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ON CERTAIN SUBORDINATION PROPERTIES OF A LINEAR OPERATOR

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Abstract. By making use of certain linear operator involving the generalized multiplier transformation, the authors introduce a new subclass of p–valent meromorphic functions with positive coefficients and investigate various subordination relationships. Relevant connections of the main results with various known results are also considered.

1. Introduction and Preliminaries

Let $\Sigma_{p,m}$ be the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \quad \left(p, m \in \mathbb{N} := \{1, 2, 3, \dots\}\right), \tag{1}$$

which are analytic and p–valent in the punctured unit disk $\mathcal{U}^* := \mathcal{U} \setminus \{0\}$, where $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. For the functions $f \in \Sigma_{p,m}$ of the form (1) and $g \in \Sigma_{p,m}$ given by $g(z) = z^{-p} + \sum\limits_{k=m}^{\infty} b_k z^k$, the *Hadamard (or convolution) product of* f *and* g is defined by

$$(f * g)(z) := z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k,\ z \in \mathcal{U}^*.$$

For $\lambda,\ l > 0$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and a function f of the form (1), H. E. Darwish et al. [4] defined the linear operator $\mathfrak{J}_p^n(\lambda, l)$ by

$$\mathfrak{J}_p^n(\lambda, l) f = \Phi^n(\lambda, l) * f,$$

where

$$\Phi^n(\lambda, l)(z) := z^{-p} + \sum_{k=m}^{\infty} \left[1 + \frac{\lambda(p+k)}{l}\right]^n z^k,\ z \in \mathcal{U}^*.$$

Thus, we have

$$\mathfrak{J}_p^n(\lambda, l) f(z) = z^{-p} + \sum_{k=m}^{\infty} \left[1 + \frac{\lambda(p+k)}{l}\right]^n a_k z^k,\ z \in \mathcal{U}^*, \tag{2}$$

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and it is easily verified from (2) that

$$\lambda z\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]'=l\mathfrak{J}_p^{n+1}(\lambda,l)f(z)-(\lambda p+l)\mathfrak{J}_p^n(\lambda,l)f(z),\ z\in\mathcal{U}^*\quad(\lambda>0).\quad(3)$$

We also note that

$$\mathfrak{J}_p^0(\lambda,l)f=f\quad\text{and}\quad\mathfrak{J}_p^1(1,1)f(z)=zf'(z)+(p+1)f(z).$$

Remark 1.1. By specializing the parameters λ, l and p, the multiplier transformation $\mathfrak{J}_p^n(\lambda,l)$ reduced to the following familiar operators:

- (i) For the choice of $\lambda=l=1$, the operator defined in (2) reduces to the operator D^n studied by Aouf et al.[2], Liu et al. [7] and Srivastava and Patel [12];
- (ii) Taking $p=1$, the multiplier transformation $\mathfrak{J}_p^n(\lambda,l)$ yields the operator $I(n,l)$ which was investigated by Cho et al. [3];
- (iii) For the choice of $p=l=1$, the operator $\mathfrak{J}_p^n(\lambda,l)$ reduces to the operator $D_{\lambda,p}^n$ studied by Al-Oboudi et al. [1];
- (iv) A special case of the operator $\mathfrak{J}_p^n(\lambda,l)$ for $p=\lambda=l=1$ gives the operator I^n investigated by Uralegaddi and Somanatha [13].

If f and g are two analytic functions in $\mathcal{U}$, we say that f *is said to be subordinate to* g, written symbolically as $f(z)\prec g(z)$, if there exists a *Schwarz function* w, which (by definition) is analytic in $\mathcal{U}$, with $w(0)=0$, and $|w(z)|<1$ for all $z\in\mathcal{U}$, such that $f(z)=g(w(z))$, $z\in\mathcal{U}$.

If the function g is univalent in $\mathcal{U}$, then we have the following equivalence (c.f [9, 10]):

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0)\quad\text{and}\quad f(\mathcal{U})\subset g(\mathcal{U}).$$

In proving our main results, we need each of the following definitions and lemmas.

Definition 1.1. [14] *A sequence* $\{b_n\}_{n\in\mathbb{N}}$ *of complex numbers is said to be a* subordination factor sequence *if for each function* $f(z)=\sum\limits_{k=0}^{\infty}a_kz^k$, $z\in\mathcal{U}$, *from the class of convex (univalent) functions in* $\mathcal{U}$, *denoted by* S^c, *we have*

$$\sum_{n=1}^{\infty}b_na_nz^n\prec f(z)\quad(\textit{where}\quad a_1=1).$$

Lemma 1.1. [14] *A sequence* $\{b_n\}_{n\in\mathbb{N}}$ *is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left(1+2\sum_{n=1}^{\infty}b_nz^n\right)>0,\ z\in\mathcal{U}.\quad(4)$$

Lemma 1.2. [9, 10] *Let the function* h *be analytic and convex (univalent) in* $\mathcal{U}$ *with* $h(0)=1$. *Suppose also that the function* ϕ *given by*

$$\phi(z)=1+c_{p+m}z^{p+m}+c_{p+m+1}z^{p+m+1}+\ldots,\ z\in\mathcal{U},\quad(5)$$

is analytic in $\mathcal{U}$. *If*

$$\phi(z)+\frac{z\phi'(z)}{\gamma}\prec h(z)\quad(\operatorname{Re}\gamma\geq0,\ \gamma\in\mathbb{C}^*),\quad(6)$$

then

$$\phi(z)\prec\psi(z)=\frac{\gamma}{p+m}z^{-\frac{\gamma}{p+m}}\int_0^z t^{\frac{\gamma}{p+m}-1}h(t)\,dt\prec h(z)$$

and ψ is the best dominant.

