

## ON THE RELATION BETWEEN NON-HOMOGENEOUS FRACTIONAL BURGERS EQUATIONS AND TIME-DEPENDENT HARMONIC OSCILLATOR

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**ABSTRACT.** In this paper we discuss the relation between non-homogeneous nonlinear fractional diffusive equations and the Schrödinger equation with time-dependent harmonic potential. It is well known that the Cole-Hopf transform allows to linearize non-homogeneous nonlinear diffusive equations (NHNDEs) into a Schrödinger-type equation with time-dependent potential. We first discuss the utility of the results about time-dependent harmonic oscillator to obtain explicit solutions for non-homogeneous nonlinear partial differential equations. In particular, we recall that, starting from a trial polynomial solution of the NHNDE, it is possible to construct other solutions by using linear invariants of the Schrödinger equation with time-dependent potential. Finally, we apply these results to find explicit solutions to a novel non-homogeneous fractional Burgers-type equation.

### 1. INTRODUCTION

One of the most known cases of linearizable nonlinear partial differential equations is the classical Burgers equation, firstly introduced by J.M Burgers in 1948 in the framework of the theory of turbulence [1]. Several studies have been devoted to the analysis of different generalizations of Burgers-type equations. In particular, two kind of generalizations have been widely studied in literature: the non-homogeneous Burgers equation and the generalized Burgers equation with time-dependent viscosity and drift. In the framework of non-homogeneous Burgers equations, Rao and Yadav [2] considered the following equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \frac{kx}{(2\beta t + 1)^2}, \quad x \in \mathbb{R}, t > 0, \quad (1)$$

subject to some unbounded initial conditions. The solutions of (1) are expressed in terms of the self-similar solutions of a linear partial differential equation with variable coefficients. In this way the large time behavior of the solutions is obtained.

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A similar equation was considered by Eule and Friedrich [3] with a more general non-homogeneous term, again linear in space. More recently, exact solutions for forced Burgers equation with time-variable coefficients have been discussed by Buyukasik and Pashaev in [4]. In [5], Darboux transformations have been applied to the time-dependent non-homogeneous Burgers equation, in relation with the Schrödinger equation. Finally an interesting relation between the Burgers equation with elastic forcing and the Ornstein-Uhlenbeck process was pointed out in [6].

In the framework of generalized Burgers equation with time-dependent viscosity, Rao and Satyanarayana in [7] discussed the large-time asymptotics for solutions of Burgers equation with variable viscosity related to nonlinear acoustics. Vaganan and Jeyalakshmi [8] considered the following more general Burgers equations with linear damping and variable viscosity

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \frac{\Delta(t)}{2} \frac{\partial^2 u}{\partial x^2} + \alpha u = 0, \quad \alpha > 0, \quad (2)$$

discussing its relation with several linear parabolic equations. In [8] the authors study the conditions on the viscosity time-dependence  $\Delta(t)$  for which (2) is linearizable. In the recent paper [9], the authors consider a forced Burgers equation with time-variable coefficients and solve the initial-boundary value problem on the half-line with inhomogeneous Dirichlet boundary conditions. We finally refer to the review paper [10] for an updated bibliography on generalizations of Burgers equations.

Here we consider the following non-homogeneous nonlinear diffusive equation (NHNDE)

$$\frac{\partial u}{\partial t} + \left( \frac{\partial u}{\partial x} \right)^2 - \frac{\partial^2 u}{\partial x^2} = V(x, t). \quad (3)$$

This NHNDE is widely used in different fields of applied sciences. For example, similar nonlinear diffusive equations arise in the analysis of the propagation of nonlinear waves in porous media (see e.g. [11] and the references therein). It is simple to prove that such equation is directly related to the non-homogeneous Burgers equation. Indeed it suffices to differentiate both sides in (3) in order to obtain

$$\frac{\partial}{\partial t} \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^3 u}{\partial x^3} = \frac{\partial V(x, t)}{\partial x}, \quad (4)$$

