_1 and z_2. We establish existence and uniqueness of solution in the case of non-Lipschitz condition on the generator. The stochastic integral used throughout the paper is the divergence type integral.

1. Introduction

Backward stochastic differential equations (BSDEs in short) were first introduced by Pardoux and Peng [9] with Lipschitz assumption under which they proved the celebrated existence and uniqueness result. This pioneer work was extensively used in many fields like stochastic interpretation of solutions of PDEs and financial mathematics. Few years later, several authors investigated BSDEs with respect to fractional Brownian motion $\left(B_t^H\right)_{t\geq 0}$ with Hurst parameter H. This process is a self-similar, i.e. B_{at}^H has the same law as $a^H B_t^H$ for any $a > 0$, it has a long range dependence for $H > \frac{1}{2}$. For $H = \frac{1}{2}$ we obtain a standard Wiener process, but for $H \neq \frac{1}{2}$, this process is not a semimartingale. These properties make this process a useful driving noise in models arising in physics, telecommunication networks, finance and other fields.

Since B^H is not a semimartingale when $H \neq \frac{1}{2}$, we cannot use the beautiful classical theory of stochastic calculus to define the fractional stochastic integral. It is a significant and challenging problem to extend the results in the classical stochastic calculus to this fractional Brownian motion. Essentially, two different types of integrals with respect to a fractional Brownian motion have been defined and studied. The first one is the pathwise Riemann-Stieltjes integral (see Young [11]). This integral has a proprieties of Stratonovich integral, which leads to difficulties in applications. The second one, introduced in Decreusefond [5] is the divergence operator (or Skorohod integral), defined as the adjoint of the derivative operator in the framework of the Malliavin calculus. Since this stochastic integral satisfies the

2010 *Mathematics Subject Classification.* 60H05, 60H07, 60G22.

Key words and phrases. Brownian motion, fractional Brownian motion, backward stochastic differential equation, Malliavin derivative, fractional Itô formula.

Submitted March 31, 2019. Revised Nov. 8, 2020.

zero mean property and it can be expressed as the limit of Riemann sums defined using Wick products, it was later developed by many authors.

Concerning the study of BSDEs in the fractional framework, the major problem is the absence of a martingale representation type theorem with respect to the fractional Brownian motion. Hu and Peng [8] overcame successfully this problem in the case $H > 1/2$ by means of the quasi-conditional expectation. The authors prove existence and uniqueness of the solution but with some restrictive assumptions on the generator.

Recently, Fei et al [6] introduced the following type of BSDE driven by both standard and fractional Brownian motions (SFBSDEs in short)

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s})ds - \int_t^T Z_{1,s} dB_s - \int_t^T Z_{2,s} dB_s^H, \quad 0 \leq t \leq T \tag{1.1}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion, $\left(B_t^H\right)_{t\geq 0}$ is a fractional Brownian motion.
In [6], the authors abtained the existence and uniqueness of the solution of SFABSDEs under Lipschitz assumption.

In this paper, inspired by the works of Fei et al [6] and Aidara and Sow [2], we are interesting in extending this result with weak assumption on the drift. To be precise, the generator function f satisfies $|f(t,x,y,z_1,z_2) - f(t,x,y',z_1',z_2')|^2 \leq \rho(t,|y-y'|^2) + K(|z_1 - z_1'|^2 + |z_2 - z_2'|^2)$, where K is a positive constant and ρ is a continuous nondecreasing concave function with additional properties. We establish an existence and uniqueness result of solutions for this kind of BSDEs by a Picard-type iteration, for which the well-known Bihari's inequality played an important role.

This paper is organized as follows. In section 2, we introduce some preliminaries. In section 3, we prove existence and uniqueness of solutions of SFBSDEs under non-Lipschitz condition.

2. Fractional Stochastic calculus

Let Ω be a non-empty set, F a $\sigma-$algebra of sets Ω, $\mathbf{P}$ a probability measure defined on $\mathcal{F}$ and $\{\mathcal{F}_t,\ t \in [0,T]\}$ a $\sigma-$algebra generated by both standard and fractional Brownian motions. The triplet $(\Omega, \mathcal{F}, \mathbf{P})$ defines a probability space and $\mathbf{E}$ the mathematical expectation with respect to the probability measure $\mathbf{P}$.

The fractional Brownian motion $\left(B_t^H\right)_{t\geq 0}$ with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process with the covariance function

$$\mathbf{E}[B_t^H B_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t,s \geq 0.$$

Suppose that the process $\left(B_t^H\right)_{t\geq 0}$ is independent of the standard Brownian motion $(B_t)_{t\geq 0}$. Throughout this paper it is assumed that $H \in (1/2, 1)$ is arbitrary but fixed.