Lemma 1.3. [11] *Let the function p be analytic in $\mathcal{U}$, such that $p(0) = 1$ and $p(z) \neq 0$ for all $z \in \mathcal{U}$. If there exists a point $z_0 \in \mathcal{U}$ such that*

$$|\arg p(z)| < \frac{\pi\delta}{2}, \quad \textit{for} \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\delta}{2} \quad (\delta > 0),$$

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\delta,$$

where

$$k \geq \frac{1}{2}\left(c + \frac{1}{c}\right), \quad \textit{when} \quad \arg p(z_0) = \frac{\pi\delta}{2}$$

and

$$k \leq -\frac{1}{2}\left(c + \frac{1}{c}\right), \quad \textit{when} \quad \arg p(z_0) = -\frac{\pi\delta}{2},$$

where

$$p(z_0)^{1/\delta} = \pm\, ic, \quad \textit{and} \quad c > 0.$$

We shall also make use of the *Gaussian hypergeometric function* ${}_2F_1$ defined by

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \; z \in \mathcal{U} \quad \left(a, b, c \in \mathbb{C}, \; c \notin \mathbb{Z}_0^- := \{0, -1, -2, \dots\}\right), \tag{7}$$

where $(d)_k$ denotes the *Pochhammer symbol* given in terms of the *Gamma function* Γ, by

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} \begin{cases} 1, & \text{if} \quad k = 0, \; d \in \mathbb{C}^*, \\ d(d+1)\dots(d+k-1), & \text{if} \quad k \in \mathbb{N}, \; d \in \mathbb{C}. \end{cases} \tag{8}$$

The series defined by (7) converges absolutely in $\mathcal{U}$, hence ${}_2F_1$ represents an analytic function in $\mathcal{U}$ [15, Ch.14].

Lemma 1.4. [15] *For the complex numbers a, b and c, with $c \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$, the following identities hold:*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a, b; c; z), \; z \in \mathcal{U}, \tag{9}$$

$$\textit{for } \operatorname{Re} c > \operatorname{Re} b > 0, \tag{10}$$

$${}_2F_1(a, b; c; z) = (1-z)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \; z \in \mathcal{U}, \tag{11}$$

and

$$(b+1)\, {}_2F_1(1, b; b+1; z) = (b+1) + bz\, {}_2F_1(1, b+1; b+2; z), \; z \in \mathcal{U}. \tag{12}$$

Now we introduce a subclass of $\Sigma_{p,m}$ by making use of the generalized multiplier transformation $\mathfrak{J}_p^n(\lambda, l)$, as follows:

Definition 1.2. *(i) For the fixed parameters A and B, with $-1 \leq B < A \leq 1$, the function $f \in \Sigma_{p,m}$ is in the class $\mathcal{R}_{n,m}(\lambda, p, l; \alpha)$, if it satisfies the following subordination condition*

$$\frac{-\left[\alpha z^{p+2}\left[\mathfrak{J}_p^n(\lambda, l)f(z)\right]'' + (1-\alpha)z^{p+1}\left[\mathfrak{J}_p^n(\lambda, l)f(z)\right]'\right]}{(1-\alpha)p - p(p+1)\alpha} \prec \frac{1+Az}{1+Bz},$$

$$(0 \leq \alpha < 1/p+2,\ \lambda > 0,\ l > 0,\ n \in \mathbb{N}_0)$$

or equivalently

$$\left|\frac{\alpha z^{p+2}\left[\mathfrak{J}_p^n(\lambda, l)f(z)\right]'' + (1-\alpha)z^{p+1}\left[\mathfrak{J}_p^n(\lambda, l)f(z)\right]' + (1-\alpha)p - (p+1)p\alpha}{\left[(1-\alpha)p - (p+1)p\alpha\right]A + \left[\alpha z^{p+2}\left[\mathfrak{J}_p^n(\lambda, l)f(z)\right]'' + (1-\alpha)z^{p+1}\left[\mathfrak{J}_p^n(\lambda, l)f(z)\right]'\right]B}\right| < 1, \tag{13}$$

Remark 1.2. Some special cases of the above defined subclass were studied by different authors, as follows:

(i) $\mathcal{R}_{n,m}(\lambda, p, l; 0) =: \Sigma_{p,m}^n(\lambda, l; A, B)$ (see Aouf et al. [5]);
(ii) $\mathcal{R}_{n,0}(1, p, 1; 0) =: R_{n,p}(A, B)$ (see Liu and Srivastava [7]);
(iii) $\mathcal{R}_{n,m}(1, p, 1; 0) =: \Sigma_{p,m}^n(A, B)$ (see Srivastava and Patel [12]);

A study of such multiplier transformations was initiated and studied systematically by Jung et al[6].The generalized multiplier transformation defined by (2) has been extensively studied by many authors [1, 2, 3, 5, 7, 12, 13] with suitable restriction on the parameters λ, p, l and for f belonging to some favoured classes of analytic functions. In particular, Liu and Srivastava [7] obtained several inclusion relationships for certain class of functions defined by the generalized multiplier transformation with $\lambda = l = 1$.

Moreover, using the principle of subordination, El-Ashwah et al. [5] proved some inclusion results and subordination theorems involving the generalized multiplier transformation defined by (2). Similar results were obtained by Srivastava and Patel [12] with restrictions on l and λ.

Our work is essentially motivated by the aforementioned works of [5] and [12]. A subordination relationship involving the class $\mathcal{R}_{n,m}(\lambda, p, l; \alpha)$ and certain subordination properties involving the linear operator defined in (2) and argument estimate results are also investigated.