that coincides with the non-homogeneous Burgers equation for  $f(x, t) = \partial u / \partial x$ . For a suitable non-homogeneous term  $V(x, t)$  (see (12) behind), we discuss the relation between NHNDE and time-dependent harmonic oscillator. In particular, we underline the utility to apply known results about Hamiltonian invariants of the time-dependent harmonic oscillator in order to construct exact solutions of NHNDEs, starting from polynomial-type trial solutions. This approach can be applied to handle more general NHNDE with variable viscosity of the form

$$\frac{\partial u}{\partial t} + \left( \frac{\partial u}{\partial x} \right)^2 - \frac{1}{2m(t)} \frac{\partial^2 u}{\partial x^2} = V(x, t), \quad (5)$$

under suitable conditions on the given functions  $V(x, t)$  and  $m(t)$ .

The aim of this paper is to apply these results to solve a new class of non-homogeneous nonlinear nonlocal diffusive equations, that is

$$\frac{\partial u}{\partial t} + \frac{\partial^\alpha}{\partial x^\alpha} \left[ \left( \frac{\partial^{1-\alpha} u}{\partial x^{1-\alpha}} \right)^2 - \frac{\partial^2}{\partial x^2} \int_0^x \frac{(x-x')^{\alpha-1}}{\Gamma(\alpha)} u(x') dx' \right] = -\frac{k(t)x^{2-\alpha}}{\Gamma(3-\alpha)}, \quad x \geq 0 \quad (6)$$

where

$$\int_0^x \frac{(x-x')^{\alpha-1}}{\Gamma(\alpha)} u(x', t) dx' = J_x^\alpha u(x, t), \quad (7)$$

is the Riemann-Liouville fractional integral of order  $\alpha \in (0, 1]$  and

$$\frac{\partial^\alpha}{\partial x^\alpha} u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-x')^{-\alpha} \frac{\partial}{\partial x'} u(x', t) dx', \quad (8)$$

is the Caputo fractional derivative with respect to  $x$ -variable (see e.g. [18, 19, 21]). This is a slight modification of the Burgers equation considered by Miskinis in [16, 17]. As far as we know, this is the unique case discussed in literature, where a nonlinear nonlocal differential equation can be reduced to the linear heat equation by means of a generalized Cole-Hopf transform. Even if this formulation of the fractional Burgers equation seems to be in some way artificial, it has the great advantage and interest to couple nonlinear and nonlocal effects in the dynamics.

The paper is organized as follows. In section 2 we discuss the relation between non-homogeneous nonlinear diffusion equations and time-dependent harmonic oscillator, showing the utility of invariant analysis to find explicit solutions. In section 3 we apply the discussed results to solve a non-homogeneous fractional Burgers-type equation.

## 2. RELATIONS BETWEEN NON-HOMOGENEOUS NONLINEAR DIFFUSION EQUATIONS AND TIME-DEPENDENT HARMONIC OSCILLATOR

Let us consider the NHNDE

$$\frac{\partial u}{\partial t} + \left( \frac{\partial u}{\partial x} \right)^2 - \frac{\partial^2 u}{\partial x^2} = V(x, t). \quad (9)$$

In order to linearize the previous equation, we use the Cole-Hopf transform

$$u(x, t) = -\ln \psi(x, t). \quad (10)$$

By substituting (10) in (9), we obtain

$$\frac{\partial \psi}{\partial t} - \frac{\partial^2 \psi}{\partial x^2} = -V(x, t)\psi. \quad (11)$$

From now on, we assume that

$$V(x, t) = \frac{1}{4}\omega^2 k(t)\hbar^2 x^2, \quad (12)$$

and with the transformation

$$\begin{aligned} \tau &= i\hbar t, \\ \xi &= \frac{i\hbar}{\sqrt{2m}}x, \end{aligned} \quad (13)$$

the Schrödinger equation with time-dependent harmonic potential is obtained

$$i\hbar \frac{\partial \psi}{\partial \tau} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial \xi^2} + \frac{1}{2}m\omega^2 \xi^2 k(\tau)\psi. \quad (14)$$