Denote $\phi(t,s) = H(2H-1)|t-s|^{2H-2},\ (t,s) \in \mathbf{R}^2$. Let ξ and η be measurable functions on $[0,T]$. Define

$$\langle \xi, \eta \rangle_t = \int_0^t \int_0^t \phi(u,v)\xi(u)\eta(v)dudv \ \text{ and } \ \|\xi\|_t^2 = \langle \xi, \xi \rangle_t.$$

Note that, for any $t \in [0,T]$, $\langle \xi, \eta \rangle_t$ is a Hilbert scalar product. Let $\mathcal{H}$ be the completion of the set of continuous functions under this Hilbert norm $\|\cdot\|_t$ and $(\xi_n)_n$ be a sequence in $\mathcal{H}$ such that $\langle \xi_i, \xi_j \rangle_T = \delta_{ij}$. Let P_T^H be the set of all polynomials of fractional Brownian motion. Namely, P_T^H contains all elements of the form

$$F(\omega) = f\left(\int_0^T \xi_1(t)dB_t^H, \int_0^T \xi_2(t)dB_t^H, \dots, \int_0^T \xi_n(t)dB_t^H\right)$$

where f is a polynomial function of n variables. The Malliavin derivative D_t^H of F is given by

$$D_s^H F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(\int_0^T \xi_1(t)dB_t^H, \int_0^T \xi_2(t)dB_t^H, \dots, \int_0^T \xi_n(t)dB_t^H\right)\xi_i(s) \quad 0 \leq s \leq T.$$

Similarly, we can define the Malliavin derivative D_tG of the Brownian functional

$$G(\omega) = f\left(\int_0^T \xi_1(t)dB_t, \int_0^T \xi_2(t)dB_t, \dots, \int_0^T \xi_n(t)dB_t\right).$$

The divergence operator D^H is closable from $L^2(\Omega, F, \mathbf{P})$ to $L^2(\Omega, F, \mathbf{P}, H)$. Hence we can consider the space $\mathbb{D}_{1,2}$ is the completion of P_T^H with the norm

$$||F||_{1,2}^2 = \mathbf{E}|F|^2 + \mathbf{E}||D_s^H F||_T^2.$$

Now we introduce the Malliavin ϕ-derivative $\mathbb{D}_t^H$ of F by

$$\mathbb{D}_t^H F = \int_0^T \phi(t,s) D_s^H F ds.$$

We have the following (see[[7], Proposition 6.25]):

Theorem 2.1. *Let $F : (\Omega, \mathcal{F}, \mathbf{P}) \longrightarrow \mathcal{H}$ be a stochastic processes such that*

$$\mathbf{E}\left(\|F\|_T^2 + \int_0^T \int_0^T |\mathbb{D}_s^H F_t|^2 dsdt\right) < +\infty.$$

Then, the Itô-Skorohod type stochastic integral denoted by $\int_0^T F_s dB_s^H$ exists in $L^2(\Omega, \mathcal{F}, \mathbf{P})$ and satisfies

$$\mathbf{E}\left(\int_0^T F_s dB_s^H\right) = 0 \quad \textit{and} \quad \mathbf{E}\left(\int_0^T F_s dB_s^H\right)^2 = \mathbf{E}\left(\|F\|_T^2 + \int_0^T \int_0^T \mathbb{D}_s^H F_t \mathbb{D}_t^H F_s dsdt\right).$$

Let us recall the fractional Itô formula (see[[6], Theorem 3.1]).

Theorem 2.2. *Let $\sigma_1 \in L^2([0,T])$ and $\sigma_2 \in \mathcal{H}$ be deterministic continuous functions.*
Assume that $\|\sigma_2\|_t$ is continuously differentiable as a function of $t \in [0,T]$. Denote

$$X_t = X_0 + \int_0^t \alpha(s)ds + \int_0^t \sigma_1(s)dB_s + \int_0^t \sigma_2(s)dB_s^H,$$

where X_0 is a constant, $\alpha(t)$ is a deterministic function with $\int_0^t |\alpha(s)|ds < +\infty$. Let $F(t,x)$ be continuously differentiable with respect to t and twice continuously *differentiable with respect to x. Then*

$$F(t,X_t)=F(0,X_0)+\int_0^t\frac{\partial F}{\partial s}(s,X_s)ds+\int_0^t\frac{\partial F}{\partial x}(s,X_s)dX_s$$
$$+\frac{1}{2}\int_0^t\frac{\partial^2 F}{\partial x^2}(s,X_s)\left[\sigma_1^2(s)+\frac{d}{ds}\left\|\sigma_2\right\|_s^2\right]ds,\quad 0\le t\le T.$$

Let us finish this section by giving a fractional Itô chain rule (see[[6], Theorem 3.2]).

Theorem 2.3. *Assume that for $i=1,2$, the processes μ_i, α_i and ϑ_i, satisfy*

$$\mathbf{E}\left[\int_0^T\mu_i^2(s)ds+\int_0^T\alpha_i^2(s)ds+\int_0^T\vartheta_i^2(s)ds\right]<\infty.$$

Suppose that $D_t\alpha_i(s)$ and $\mathbb{D}_t^H\vartheta_i(s)$ are continuously differentiable with respect to $(s,t)\in[0,T]^2$ for almost all $\omega\in\Omega$. Let X_t and Y_t be two processes satisfying

$$X_t=X_0+\int_0^t\mu_1(s)ds+\int_0^t\alpha_1(s)dB_s+\int_0^t\vartheta_1(s)dB_s^H,\qquad 0\le t\le T,$$
$$Y_t=Y_0+\int_0^t\mu_2(s)ds+\int_0^t\alpha_2(s)dB_s+\int_0^t\vartheta_2(s)dB_s^H,\qquad 0\le t\le T.$$