2. Coefficient estimates and subordination results for the class $\mathcal{R}_{n,m}(\lambda, p, l; \alpha)$

First, we will prove the following lemma which gives a sufficient condition for functions belonging to the class $\mathcal{R}_{n,m}(\lambda, p, l; \alpha)$.

Lemma 2.1. *A sufficient condition for a function f of the form (1) to be in the class $\mathcal{R}_{n,m}(\lambda, p, l; \alpha)$ is*

$$\sum_{k=m}^{\infty} \omega_k |a_k| \leq p(A-B)\left[(1-\alpha) - (p+1)\alpha\right], \tag{14}$$

where

$$\omega_k = k\left[\frac{l + \lambda(p+k)}{l}\right]^n \left[\alpha(k-1) + (1-\alpha)\right](1+|B|), \quad (k \geq m). \tag{15}$$

Proof. A function f of the form (1) belongs to the class $\mathcal{R}_{n,m}(\lambda,p,l;\alpha)$ if and only if there exists a *Schwarz function* w, such that

$$\frac{-\left[\alpha z^{p+2}\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]''+(1-\alpha)z^{p+1}\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]'\right]}{(1-\alpha)p-p(p+1)\alpha}=\frac{1+Aw(z)}{1+Bw(z)},\ z\in\mathcal{U}.$$

Since $|w(z)|\leq|z|$ for all $z\in\mathcal{U}$, the above relation is equivalent to (13). Thus, it is sufficient to prove that

$$\left|\alpha z^{p+2}\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]''+(1-\alpha)z^{p+1}\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]'+(1-\alpha)p-(p+1)p\alpha\right|$$
$$-\left|\left[(1-\alpha)p-(p+1)p\alpha\right]A+\left[\alpha z^{p+2}\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]''+(1-\alpha)z^{p+1}\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]'\right]B\right|<0,$$

Indeed, letting $|z|=r$ $(0<r<1)$ and using (14), we have

$$\left|\alpha z^{p+2}\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]''+(1-\alpha)z^{p+1}\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]'+(1-\alpha)p-(p+1)p\alpha\right|$$
$$-\left|\left[(1-\alpha)p-(p+1)p\alpha\right]A+\left[\alpha z^{p+2}\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]''+(1-\alpha)z^{p+1}\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]'\right]B\right|$$
$$=\left|\sum_{k=m}^{\infty}a_k\left[\frac{l+\lambda(k+p)}{l}\right]^n k\left[\alpha(k-1)+(1-\alpha)\right]z^{k+p}\right|$$
$$-\left|p(A-B)\left[(1-\alpha)-(p+1)\alpha\right]+B\sum_{k=m}^{\infty}a_k\left[\frac{l+\lambda(k+p)}{l}\right]^n k\left[\alpha(k-1)+(1-\alpha)\right]z^{k+p}\right|$$
$$\leq\sum_{k=m}^{\infty}|a_k|\left[\frac{l+\lambda(k+p)}{l}\right]^n k\left[\alpha(k-1)+(1-\alpha)\right]r^{k+p}-p(A-B)\left[(1-\alpha)-(p+1)\alpha\right]$$
$$+|B|\sum_{k=m}^{\infty}|a_k|\left[\frac{l+\lambda(k+p)}{l}\right]^n k\left[\alpha(k-1)+(1-\alpha)\right]r^{k+p}$$
$$\leq\sum_{k=m}^{\infty}|a_k|\omega_k r^{k+p}-p(A-B)\left[(1-\alpha)-(p+1)\alpha\right]<0,$$

hence $f\in\mathcal{R}_{n,m}(\lambda,p,l;\alpha)$. □

Our next result provides a sharp subordination result involving the functions of the class $\mathcal{R}_{n,m}(\lambda,p,l;\alpha)$.

Theorem 2.1. *Let the sequence $\{\omega_k\}_{k\in\mathbb{N}}$ defined by (15) be a nondecreasing sequence. If the function f of the form (1) belongs to the class $\mathcal{R}_{n,m}(\lambda,p,l;\alpha)$ and $h\in\mathcal{S}^c$, then*

$$\left(\chi\left(z^{p+1}f\right)*h\right)(z)\prec h(z), \tag{16}$$

and

$$\operatorname{Re}\left(z^{p+1}f(z)\right)>-\frac{1}{2\chi},\ z\in\mathcal{U}, \tag{17}$$

whenever

$$\chi=\frac{\omega_m}{2\left\{p(A-B)\left[(1-\alpha)-(p+1)\alpha\right]+\omega_m\right\}}.$$

Moreover, the number χ cannot be replaced by a larger number for odd p and m.

Proof. Supposing that the function $h\in\mathcal{S}^c$ is of the form

$$h(z)=\sum_{k=1}^{\infty}b_k z^k,\ z\in\mathcal{U}\quad(\text{where}\quad b_1=1),$$

then

$$\sum_{k=1}^{\infty} d_k b_k z^k = \left(\chi\left(z^{p+1} f\right) * h\right)(z) \prec h(z),$$

where

$$d_k = \begin{cases} \chi, & \text{if} \quad k = 1, \\ 0, & \text{if} \quad 2 \leq k \leq m+p, \\ \chi a_{k+p+1}, & \text{if} \quad k > m+p. \end{cases}$$

Now, using the Definition 1.1, the subordination result in (16) holds if $\{d_k\}_{k\in\mathbb{N}}$ is a subordinating factor sequence.