We will now follow the argument of Salusti and Zirilli in [12], where the authors have proved that a trial solution for (14) of the form

$$\psi(\xi, \tau) = \exp \left[ \frac{i}{2\hbar} (a(\tau)\xi^2 + 2b(\tau)\xi + c(\tau)) \right] \tag{15}$$

can be obtained. This leads to the following system

$$\begin{cases} \dot{a} = -\frac{1}{m}a^2 - m\omega^2k, \\ \dot{b} = -\frac{1}{m}ab \\ \dot{c} = \frac{i\hbar}{m}a - \frac{1}{m}b^2. \end{cases} \tag{16}$$

Going back to the original problem, i.e. by using the transformation (10), we obtain a solution of the non-homogeneous nonlinear diffusion equation (9) in the form

$$u(\xi, \tau) = \frac{i}{2\hbar} (a(\tau)\xi^2 + 2b(\tau)\xi + c(\tau)), \tag{17}$$

and finally, going back to the original variables  $(x, t)$ , we have a polynomial solution of equation (9). The existence of this kind of solutions with separate variables is not surprising. Indeed, a similar relation can be obtained by using the Invariant Subspace Method, introduced by Galaktionov [13], which allows to solve exactly nonlinear equations by separation of variables.

We recall the main idea of this method: consider a scalar evolution equation

$$\frac{\partial u}{\partial t} = F \left[ u, \frac{\partial u}{\partial x}, \dots \right], \tag{18}$$

where  $u = u(x, t)$  and  $F[\cdot]$  is a nonlinear differential operator. Given  $n$  linearly independent functions

$$f_1(x), f_2(x), \dots, f_n(x),$$

we call  $W_n$  the  $n$ -dimensional linear space

$$W_n = \langle f_1(x), \dots, f_n(x) \rangle.$$

This space is called invariant under the given operator  $F[\cdot]$ , if  $F[y] \in W_n$  for any  $y \in W_n$ . This means that there exist  $n$  functions  $\Phi_1, \Phi_2, \dots, \Phi_n$  such that

$$F[C_1f_1(x) + \dots + C_nf_n(x)] = \Phi_1(C_1, \dots, C_n)f_1(x) + \dots + \Phi_n(C_1, \dots, C_n)f_n(x),$$

where  $C_1, C_2, \dots, C_n$  are arbitrary constants.

Once the set of functions  $f_i(x)$  that form the invariant subspace has been determined, we can find an explicit solution of (18) in the invariant subspace in the form

$$u(x, t) = \sum_{i=1}^n u_i(t) f_i(x). \tag{19}$$

where  $f_i(x) \in W_n$ . In this way, we arrive to a system of ODEs. In many cases, this problem is simpler than the original one and we can find an explicit solution for the nonlinear problem by separation of variables (see [13]).

We can now show by direct calculations that (9) admits  $W_3 = \langle 1, x, x^2 \rangle$  as invariant subspace. Indeed, in this case we have that

$$F \left[ u, \frac{\partial u}{\partial x}, x, t \right] = - \left( \frac{\partial u}{\partial x} \right)^2 + \frac{\partial^2 u}{\partial x^2} - \frac{1}{4} \omega^2 k(t) \hbar^2 x^2, \tag{20}$$

so that

$$F[c_1x^2 + c_2x + c_3] = -4c_1^2x^2 - c_2^2 - 4c_1c_2x + 2c_1 - \frac{1}{4}\omega^2k(t)\hbar^2x^2, \quad (21)$$

as claimed.

**2.1. Linear Invariants.** In the algebraic approach, we want to find an operator  $\hat{A}(t)$  such that, if  $\psi(x, t)$  is a solution of equation (11), also  $\hat{A}\psi(x, t)$  is a solution. Recalling that (14), is related to the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \xi^2} + \frac{1}{2}m\omega^2k(t)\xi^2, \quad (22)$$

we want to determine an operator  $\hat{A}$  such that

$$i\hbar \frac{\partial}{\partial t} \hat{A}\psi(x, t) = \hat{H} \hat{A}\psi(x, t). \quad (23)$$