If for $i=1,2$, the following conditions hold:

$$\mathbf{E}\left[\int_0^T|D_t\alpha_i(s)|^2dsdt\right]<+\infty,\qquad \mathbf{E}\left[\int_0^T|\mathbb{D}_t^H\vartheta_i(s)|^2dsdt\right]<+\infty,$$

then

$$X_tY_t=X_0Y_0+\int_0^tX_sdY_s+\int_0^tY_sdX_s$$
$$+\int_0^t\left[\alpha_1(s)D_sY_s+\alpha_2(s)D_sX_s+\vartheta_1(s)\mathbb{D}_s^HY_s+\vartheta_2(s)\mathbb{D}_s^HX_s\right]ds,$$

which may be written formally as

$$d\left(X_tY_t\right)=X_tdY_t+Y_tdX_t+\left[\alpha_1(t)D_tY_t+\alpha_2(t)D_tX_t+\vartheta_1(t)\mathbb{D}_t^HY_t+\vartheta_2(t)\mathbb{D}_t^HX_t\right]dt.$$

We are now in position to move on to study our main subject.

3. SFBSDEs with non-Lipschitz Conditions

In this section, our objective is to study the existence and uniqueness of the solution to the following SFBSDE

$$Y_t=\xi+\int_t^Tf(s,\eta_s,Y_s,Z_{1,s},Z_{2,s})ds-\int_t^TZ_{1,s}dB_s-\int_t^TZ_{2,s}dB_s^H,\quad 0\le t\le T.\tag{3.1}$$

3.1. **Definitions and notations.** Let us consider

$$\eta_t = \eta_0 + b(t) + \int_0^t \sigma_1(s)dB_s + \int_0^t \sigma_2(s)dB_s^H, \quad 0 \le t \le T$$

where the coefficients η_0, b, σ_1 and σ_2 satisfy:

- η_0 is a given constant,
- $b, \sigma_1, \sigma_2 : [0,T] \to \mathbf{R}$ are deterministic continuous functions, σ_1 and σ_2 are differentiable and $\sigma_1(t) \neq 0,\ \sigma_2(t) \neq 0$ such that

$$|\sigma|_t^2 = \int_0^t \sigma_1^2(s)ds + \|\sigma_2\|_t^2, \quad 0 \le t \le T, \tag{3.2}$$

where $$\|\sigma_2\|_t^2 = H(2H-1)\int_0^t\int_0^t |u-v|^{2H-2}\sigma_2(u)\sigma_2(v)dudv.$$

Let $$\hat{\sigma}_2(t) = \int_0^t \phi(t,v)\sigma_2(v)dv, \quad 0 \le t \le T.$$

The next Remark will be useful in the sequel.

Remark 3.1. *The function $|\sigma|_t^2$ defined by eq.(3.2) is continuously differentiable with respect to t on $[0,T]$, and*

a) $\frac{d}{dt}|\sigma|_t^2 = \sigma_1^2(t) + \frac{d}{dt}\|\sigma_2\|_t^2 = \sigma_1^2(t) + \sigma_2(t)\hat{\sigma}_2(t) > 0, \quad 0 \le t \le T.$

b) *for a suitable constant $C_0 > 0$,* $\inf_{0\le t\le T} \frac{\hat{\sigma}_2(t)}{\sigma_2(t)} \ge C_0$.

Before giving the definition of the solution for the above equation, we introduce the following sets:

- $C^{1,2}_{\text{pol}}([0,T]\times\mathbf{R})$ is the space of all $C^{1,2}$-functions over $[0,T]\times\mathbf{R}$, which together with their derivatives are of polynomial growth,
- $V_{[0,T]} = \left\{ Y = \psi(\cdot,\eta) :\ \psi \in C^{1,2}_{\text{pol}}([0,T]\times\mathbf{R}), \frac{\partial\psi}{\partial t} \text{ is bounded},\ t\in[0,T]\right\}$,
- $V^\beta_{[0,T]}$ the completion of $V_{[0,T]}$ under the following norm ($\beta > 0$)

$$\|Y\|_\beta = \left(\int_0^T e^{\beta t}\mathbf{E}|Y_t|^2dt\right)^{1/2} = \left(\int_0^T e^{\beta t}\mathbf{E}|\psi(t,\eta_t)|^2dt\right)^{1/2}.$$

Definition 3.2. *A triplet of processes $(Y_t, Z_{1,t}, Z_{2,t})_{0\le t\le T}$ is called a solution to SFBSDE* (3.1)*, if $(Y_t, Z_{1,t}, Z_{2,t})_{0\le t\le T} \in V^\beta_{[0,T]}\times V^\beta_{[0,T]}\times V^\beta_{[0,T]}$ and satisfies eq.*(3.1)*.*

We have the following (see [[6], Theorem 5.3])

Theorem 3.3. *Assume that σ_1 and σ_2 are continuous and $|\sigma|_t^2$ defined by eq.*(3.2) *is a strictly increasing function of t. Let the SFBSDE* (3.1) *has a solution of the form*
$(Y_t = \psi(t,\eta_t),\ Z_{1,t} = -\varphi_1(t,\eta_t),\ Z_{2,t} = -\varphi_2(t,\eta_t))$*, where* $\psi \in C^{1,2}([0,T]\times\mathbf{R})$.
Then

$$\varphi_1(t,x) = \sigma_1(t)\psi'_x(t,x), \quad \varphi_2(t,x) = \sigma_2(t)\psi'_x(t,x).$$

The next proposition will be useful in the sequel.