Since $\{\omega_k\}_{k\in\mathbb{N}}$ is a nondecreasing sequence we have

$$\operatorname{Re}\left(1 + 2\sum_{k=1}^{\infty} d_k z^k\right) = \operatorname{Re}\left(1 + \frac{\omega_m}{p(A-B)\left[(1-\alpha)-(p+1)\alpha\right] + \omega_m} z + \right. \tag{18}$$
$$\left. \sum_{k=m}^{\infty} \frac{\omega_m}{p(A-B)\left[(1-\alpha)-(p+1)\alpha\right] + \omega_m} a_k z^{k+p}\right) \geq$$
$$1 - \frac{\omega_m}{p(A-B)\left[(1-\alpha)-(p+1)\alpha\right] + \omega_m} r$$
$$- \frac{r}{p(A-B)\left[(1-\alpha)-(p+1)\alpha\right] + \omega_m} \sum_{k=m}^{\infty} \omega_k |a_k|, \ |z| = r < 1.$$

Thus, by using Lemma 2.1 in (18) we obtain

$$\operatorname{Re}\left(1 + 2\sum_{k=1}^{\infty} d_k z^k\right) \geq 1 - \frac{\omega_m}{p(A-B)\left[(1-\alpha)-(p+1)\alpha\right] + \omega_m} r$$
$$- \frac{r}{p(A-B)\left[(1-\alpha)-(p+1)\alpha\right] + \omega_m} p(A-B)\left[(1-\alpha)-(p+1)\alpha\right] > 0, \ z \in \mathcal{U},$$

which proves the inequality (4), hence also the subordination result asserted by (16).

The inequality (17) asserted by Theorem 2.1 would follow from (16) upon setting

$$h(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n, \ z \in \mathcal{U}.$$

We also observe that, whenever the functions of the form

$$f_k(z) = z^{-p} + \frac{p(A-B)\left[(1-\alpha)-(p+1)\alpha\right]}{k\left[\frac{l+\lambda(p+k)}{l}\right]^n \left[\alpha(k-1) + (1-\alpha)\right](1+|B|)} z^k, \ z \in \mathcal{U}^* \quad (k \geq m),$$

belong to the class $\mathcal{R}_{n,m}(\lambda, p, l; \alpha)$, for p and m odd numbers, we have

$$z^{p+1} f_m(z)\Big|_{z=-1} = \frac{-1}{2\chi},$$

and the constant χ is the best estimate. □

3. Subordination properties of the operator $\mathfrak{J}_p^n(\lambda,l)$ and Argument estimates

In this section we obtain certain subordination properties involving the operator $\mathfrak{J}_p^n(\lambda,l)$.

Theorem 3.1. *For $f\in\Sigma_{p,m}$ let the operator $\mathcal{T}$ be defined by*

$$\mathcal{T}f(z):=\left[1-\beta-\left(p+\frac{l}{\lambda}\right)\beta\right]\mathfrak{J}_p^n(\lambda,l)f(z)+\frac{\beta l}{\lambda}\,\mathfrak{J}_p^{n+1}(\lambda,l)f(z), \tag{19}$$

for $\lambda,\ l>0$ and $0<\beta<\dfrac{1}{p+1}$.

(i) If

$$\frac{\mathcal{T}^{(j)}f(z)}{(-1)^j z^{-p-j}(p)_j}\prec(1-\beta-\beta p)\frac{1+Az}{1+Bz}\quad(j\in\mathbb{N}_0), \tag{20}$$

and $(p)_j$ is defined by (8), then

$$\frac{\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]^{(j)}}{(-1)^j z^{-p-j}(p)_j}\prec\widetilde{q}(z)\prec\frac{1+Az}{1+Bz}, \tag{21}$$

where the function $\widetilde{q}$ is given by

$$\widetilde{q}(z)=\begin{cases}\dfrac{A}{B}+\left(1-\dfrac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\dfrac{1-\beta-\beta p}{\beta(p+m)}+1;\dfrac{Bz}{1+Bz}\right), & \text{if}\quad B\neq0,\\[2mm] 1+\dfrac{A(1-\beta-\beta p)}{1-\beta+\beta m}z, & \text{if}\quad B=0,\end{cases}$$

and it is the best dominant of (21).

(ii) Moreover,

$$\operatorname{Re}\frac{\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]^{(j)}}{(-1)^j z^{-p-j}}>(p)_j\,\sigma_1,\ z\in\mathcal{U}, \tag{22}$$

where

$$\sigma_1=\begin{cases}\dfrac{A}{B}+\left(1-\dfrac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1,1;\dfrac{1-\beta-\beta p}{\beta(p+m)}+1;\dfrac{B}{B-1}\right), & \text{if}\quad B\neq0,\\[2mm] 1-\dfrac{A(1-\beta-\beta p)}{1-\beta+\beta m}, & \text{if}\quad B=0.\end{cases}$$

The inequality (22) is the best possible.

Proof. From (19) and (3) we easily obtain

$$\mathcal{T}^{(j)}f(z)=(1-\beta+\beta j)\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]^{(j)}+\beta z\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]^{(j+1)},\ z\in\mathcal{U}^*. \tag{23}$$

Letting

$$q(z):=\frac{\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]^{(j)}}{(-1)^j z^{-p-j}(p)_j}.$$

with $f\in\Sigma_{p,m}$, then q is is analytic in $\mathcal{U}$ and has the form (5). Also, note that

$$(1-\beta-\beta p)\left[q(z)+\frac{\beta}{1-\beta-\beta p}\,zq'(z)\right]=\frac{\mathcal{T}^{(j)}f(z)}{(-1)^j z^{-p-j}(p)_j}. \tag{24}$$