The definition of invariants, according to (23), was firstly introduced by Lewis and Riesenfeld in [14] and gave rise to a number of papers devoted to the analysis and application of the Schrödinger equation with time-dependent harmonic potential. This requirement leads to the following equation (see e.g. [12])

$$i\hbar \dot{\hat{A}} = [\hat{H}, \hat{A}]. \quad (24)$$

Salusti and Zirilli have shown that the Ansatz

$$\hat{A} = \alpha(t)x + \beta(t) \frac{\partial}{\partial x} + \gamma(t) \quad (25)$$

satisfies equation (24). This argument leads to the system

$$\begin{cases} i\hbar \dot{\alpha} = -m\omega^2k\beta \\ i\hbar \dot{\beta} = -\frac{\hbar^2}{m}\alpha \\ \dot{\gamma} = 0. \end{cases} \quad (26)$$

We can decouple system (26) by deriving the second equation with respect to  $t$  and, by substitution, we obtain

$$\ddot{\beta} = -k(t)\omega^2\beta, \quad (27)$$

that is a time-dependent harmonic oscillator.

We conclude that, once fixed the function  $k(t)$  in (14), we can solve equation (27) and thus, by (26), we obtain the explicit form of the operator  $\hat{A}$ . Finally, we can find other solutions to the non-homogeneous nonlinear equation (9). We remark that this procedure can be applied in an iterative way to construct a family of exact solutions starting from the simple polynomial one.

For instance by taking  $k(t) = t$ , we can find an explicit solution in terms of a cumbersome combination of Airy functions, as can be seen by using simple mathematical routines.

### 3. THE NON-HOMOGENEOUS NONLINEAR FRACTIONAL DIFFUSIVE EQUATION

As mentioned in the introduction, we now apply the results discussed in the previous sections in order to solve the following non-homogeneous nonlinear fractional Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial^\alpha}{\partial x^\alpha} \left[ \left( \frac{\partial^{1-\alpha} u}{\partial x^{1-\alpha}} \right)^2 - \frac{\partial^2}{\partial x^2} J_x^\alpha u \right] = -\frac{k(t)x^{2-\alpha}}{\Gamma(3-\alpha)}, \quad (28)$$

where the fractional derivatives are in the sense of Caputo (see [18]) and  $\alpha \in (0, 1]$ . Observe that for  $\alpha = 1$  this equation coincides with a particular case of the non-homogeneous Burgers equation, i.e.

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} - \frac{\partial^2 u}{\partial x^2} = -\frac{k(t)x}{2}, \tag{29}$$

Here we study a class of explicit solutions of this equation, considering the problem in the semi-line  $x \geq 0$ , with an appropriate boundary condition in  $x = 0$  that will be discussed in what follows. Observe that in the right hand side term we can write (see e.g. [20, 21, 22] for the main useful formulas and properties of fractional derivatives)

$$\frac{k(t)}{2} \frac{\partial^\alpha x^2}{\partial x^\alpha} = \frac{k(t)x^{2-\alpha}}{\Gamma(3-\alpha)} \tag{30}$$

so that equation (28) becomes

$$\frac{\partial u}{\partial t} + \frac{\partial^\alpha}{\partial x^\alpha} \left[ \left( \frac{\partial^{1-\alpha} u}{\partial x^{1-\alpha}} \right)^2 - \frac{\partial^2}{\partial x^2} J_x^\alpha u + \frac{1}{2} k(t)x^2 \right] = 0. \tag{31}$$

We consider the following Cole-Hopf transform

$$u(x, t) = -\frac{\partial^\alpha}{\partial x^\alpha} \ln \psi(x, t). \tag{32}$$

We now observe that, by using Caputo fractional derivatives, the following equality holds (see [20], pag.6 and pag.10)

$$\frac{\partial^{1-\alpha}}{\partial x^{1-\alpha}} \frac{\partial^\alpha u}{\partial x^\alpha} = \frac{\partial u}{\partial x}. \tag{33}$$

Moreover, it can be proved that (see [20], pag. 7)

$$J_x^\alpha \frac{\partial^\alpha}{\partial x^\alpha} u(x, t) = u(x, t) - u(0, t), \quad \alpha \in (0, 1], x > 0 \tag{34}$$