Proposition 3.4. *Let $(Y_t, Z_{1,t}, Z_{2,t})_{0\le t\le T}$ be a solution of the SFBSDE* (3.1)*. Then for almost $t \in [0,T]$,*

$$D_tY_t = Z_{1,t}, \quad \text{and} \quad \mathbb{D}_t^H Y_t = \frac{\hat{\sigma}_2(t)}{\sigma_2(t)}Z_{2,t}.$$

Proof. Since $(Y_t, Z_{1,t}, Z_{2,t})$ satisfies the SFBSDE (3.1) then we have $Y = \psi(\cdot, \eta)$ where
$\psi \in C^{1,2}([0,T] \times \mathbf{R})$. From Theorem 3.3, we have

$$Z_{1,t} = \sigma_1(t)\psi'_x(t,x), \quad Z_{2,t} = \sigma_2(t)\psi'_x(t,x).$$

Then we can write $D_t Y_t = \sigma_1(t)\psi'_x(t,x) = Z_{1,t}$ and

$$\begin{aligned}\mathbb{D}_t^H Y_t = \int_0^T \phi(t,s) D_s^H \psi(t,\eta_t) ds &= \psi'_x(t,\eta_t) \int_0^T \phi(t,s)\sigma_2(s) ds \\ &= \widehat{\sigma}_2(t)\psi'_x(t,\eta_t) = \frac{\widehat{\sigma}_2(t)}{\sigma_2(t)} Z_{2,t}.\end{aligned}$$

□

3.2. **Existence and Uniqueness of solution.** We say that the coefficient f satisfies assumptions **(H)** if the following holds:

- **(H1)** $\xi = h(\eta_T)$, where $h : \mathbf{R} \to \mathbf{R}$ is a differentiable function and $\mathbf{E}\left[e^{\beta T}|\xi|^2\right] < +\infty$.
- **(H2)** $f : [0,T] \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a continuous function and there exists a constant $K > 0$ such that for all $t \in [0,T]$, $x \in \mathbf{R}$, $(y, y') \in \mathbf{R}^2$, $(z_1, z_1') \in \mathbf{R}^2$, $(z_2, z_2') \in \mathbf{R}^2$,

$$|f(t,x,y,z_1,z_2) - f(t,x,y',z_1',z_2')|^2 \leq \rho(t, |y-y'|^2) + K(|z_1 - z_1'|^2 + |z_2 - z_2'|^2)$$

where $\rho(t,\nu) : [0,T] \times \mathbf{R}_+ \to \mathbf{R}_+$ satisfies

- For fixed $t \in [0,T]$, $\rho(t,\cdot)$ is a continuous, concave and nondecreasing s.t.
$$\rho(t,0) = 0, \qquad \text{and} \quad \forall \alpha > 0 \quad \alpha\rho(t,\nu) = \rho(t,\alpha\nu).$$
- The ordinary differential equation
$$\nu'(t) = -\rho(t,\nu(t)), \quad v(T) = 0, \tag{3.3}$$
has a unique solution $\nu(t) = 0, \quad 0 \leq t \leq T$.
- There exists $a(\cdot), b(\cdot) : [0,T] \to \mathbf{R}_+$ such that
$$\rho(t,\nu) \leq a(t) + b(t)\nu \quad \text{and} \quad \int_0^T [a(t) + b(t)] dt < +\infty.$$

Let us mention that assumptions **(H)** are weaker than Lipschitz conditions required on the coefficient f in [6].

Example 3.5. *If $f(t,x,y,z_1,z_2) = \frac{y}{\sqrt[4]{t}} + K(x + z_1 + z_2)$ and $\rho(t,u) = \frac{2u}{\sqrt{t}}$, then it is easy to check that f satisfies* **(H2)**.

Let us recall the following result given in [[6], Theorem 5.5].

Proposition 3.6. *Assume that f is Lipschitzian. Then eq.(3.1) has a unique solution $(Y_t, Z_{1,t}, Z_{2,t})_{0 \leq t \leq T} \in V_{[0,T]}^{\beta} \times V_{[0,T]}^{\beta} \times V_{[0,T]}^{\beta}$.*

The main result of this section is the following theorem:

Theorem 3.7. *Let the assumption* **(H)** *be satisfied. Then the SFBSDE (3.1) has a unique solution $(Y_t, Z_{1,t}, Z_{2,t})_{0 \leq t \leq T} \in V_{[0,T]}^{\beta} \times V_{[0,T]}^{\beta} \times V_{[0,T]}^{\beta}$.*

We can construct the Picard approximate sequence of eq.(3.1) as follows

$$\begin{cases} Y_t^0 = 0, \\ Y_t^n = \xi + \int_t^T f(s, \eta_s, Y_s^{n-1}, Z_{1,s}^n, Z_{2,s}^n) ds - \int_t^T Z_{1,s}^n dB_s - \int_t^T Z_{2,s}^n dB_s^H, \quad n \geq 1. \end{cases} \tag{3.4}$$

Thanks to Proposition 3.6, this sequence is well defined. In order to prove Theorem 3.7, we need two lemmas.