Then, by (20) we have

$$q(z)+\frac{\beta}{1-\beta-\beta p}\,zq'(z)\prec\frac{1+Az}{1+Bz}.$$

Now, by using Lemma 1.2 for $\gamma = \dfrac{1-\beta-\beta p}{\beta}$ and whenever $\gamma > 0$, by a changing of variables followed by the use of the identities (10), (11) and (12), we deduce that

$$\frac{\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]^{(j)}}{(-1)^j z^{-p-j}(p)_j} \prec \widetilde{q}(z) = \frac{(1-\beta-\beta p)}{\beta(p+m)} z^{-\frac{(1-\beta-\beta p)}{\beta(p+m)}} \int\limits_0^z t^{\frac{(1-\beta-\beta p)}{\beta(p+m)}-1} \frac{1+At}{1+Bt}\, dt$$

$$= \begin{cases} \dfrac{A}{B} + \left(1-\dfrac{A}{B}\right)(1+Bz)^{-1}\, {}_2F_1\left(1,1;\dfrac{1-\beta-\beta p}{\beta(p+m)}+1;\dfrac{Bz}{1+Bz}\right), & \text{if} \quad B \neq 0, \\ 1+\dfrac{A(1-\beta-\beta p)}{1-\beta+\beta m} z, & \text{if} \quad B = 0, \end{cases}$$

which proves the assertion (21) of our theorem.

Next, in order to prove the assertion (22), it sufficies to show that

$$\inf\left\{\operatorname{Re}\widetilde{q}(z) : z \in \mathcal{U}\right\} = \widetilde{q}(-1). \tag{25}$$

Indeed, for $|z| \leq r < 1$ we have

$$\operatorname{Re}\frac{1+Az}{1+Bz} \geq \frac{1-Ar}{1-Br},$$

and setting

$$\mathcal{E}(s,z) = \frac{1+Asz}{1+Bsz} \quad \text{and} \quad d\mu(s) = \frac{1-\beta-\beta p}{\beta(p+m)} s^{\frac{1-\beta-\beta p}{\beta(p+m)}-1}\, ds \quad (0 \leq s \leq 1)$$

which is a positive measure on the closed interval $[0,1]$ whenever $0 < \beta < \dfrac{1}{p+1}$, we get

$$\widetilde{q}(z) = \int_0^1 \mathcal{E}(s,z)\, d\mu(s),$$

and

$$\operatorname{Re}\widetilde{q}(z) \geq \int_0^1 \frac{1-Asr}{1-Bsr}\, d\mu(s) = \widetilde{q}(-r),\ |z| \leq r < 1.$$

Letting $r \to 1^-$ in the above inequality we obtain the assertion (25) of our theorem. The estimate in (22) is the best possible since the function $\widetilde{q}$ is the best dominant of (21). □

Taking $n = 0$, $l = m = \lambda = 1$, $A = 1 - \dfrac{2\alpha}{(1-\beta-\beta p)(p)_j}$ and $B = -1$ in Theorem 3.1 we get the following result:

Corollary 3.1. *Let* $\mathcal{T}f(z) = (1-\beta)f(z) + \beta z f'(z)$, *where* $f \in \Sigma_{p,1}$. *If* $0 < \beta < \dfrac{1}{p+1}$, *then*

$$\operatorname{Re}\frac{\mathcal{T}^{(j)}f(z)}{(-1)^j z^{-p-j}} > \alpha,\ z \in \mathcal{U} \quad \left(0 \leq \alpha < (p)_j\,(1-\beta-\beta p),\ j \in \mathbb{N}_0\right),$$

implies that

$$\operatorname{Re}\frac{f^{(j)}(z)}{(-1)^j z^{-p-j}} > \frac{\alpha}{1-\beta-\beta p} +$$

$$\left[(p)_j - \frac{\alpha}{1-\beta-\beta p}\right]\left[{}_2F_1\left(1,1;\frac{1-\beta-\beta p}{\beta(p+1)}+1;\frac{1}{2}\right)-1\right],\ z \in \mathcal{U}.$$

The above inequality is the best possible.

Theorem 3.2. *For $f \in \Sigma_{p,m}$ let the operator $\mathcal{T}$ be given by (19), and let $0 < \beta < \dfrac{1}{p+1}$.*

(*i*) *If*

$$\operatorname{Re} \frac{\left[\mathfrak{J}_p^n(\lambda, l) f(z)\right]^{(j)}}{(-1)^j z^{-p-j}} > \alpha,\ z \in \mathcal{U} \quad \Big(\alpha < (p)_j,\ j \in \mathbb{N}_0\Big),$$

then

$$\operatorname{Re} \frac{\mathcal{T}^{(j)} f(z)}{(-1)^j z^{-p-j}} > \alpha(1-\beta-\beta p),\ |z| < R_1,$$

where

$$R_1 = \left[\sqrt{1+\left(\frac{\beta(p+m)}{1-\beta-\beta p}\right)^2} - \frac{\beta(p+m)}{1-\beta-\beta p}\right]^{\frac{1}{p+m}}. \tag{26}$$

(*ii*) *If*

$$\operatorname{Re} \frac{\left[\mathfrak{J}_p^n(\lambda, l) f(z)\right]^{(j)}}{(-1)^j z^{-p-j}} < \alpha,\ z \in \mathcal{U} \quad \Big(\alpha > (p)_j,\ j \in \mathbb{N}_0\Big),$$

then

$$\operatorname{Re} \frac{\mathcal{T}^{(j)} f(z)}{(-1)^j z^{-p-j}} < \alpha(1-\beta-\beta p),\ |z| < R_1.$$

The bound R_1 is the best possible.