Considering these preliminaries, we now assume that  $\ln \psi(x, t) \Big|_{x=0} = \partial_x \ln \psi(x, t) \Big|_{x=0} = 0$ . This is equivalent to take  $u(0, t) = 0$  as boundary condition. Under these assumption, we have

$$\frac{\partial^\alpha}{\partial x^\alpha} \left[ -\frac{\partial_t \psi}{\psi} + \left( \frac{\partial_x \psi}{\psi} \right)^2 - \left( \frac{\partial_x \psi}{\psi} \right)^2 + \frac{\partial_{xx} \psi}{\psi} \right] = -\frac{1}{2} k(t) \frac{\partial^\alpha x^2}{\partial x^\alpha}, \tag{35}$$

thus,

$$\frac{\partial^\alpha}{\partial x^\alpha} \left[ -\frac{\partial_t \psi}{\psi} + \frac{\partial_{xx} \psi}{\psi} + \frac{1}{2} k(t)x^2 \right] = 0. \tag{36}$$

This means that

$$\frac{\partial_t \psi}{\psi} - \frac{\partial_{xx} \psi}{\psi} - \frac{1}{2} k(t)x^2 = f(t). \tag{37}$$

Now, suppose  $f(t) = 0$ , we arrive to the following equation

$$\frac{\partial_t \psi}{\psi} - \frac{\partial_{xx} \psi}{\psi} = \frac{1}{2} k(t)x^2 \tag{38}$$

that is the analogous of equation (11). This means that we can apply exactly the same reasoning discussed in the previous sections in order to construct exact

solutions for the equation (28) starting from a polynomial trial solution. Indeed, we can start again from the Gaussian ansatz

$$\psi(x, t) = \exp[a(t)x^2 + b(t)x + c(t)], \quad (39)$$

as a trial solution for (38). Then, by using (32) we obtain the following solution of equation (31)

$$\begin{aligned} u(x, t) &= -\frac{\partial^\alpha}{\partial x^\alpha}(a(t)x^2 + b(t)x + c(t)) \\ &= -\left(\frac{b(t)x^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{2a(t)x^{2-\alpha}}{\Gamma(3-\alpha)}\right), \end{aligned} \quad (40)$$

where  $a(t)$  and  $b(t)$  satisfy the following system of differential equations

$$\begin{aligned} \dot{a} &= 4a^2 + \frac{k(t)}{2}, \\ \dot{b} &= 4ab, \end{aligned} \quad (41)$$

whose solutions clearly depend by the particular choice of  $k(t)$ . Then, by using again the relations with the time-dependent Schrödinger equation and the analysis of invariants, it is possible to find other explicit solutions starting from (40).

We finally observe that the solution (40) corresponds to the fact that the equation (28) admits as invariant subspace  $W_2 = \langle x^{1-\alpha}, x^{2-\alpha} \rangle$ , for all  $\alpha \in (0, 1]$ .

**3.1. The case  $k(t) = t$ .** We solve the system of differential equations (41) considering the particular choice  $k(t) = t$ . The first equation of the system become:

$$\dot{a} = 4a^2 + \frac{t}{2} \quad (42)$$

that is a Special Riccati equation with solution

$$a(t) = \frac{1}{4} \frac{\dot{\phi}(t)}{\phi(t)} \quad (43)$$

where

$$\phi(t) = \sqrt{t} [C_1 J_{1/3}(\frac{2\sqrt{2}}{3}t) + C_2 Y_{1/3}(\frac{2\sqrt{2}}{3}t)], \quad (44)$$

where  $J_\nu(t)$  and  $Y_\nu(t)$  are the Bessel functions of the first and second kind, respectively. From the second equation of the system and (43), we obtain

$$\frac{\dot{b}}{b} = \frac{\dot{\phi}}{\phi}, \quad (45)$$

namely:

$$b(t) = \phi(t) + C_3. \quad (46)$$

## 4. CONCLUSIONS

In this paper we have discussed a conceptual link between the construction of exact solutions to the Schrödinger equation with time-dependent harmonic potential by means of the invariants of the related Hamiltonian and the problem to construct exact solutions to nonlinear non homogeneous diffusive equations. Here we consider for simplicity the construction of solutions by using linear invariants, but following our ideas there is a bridge between the numerous results about Time-Dependent Harmonic Oscillator and the solution of nonlinear diffusive equation with a fixed non-homogeneous term.

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