Lemma 3.8. *Let the assumption* **(H)** *be satisfied. There exists a constant* $C > 0$ *such that for all* $0 \leq t \leq T, \ n, m \geq 1$, *we have*

$$\mathbf{E}\left[e^{\beta t}|Y_t^{n+m} - Y_t^n|^2\right] \leq \frac{1}{C} e^{C(T-t)} \int_t^T \rho\left(s, \mathbf{E}\left[e^{\beta s}|Y_s^{n+m-1} - Y_s^{n-1}|^2\right]\right) ds.$$

Proof. Let us define for a process $\delta \in \{Y, Z_1, Z_2\}, \ n, m \geq 1, \ \bar{\delta}^{n,m} = \delta^{n+m} - \delta^n$ and the function

$$\Delta f^{(n,m)}(s) = f(s, \eta_s, Y_s^{n+m-1}, Z_{1,s}^{n+m}, Z_{2,s}^{n+m}) - f(s, \eta_s, Y_s^{n-1}, Z_{1,s}^n, Z_{2,s}^n).$$

Then, it is obvious that $(\bar{Y}^{n,m}, \bar{Z}_1^{n,m}, \bar{Z}_2^{n,m})$ solves the SFBSDE

$$\bar{Y}_t^{n,m} = \int_t^T \Delta f^{(n,m)}(s) ds - \int_t^T \bar{Z}_{2,s}^{n,m} dB_s - \int_t^T \bar{Z}_{2,s}^{n,m} dB_s^H, \quad 0 \leq t \leq T.$$

By the fractional Itô chain rule, we have

$$\begin{aligned} |\bar{Y}_t^{n,m}|^2 = & \, 2\int_t^T \bar{Y}_s^{n,m} \Delta f^{(n,m)}(s) ds - 2\int_t^T \bar{Z}_{1,s}^{n,m} D_s \bar{Y}_s^{n,m} ds - 2\int_t^T \bar{Z}_{2,s}^{n,m} \mathbb{D}_s^H \bar{Y}_s^{n,m} ds \\ & - 2\int_t^T \bar{Y}_s^{n,m} \bar{Z}_{1,s}^{n,m} dB_s - 2\int_t^T \bar{Y}_s^{n,m} \bar{Z}_{2,s}^{n,m} dB_s^H. \end{aligned}$$

Applying Itô formula to $e^{\beta t}|\bar{Y}_t^{n,m}|^2$, we obtain that

$$\begin{aligned} e^{\beta t}|\bar{Y}_t^{n,m}|^2 = & \, 2\int_t^T e^{\beta s} \bar{Y}_s^{n,m} \Delta f^{(n,m)}(s) ds - 2\int_t^T e^{\beta s} \bar{Z}_{1,s}^{n,m} D_s \bar{Y}_s^{n,m} ds - 2\int_t^T e^{\beta s} \bar{Z}_{2,s}^{n,m} \mathbb{D}_s^H \bar{Y}_s^{n,m} ds \\ & - 2\int_t^T e^{\beta s} \bar{Y}_s^{n,m} \bar{Z}_{1,s}^{n,m} dB_s - 2\int_t^T e^{\beta s} \bar{Y}_s^{n,m} \bar{Z}_{2,s}^{n,m} dB_s^H - \beta \int_t^T e^{\beta s} |\bar{Y}_s^{n,m}|^2 ds. \end{aligned}$$

By Proposition 3.4, we have that

$$\begin{aligned} \mathbf{E}\left[e^{\beta t}|\bar{Y}_t^{n,m}|^2\right] + \beta \mathbf{E}\int_t^T e^{\beta s} |\bar{Y}_s^{n,m}|^2 ds + 2\mathbf{E}\int_t^T e^{\beta s} |\bar{Z}_{1,s}^{n,m}|^2 ds + 2\mathbf{E}\int_t^T e^{\beta s} \frac{\hat{\sigma}_2(s)}{\sigma_2(s)} |\bar{Z}_{2,s}^{n,m}|^2 ds \\ = 2\mathbf{E}\int_t^T e^{\beta s} \bar{Y}_s^{n,m} \Delta f^{(n,m)}(s) ds. \end{aligned}$$