Proof. (*i*) Defining the function ϕ by

$$\frac{\left[\mathfrak{J}_p^n(\lambda, l) f(z)\right]^{(j)}}{(-1)^j z^{-p-j}} =: \alpha + \left[(p)_j - \alpha\right] \phi(z), \tag{27}$$

then ϕ is an analytic function with positive real part in $\mathcal{U}$. Differentiating (27) with respect to z and using (23) we have

$$\frac{\mathcal{T}^{(j)} f(z)}{(-1)^j z^{-p-j}} - \alpha(1-\beta-\beta p) = \left[(p)_j - \alpha\right]\left[(1-\beta-\beta p)\phi(z) + \beta z \phi'(z)\right]. \tag{28}$$

Now, by applying in (28) the following well-known estimate [8]

$$\frac{|z\phi'(z)|}{\operatorname{Re}\phi(z)} \leq \frac{2(p+m)r^{p+m}}{1-r^{2(p+m)}},\ |z| = r < 1, \tag{29}$$

we have

$$\operatorname{Re}\left[\frac{\mathcal{T}^{(j)} f(z)}{(-1)^j z^{-p-j}} - \alpha(1-\beta-\beta p)\right] \geq \tag{30}$$
$$\operatorname{Re}\phi(z)\left[(p)_j - \alpha\right]\left[(1-\beta-\beta p) - \frac{2\beta(p+m)r^{p+m}}{1-r^{2(p+m)}}\right],\ |z| = r < 1.$$

Now, it is easy to see that the right hand side of (30) is positive whenever $r < R_1$, where R_1 is given by (26). In order to show that the bound R_1 is the best possible, we consider the function $f \in \Sigma_{p,m}$ defined by

$$\frac{\left[\mathfrak{J}_p^n(\lambda, l) f(z)\right]^{(j)}}{(-1)^j z^{-p-j}} = \alpha + \left[(p)_j - \alpha\right]\frac{1+z^{p+m}}{1-z^{p+m}}.$$

Then,

$$\frac{\mathcal{T}^{(j)}f(z)}{(-1)^j z^{-p-j}} - \alpha(1-\beta-\beta p) =$$

$$\frac{(p)_j-\alpha}{(1-z^{p+m})^2}\left[(1-\beta-\beta p)\left(1-z^{2(p+m)}\right)+2\beta(p+m)z^{p+m}\right]=0,$$

for $z = R_1 \exp^{\frac{i\pi}{p+m}}$, and the first part of the theorem is proved.

(ii) For the proof of the second part, we define the function ϕ by

$$\frac{\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]^{(j)}}{(-1)^j z^{-p-j}} =: \alpha - [\alpha-(p)_j]\,\phi(z). \tag{31}$$

Thus, the function ϕ is analytic and has positive real part in $\mathcal{U}$. Differentiating (31) with respect to z and using (23) we have

$$\frac{\mathcal{T}^{(j)}f(z)}{(-1)^j z^{-p-j}} - \alpha(1-\beta-\beta p) = [\alpha-(p)_j)]\left[-(1-\beta-\beta p)\phi(z)-\beta z\phi'(z)\right]. \tag{32}$$

From the inequality (29) we get

$$\operatorname{Re} z\phi'(z) \geq -|z\phi'(z)| \geq -\frac{2(p+m)r^{p+m}}{1-r^{2(p+m)}}\operatorname{Re}\phi(z),\ |z|=r<1,$$

and from (32) we deduce that

$$\operatorname{Re}\left[\frac{\mathcal{T}^{(j)}f(z)}{(-1)^j z^{-p-j}} - \alpha(1-\beta-\beta p)\right] \leq \tag{33}$$

$$\operatorname{Re}\phi(z)\,[\alpha-(p)_j]\left[-(1-\beta-\beta p)+\frac{2\beta(p+m)r^{p+m}}{1-r^{2(p+m)}}\right],\ |z|=r<1.$$

Now, we see that the right hand side of (33) is negative provided that $r < R_1$, where R_1 is given by (26). To show that the bound R_1 is the best possible, let consider the function $f \in \Sigma_{p,m}$ defined by

$$\frac{\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]^{(j)}}{(-1)^j z^{-p-j}} = \alpha - [\alpha-(p)_j]\,\frac{1+z^{p+m}}{1-z^{p+m}}.$$

Then,

$$\frac{\mathcal{T}^{(j)}f(z)}{(-1)^j z^{-p-j}} - \alpha(1-\beta-\beta p) =$$

$$\frac{\alpha-(p)_j}{(1-z^{p+m})^2}\left[-(1-\beta-\beta p)\left(1-z^{2(p+m)}\right)-2\beta(p+m)z^{p+m}\right]=0,$$

for $z = R_1 \exp^{\frac{i\pi}{p+m}}$, which proves the second part of our theorem. □

Example 3.1. *We provide an example for the function* ϕ *defined in* (27)*. For* $p = 2, m = 2, \lambda = j = n = a_2 = 1,$ *and* $l = 6$ *we have*

$$f(z) = z^{-10} + z^2$$

and

$$\mathfrak{J}_{10}^1(1,6)f(z) = z^{-10} + 3z^2$$

hence

$$\phi(z) = 1 - \frac{6}{8}z^{12},$$

which has a positive real part in $\mathcal{U}$.

For a function $f \in \Sigma_{p,m}$ let define the integral operator $\mathrm{F}_{\mathrm{p,s}}$ by

$$\mathrm{F}_{\mathrm{p,s}} f(z) := \frac{s}{z^{p+s}} \int_0^z t^{p+s-1} f(t)\, dt \quad (s > 0). \tag{34}$$

By using the integral operator defined in (34) we will obtain certain subordination properties, as follows:

Theorem 3.3. *If* $f \in \Sigma_{p,m}$, *then*

$$\frac{\left[\mathfrak{I}_p^n(\lambda, l) f(z)\right]^{(j+1)}}{(-1)^{j+1} z^{-p-j-1} (p)_{j+1}} \prec \frac{1 + Az}{1 + Bz} \quad (j \in \mathbb{N}_0), \tag{35}$$

implies that

$$\frac{\left[\mathfrak{I}_p^n(\lambda, l) \mathrm{F}_{\mathrm{p,s}} f(z)\right]^{(j+1)}}{(-1)^{j+1} z^{-p-j-1} (p)_{j+1}} \prec \widetilde{Q}(z) \prec \frac{1 + Az}{1 + Bz},$$

where $\widetilde{Q}$ *is given by*

$$\widetilde{Q}(z) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right) (1 + Bz)^{-1} {}_2F_1\left(1, 1; \dfrac{s}{p+m} + 1; \dfrac{Bz}{1 + Bz}\right), & \textit{if} \quad B \neq 0, \\ 1 + \dfrac{As}{p + s + m} z, & \textit{if} \quad B = 0. \end{cases}$$

Moreover,

$$\mathrm{Re}\, \frac{\left[\mathfrak{I}_p^n(\lambda, l) \mathrm{F}_{\mathrm{p,s}} f(z)\right]^{(j+1)}}{(-1)^{j+1} z^{-p-j-1}} > (p)_{j+1}\, \sigma_2, \ z \in \mathcal{U}, \tag{36}$$

where

$$\sigma_2 = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right) (1 - B)^{-1} {}_2F_1\left(1, 1; \dfrac{s}{p+m} + 1; \dfrac{B}{1 - B}\right), & \textit{if} \quad B \neq 0, \\ 1 - \dfrac{As}{p + s + m}, & \textit{if} \quad B = 0. \end{cases}$$

The inequality (36) *is the best possible.*

Proof. Setting

$$q(z) := \frac{\left[\mathfrak{I}_p^n(\lambda, l) \mathrm{F}_{\mathrm{p,s}} f(z)\right]^{(j+1)}}{(-1)^{j+1} z^{-p-j-1} (p)_{j+1}}, \tag{37}$$

where $f \in \Sigma_{p,m}$, then q is is analytic in $\mathcal{U}$ and has the form (5). Using in (37) the following identity

$$z \left[\mathfrak{I}_p^n(\lambda, l) \mathrm{F}_{\mathrm{p,s}} f(z)\right]^{(j+1)} = s \left[\mathfrak{I}_p^n(\lambda, l) f(z)\right]^{(j)} - (p+s+j) \left[\mathfrak{I}_p^n(\lambda, l) \mathrm{F}_{\mathrm{p,s}} f(z)\right]^{(j)}, \ z \in \mathcal{U},$$

and differentiating the resulting relation with respect to z, we obtain

$$\frac{\left[\mathfrak{I}_p^n(\lambda, l) f(z)\right]^{(j+1)}}{(-1)^{j+1} z^{-p-j-1} (p)_{j+1}} = q(z) + \frac{1}{s} z q'(z). \tag{38}$$

Then, by (35) we have

$$q(z) + \frac{1}{s} z q'(z) \prec \frac{1 + Az}{1 + Bz}.$$

Now, the remaining part of the proof follows the same techniques as in Theorem 3.1, and hence it will be omitted. □

Remark 3.1. Taking $j = 0$ in Theorem 3.3 we obtain the result of El-Ashwah et al. [5, Theorem 3.9].

For the special case $A = 1 - \dfrac{2\alpha}{(p)_{j+1}}$ and $B = -1$, Theorem 3.3 gives us the following corollary:

Corollary 3.2. *If $f \in \Sigma_{p,m}$ satisfies the inequality*

$$\operatorname{Re} \frac{\left[\mathfrak{I}_p^n(\lambda, l)f(z)\right]^{(j+1)}}{(-1)^{j+1}z^{-p-j-1}} > \alpha,\ z \in \mathcal{U} \quad \Big(0 \le \alpha < (p)_{j+1},\ j \in \mathbb{N}_0\Big),$$

then

$$\operatorname{Re} \frac{\left[\mathfrak{I}_p^n(\lambda, l)\mathrm{F}_{\mathrm{p,s}}f(z)\right]^{(j+1)}}{(-1)^{j+1}z^{-p-j-1}} > \alpha + [(p)_{j+1} - \alpha]\left[{}_2F_1\left(1, 1; \frac{s}{p+m} + 1; \frac{1}{2}\right) - 1\right],\ z \in \mathcal{U},$$

and the inequality is the best possible.

Remark 3.2. For $m = j = 0$ the Corollary 3.2 reduces to the result of El-Ashwah et al. [5, Corollary 3.11].

The following theorem is similar to Theorem 3.2, and hence we omit its proof:

Theorem 3.4. *Let the operator* $\mathrm{F}_{\mathrm{p,s}}$ *be defined by* (34) *and* $f \in \Sigma_{p,m}$.
 (*i*) *If*

$$\operatorname{Re} \frac{\left[\mathfrak{I}_p^n(\lambda, l)\mathrm{F}_{\mathrm{p,s}}f(z)\right]^{(j+1)}}{(-1)^{j+1}z^{-p-j-1}} > \alpha,\ z \in \mathcal{U} \quad \Big(\alpha < (p)_{j+1},\ j \in \mathbb{N}_0\Big),$$

then

$$\operatorname{Re} \frac{\left[\mathfrak{I}_p^n(\lambda, l)f(z)\right]^{(j+1)}}{(-1)^{j+1}z^{-p-j-1}} > \alpha,\ |z| < R_2,$$

where

$$R_2 = \left[\sqrt{1 + \left(\frac{p+m}{s}\right)^2} - \frac{p+m}{s}\right]^{\frac{1}{p+m}}$$

(*ii*) *If*

$$\operatorname{Re} \frac{\left[\mathfrak{I}_p^n(\lambda, l)\mathrm{F}_{\mathrm{p,s}}f(z)\right]^{(j+1)}}{(-1)^{j+1}z^{-p-j-1}} < \alpha,\ z \in \mathcal{U} \quad \Big(\alpha > (p)_{j+1}\Big),$$

then

$$\operatorname{Re} \frac{\left[\mathfrak{I}_p^n(\lambda, l)f(z)\right]^{(j+1)}}{(-1)^{j+1}z^{-p-j-1}} < \alpha,\ |z| < R_2.$$

The bound R_2 is the best possible.