Using standard estimates, asumption **(H2)** and Remark 3.1, we obtain that

$$\begin{aligned}\mathbf{E}\left[e^{\beta t}|\bar{Y}_t^{n,m}|^2\right] + 2\mathbf{E}\int_t^T e^{\beta s}|\bar{Z}_{1,s}^{n,m}|^2 ds + 2C_0\mathbf{E}\int_t^T e^{\beta s}|\bar{Z}_{2,s}^{n,m}|^2 ds \\ \leq C\mathbf{E}\int_t^T e^{\beta s}|\bar{Y}_s^{n,m}|^2 ds + \frac{1}{C}\mathbf{E}\int_t^T e^{\beta s}|\Delta f^{(n,m)}(s)|^2 ds \\ \leq C\mathbf{E}\int_t^T e^{\beta s}|\bar{Y}_s^{n,m}|^2 ds + \frac{1}{C}\mathbf{E}\int_t^T \rho(s, e^{\beta s}|Y_s^{n+m-1} - Y_s^{n-1}|^2) ds \\ + \frac{K}{C}\mathbf{E}\int_t^T e^{\beta s}|\bar{Z}_{1,s}^{n,m}|^2 ds + \frac{K}{C}\mathbf{E}\int_t^T e^{\beta s}|\bar{Z}_{2,s}^{n,m}|^2 ds.\end{aligned}$$

Choosing C such that $\min\left\{2 - \frac{K}{C},\, 2C_0 - \frac{K}{C}\right\} \geq 1$, we deduce that

$$\begin{aligned}\mathbf{E}\left[e^{\beta t}|\bar{Y}_t^{n,m}|^2\right] + \mathbf{E}\int_t^T e^{\beta s}|\bar{Z}_{1,s}^{n,m}|^2 ds + \mathbf{E}\int_t^T e^{\beta s}|\bar{Z}_{1,s}^{n,m}|^2 ds \\ \leq C\mathbf{E}\int_t^T e^{\beta s}|\bar{Y}_s^{n,m}|^2 ds + \frac{1}{C}\mathbf{E}\int_t^T \rho(s, e^{\beta s}|Y_s^{n+m-1} - Y_s^{n-1}|^2) ds.\end{aligned} \tag{3.5}$$

Applying Gronwalls lemma and Jensen inequality (since $\rho(t,)$ is concave), we obtain

$$\mathbf{E}\left[e^{\beta t}|Y_t^{n+m} - Y_t^n|^2\right] \leq \frac{1}{C}e^{C(T-t)}\int_t^T \rho\left(s, \mathbf{E}\left[e^{\beta s}|Y_s^{n+m-1} - Y_s^{n-1}|^2\right]\right) ds, \quad 0 \leq t \leq T.$$

□

Lemma 3.9. *Let the assumption* **(H)** *be satisfied. Then there exists a constant* $M \geq 0$ *and* $0 \leq T_1 < T$ *not depending on* ξ *and such that*

$$\forall n \geq 1, \quad \mathbf{E}\left[|Y_t^n|^2\right] \leq M, \qquad T_1 \leq t \leq T.$$

Proof. Using the same method as in the proof of Lemma 3.8, we obtain that

$$\begin{aligned}\mathbf{E}\left[e^{\beta t}|Y_t^n|^2\right] + \beta\mathbf{E}\int_t^T e^{\beta s}|Y_s^n|^2 ds + 2\mathbf{E}\int_t^T e^{\beta s}|Z_{1,s}^n|^2 ds + 2\mathbf{E}\int_t^T e^{\beta s}\frac{\hat{\sigma}_2(s)}{\sigma_2(s)}|Z_{2,s}^n|^2 ds \\ = \mathbf{E}\left[e^{\beta T}|\xi|^2\right] + 2\mathbf{E}\int_t^T e^{\beta s}Y_s^n f(s, \eta_s, Y_s^{n-1}, Z_{1,s}^n, Z_{2,s}^n) ds.\end{aligned}$$

The same computations as before imply

$$\begin{aligned}\mathbf{E}\left[e^{\beta t}|Y_t^n|^2\right] \leq \mathbf{E}\left[e^{\beta T}|\xi|^2\right] + C\mathbf{E}\int_t^T e^{\beta s}|Y_s^n|^2 ds + \frac{1}{C}\mathbf{E}\int_t^T e^{\beta s}|f(s, \eta_s, 0, 0, 0)|^2 ds \\ + \frac{1}{C}\mathbf{E}\int_t^T \rho(s, e^{\beta s}|Y_s^{n-1}|^2) ds\end{aligned}$$

Applying once again Gronwall's lemma and Jensen inequality, we deduce that

$$\begin{aligned}\mathbf{E}\left[e^{\beta t}|Y_t^n|^2\right] \leq \frac{1}{C}e^{2C(T-t)}\left(C\mathbf{E}\left[e^{\beta T}|\xi|^2\right] + \mathbf{E}\int_t^T e^{\beta s}|f(s, \eta_s, 0, 0, 0)|^2 ds\right) \\ + \frac{1}{C}e^{2C(T-t)}\int_t^T \rho(s, \mathbf{E}[e^{\beta s}|Y_s^{n-1}|^2]) ds.\end{aligned}$$

Let $\bar{T}_1 = \max\left\{T - \frac{1}{2C}\ln(C), 0\right\}$, then we have

$$\mathbf{E}\left[e^{\beta t}|Y_t^n|^2\right] \leq \mu_t + \int_t^T \rho(s, \mathbf{E}[e^{\beta s}|Y_s^{n-1}|^2])ds, \qquad \bar{T}_1 \leq t \leq T, \tag{3.6}$$

where $\mu_t = \Big(C\mathbf{E}\left[e^{\beta T}|\xi|^2\right] + \mathbf{E}\int_t^T e^{\beta s}|f(s, \eta_s, 0, 0, 0)|^2 ds\Big)$.

Let
$$M = 2\mu_0 + 2\int_0^T a(s)ds \geq 0. \tag{3.7}$$

Arguing as in [[10], Lemma 2], we choose $\bar{T}_2$ such that

$$\mu_0 + \int_t^T \rho(s, M)ds \leq M, \qquad \bar{T}_2 \leq t \leq T. \tag{3.8}$$

Let $T_1 = \max\left\{\bar{T}_1, \bar{T}_2\right\}$, then by inequality (3.6) and (3.8), we have for $T_1 \leq t \leq T$,

$$\begin{aligned}
\mathbf{E}\left[e^{\beta t}\left|Y_t^1\right|^2\right] &\leq \mu_t + \int_t^T \rho\left(s, 0\right) ds \leq \mu_0 \leq M, \\
\mathbf{E}\left[e^{\beta t}\left|Y_t^2\right|^2\right] &\leq \mu_t + \int_t^T \rho\left(s, \mathbf{E}[e^{\beta s}|Y_s^1|^2]\right) ds \leq \mu_0 + \int_t^T \rho\left(s, M\right) ds \leq M, \\
\mathbf{E}\left[e^{\beta t}\left|Y_t^3\right|^2\right] &\leq \mu_t + \int_t^T \rho\left(s, \mathbf{E}[e^{\beta s}|Y_s^2|^2]\right) ds \leq \mu_0 + \int_t^T \rho\left(s, M\right) ds \leq M.
\end{aligned}$$

Hence by induction, one can prove that for all $n \geq 1$,

$$\mathbf{E}\left[|Y_t^n|^2\right] \leq M, \quad T_1 \leq t \leq T.$$

□

We are now in a position to give the proof of Theorem 3.7.

Proof. **of Theorem 3.7**
(i) Existence. Let use consider the sequence $(\varphi_n)_{n\geq 1}$ given by

$$\varphi_0(t) = \int_t^T \rho\left(s, M\right) ds, \qquad \varphi_{n+1}(t) = \int_t^T \rho\left(s, \varphi_n(s)\right) ds.$$

Then for all $t \in [T_1, T]$, from the proof of Lemma 3.9, one can deduce that

$$\begin{aligned}
\varphi_0(t) &= \int_t^T \rho\left(s, M\right) ds \leq M, \\
\varphi_1(t) &= \int_t^T \rho\left(s, \varphi_0(s)\right) ds \leq \int_t^T \rho\left(s, M\right) ds = \varphi_0(t) \leq M, \\
\varphi_2(t) &= \int_t^T \rho\left(s, \varphi_1(s)\right) ds \leq \int_t^T \rho\left(s, \varphi_0(s)\right) ds = \varphi_1(t) \leq M.
\end{aligned}$$

By induction, one can prove that for all $n \geq 1$, $\varphi_n(t)$ satisfies

$$0 \leq \varphi_{n+1}(t) \leq \varphi_n(t) \leq \cdots \leq \varphi_1(t) \leq \varphi_0(t) \leq M.$$

Then $\{\varphi_n(t), t \in [T_1, T]\}_{n\geq 1}$ is uniformly bounded. On the other hand, for all $n \geq 1$ and $t_1, t_2 \in [T_1, T]$, we obtain

$$|\varphi_n(t_1) - \varphi_n(t_2)| = \left|\int_{t_1}^{t_2} \rho\left(s, \varphi_{n-1}(s)\right) ds\right| \leq \left|\int_{t_1}^{t_2} \rho\left(s, M\right) ds\right|.$$

Since, for fixed v, $\int_0^T \rho(s,v)\,ds < +\infty$. So

$$\sup_n |\varphi_n(t_1) - \varphi_n(t_2)| \to 0 \quad as \quad |t_1 - t_2| \to 0,$$

which means that $\{\varphi_n(t), t \in [T_1, T]\}_{n\geq 1}$ is an equicontinuous family of function. Therefore, by the Ascoli-Arzela theorem, we can define by $\varphi(t)$ the limit function of $(\varphi_n(t))_{n\geq 1}$.

By (3.3), one knows that $\varphi(t) = 0,\ t \in [T_1, T]$.