We obtain certain argument estimates involving the operator $\mathcal{J}_p^n(\lambda, l)$ and connected with the linear operator $\mathcal{T}$, and the integral operator $\mathrm{F}_{\mathrm{p,s}}$ defined in (34), respectively.

Theorem 3.5. *For $f \in \Sigma_{p,m}$ let the operator $\mathcal{T}$ be defined by* (19), *and let* $0 \le \beta < \dfrac{1}{p+1}$. *If*

$$\left|\arg \frac{\mathcal{T}^{(j)}f(z)}{(-1)^j z^{-p-j}}\right| < \frac{\pi\delta}{2},\ z \in \mathcal{U} \quad \Big(\delta > 0,\ j \in \mathbb{N}_0\Big), \tag{39}$$

then

$$\left|\arg \frac{\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]^{(j)}}{(-1)^j z^{-p-j}}\right| < \frac{\pi\delta}{2},\ z\in\mathcal{U}.$$

Proof. For $f\in\Sigma_{p,m}$, if we let

$$q(z) := \frac{\left[\mathfrak{J}_p^n(\lambda,l)f(z)\right]^{(j)}}{(-1)^j z^{-p-j}(p)_j},$$

then q is of the form (5) and it is analytic in $\mathcal{U}$. If there exists a point $z_0\in\mathcal{U}$ such that

$$|\arg q(z)| < \frac{\pi\delta}{2},\ |z|<|z_0| \quad \text{and} \quad |\arg q(z_0)| = \frac{\pi\delta}{2} \quad (\delta>0),$$

then, according to Lemma 1.3 we have

$$\frac{z_0q'(z_0)}{q(z_0)} = ik\delta \quad \text{and} \quad q(z_0)^{1/\delta} = \pm\, ic \quad (c>0).$$

Also, from the equality (24) we get

$$\frac{\mathcal{T}^{(j)}f(z_0)}{(-1)^j z_0^{-p-j}} = (p)_j\,(1-\beta-\beta p)q(z_0)\left[1+\frac{\beta}{1-\beta-\beta p}\frac{z_0q'(z_0)}{q(z_0)}\right].$$

If $\arg q(z_0) = \frac{\pi\delta}{2}$, according to the above relation we get

$$\frac{\mathcal{T}^{(j)}f(z_0)}{(-1)^j z_0^{-p-j}} = (p)_j\,(1-\beta-\beta p)\,c^\delta e^{\frac{i\pi\delta}{2}}\left(1+\frac{\beta}{1-\beta-\beta p}\,ik\delta\right),$$

which implies

$$\arg\frac{\mathcal{T}^{(j)}f(z_0)}{(-1)^j z_0^{-p-j}} = \frac{\pi\delta}{2}+\arg\left(1+\frac{\beta}{1-\beta-\beta p}\,ik\delta\right) = \frac{\pi\delta}{2}+\tan^{-1}\left(\frac{\beta}{1-\beta-\beta p}\,k\delta\right) \geq \frac{\pi\delta}{2},$$

whenever $k\geq\frac{1}{2}\left(c+\frac{1}{c}\right)$ and $0\leq\beta<\frac{1}{1+p}$, and this last inequality contradicts the assumption (39).

Similarly, if $\arg q(z_0) = -\frac{\pi\delta}{2}$, then we obtain

$$\arg\frac{\mathcal{T}^{(j)}f(z_0)}{(-1)^j z_0^{-p-j}} = -\frac{\pi\delta}{2}+\arg\left(1+\frac{\beta}{1-\beta-\beta p}\,ik\delta\right) = -\frac{\pi\delta}{2}+\tan^{-1}\left(\frac{\beta}{1-\beta-\beta p}\,k\delta\right) \leq -\frac{\pi\delta}{2},$$

whenever $k\leq-\frac{1}{2}\left(c+\frac{1}{c}\right)$ and $0\leq\beta<\frac{1}{1+p}$, which also contradicts the assumption (39).

Consequently, the function q need to satisfy the inequality $|\arg q(z)|<\frac{\pi\delta}{2}$, $z\in\mathcal{U}$, i.e. the conclusion of our theorem. □

The proof of the following Theorem is much akin to Theorem 3.5, and hence we omit it,

Theorem 3.6. *For $f\in\Sigma_{p,m}$ let operator $\mathrm{F_{p,s}}$ is defined by* (34)*. If*

$$\left|\arg \frac{\left[\mathcal{J}_p^n(\lambda,l)f(z)\right]^{(j+1)}}{(-1)^{j+1} z^{-p-j-1}}\right| < \frac{\pi\gamma}{2},\ z\in\mathcal{U} \quad \left(\gamma>0,\ j\in\mathbb{N}_0\right),$$

then

$$\left|\arg \frac{\left[\mathfrak{I}_p^n(\lambda,l)\mathrm{F}_{\mathrm{p,s}}f(z)\right]^{(j+1)}}{(-1)^{j+1}z^{-p-j-1}}\right| < \frac{\pi\gamma}{2},\ z\in\mathcal{U}.$$

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