Now for all $t \in [T_1, T],\ n, m \geq 1$, in view of Lemmas 3.8 and 3.9, we have

$$\mathbf{E}\left[e^{\beta t}|Y_t^n|^2\right] \leq M,$$

$$\mathbf{E}\left[e^{\beta t}\left|Y_t^{1+m} - Y_t^1\right|^2\right] \leq \int_t^T \rho\left(s, \mathbf{E}[e^{\beta s}|Y_s^m|^2]\right) ds \leq \int_t^T \rho(s, M)\,ds = \varphi_0(t) \leq M,$$

$$\mathbf{E}\left[e^{\beta t}\left|Y_t^{2+m} - Y_t^2\right|^2\right] \leq \int_t^T \rho\left(s, \mathbf{E}\left[e^{\beta s}\left|Y_s^{1+m} - Y_s^1\right|^2\right]\right) ds \leq \varphi_1(t) \leq M,$$

$$\mathbf{E}\left[e^{\beta t}\left|Y_t^{3+m} - Y_t^3\right|^2\right] \leq \int_t^T \rho\left(s, \mathbf{E}\left[e^{\beta s}\left|Y_s^{2+m} - Y_s^2\right|^2\right]\right) ds \leq \varphi_2(t) \leq M.$$

By induction, we can derive that

$$m \geq 1, \quad \mathbf{E}\left[e^{\beta t}\left|Y_t^{n+m} - Y_t^n\right|^2\right] \leq \varphi_{n-1}(t), \quad T_1 \leq t \leq T.$$

Therefore we have

$$\sup_{T_1\leq t\leq T} \mathbf{E}\left[e^{\beta t}\left|Y_t^{n+m} - Y_t^n\right|^2\right] \leq \sup_{T_1\leq t\leq T} \varphi_{n-1}(t) = \varphi_{n-1}(T_1) \to 0 \quad n \to \infty.$$

We see immediately that $\{Y_t^n\}_{n\geq 1}$ is a Cauchy sequence in $V^\beta_{[T_1,T]}$. We also know from (3.5) that $\{Z_{1,t}^n\}_{n\geq 1}$ and $\{Z_{2,t}^n\}_{n\geq 1}$ are a Cauchy sequence in $V^\beta_{[T_1,T]}$. Then there exists $(Y, Z_1, Z_2) \in V^\beta_{[T_1,T]} \times V^\beta_{[T_1,T]} \times V^\beta_{[T_1,T]}$ being a limit of (Y^n, Z_1^n, Z_2^n). Letting $n \to +\infty$ in eq.(3.4), we obtain

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s})ds - \int_t^T Z_{1,s}dB_s - \int_t^T Z_{2,s}dB_s^H, \quad T_1 \leq t \leq T.$$

In other words, we have shown the existence of the solution to SFBSDE (3.1) on $[T_1, T]$. Finally, by iteration, one can deduce the existence on $[T - \lambda(T - T_1), T]$, for each λ, and therefore the existence on the whole $[0, T]$.

(ii) **Uniqueness**. Let $\left(Y_t^i, Z_{1,t}^i, Z_{2,t}^i\right)_{0\leq t\leq T}$, $i = 1, 2$, be two solutions of SFBSDE (3.1).

Using the same method as in the proof of Lemma (3.8), we have

$$\begin{aligned}\mathbf{E}\left[e^{\beta t}|Y_t^1 - Y_t^2|^2\right] + \mathbf{E}\int_t^T e^{\beta s}|Z_{1,s}^1 - Z_{1,s}^2|^2 ds + \mathbf{E}\int_t^T e^{\beta s}|Z_{2,s}^1 - Z_{2,s}^2|^2 ds \\ \leq C\mathbf{E}\int_t^T e^{\beta s}|Y_s^1 - Y_s^2|^2 ds + \frac{1}{C}\int_t^T \rho(s, \mathbf{E}\left[e^{\beta s}|Y_s^1 - Y_s^2|^2\right])ds,\end{aligned} \tag{3.9}$$

By virtue of the Gronwall inequality, we can derive that

$$\mathbf{E}\left[e^{\beta t}|Y_t^1 - Y_t^2|^2\right] \leq \frac{1}{C}e^{C(T-t)}\int_t^T \rho(s, \mathbf{E}\left[e^{\beta s}|Y_s^1 - Y_s^2|^2\right])ds, \quad 0 \leq t \leq T.$$

Define $\delta = \frac{1}{C}\ln(C)$ and $N = [T/\delta] + 1$. If $(t_j)_{0\leq j\leq N}$ denotes the uniform subdivision of $[0,T]$ given by $T_0 = 0, \quad T_j = T-(N-j)\delta,\ j \geq 1$, we have

$$\mathbf{E}\left[e^{\beta t}|Y_t^1 - Y_t^2|^2\right] \leq \int_t^T \rho(s, \mathbf{E}\left[e^{\beta s}|Y_s^1 - Y_s^2|^2\right])ds, \quad T_{N-1} \leq t \leq T.$$

From the comparison theorem of ODE, we know that $\mathbf{E}\left[e^{\beta t}|Y_t^1 - Y_t^2|^2\right] \leq r(t)$, where $r(t)$ is the maximum of solution of (3.3) on $[T_{N-1}, T]$. As a consequence, we have $Y_t^1 = Y_t^2$ for $t \in [T_{N-1}, T]$. From (3.9), we deduce $\left(Z_{1,t}^1, Z_{2,t}^1\right) = \left(Z_{1,t}^2, Z_{2,t}^2\right)$ for $t \in [T_{N-1}, T]$. Then we can use the same argument to prove that uniqueness of the solution also holds on $[T_j, T_{j+1}]$, $j = 0, \ldots, N-2$. This completes the proof. $\square$

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