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ON THE COMPOSITION OF ANALYTIC FUNCTIONS IN THE UNIT DISC

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ABSTRACT. In this article we proved some results on the composition of two analytic functions defined in the unit disc in terms of their maximum modulus and type. Also we introduced hyper exponent of convergence of zeros of such analytic functions and studied some growth properties of hyper exponent of convergence of zeros of composite analytic function in the unit disc.

1. INTRODUCTION

Let $f(z) = \sum_{n=0}^{\infty} a_n z^{t_n}$ be an analytic in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$, $\{t_n\}$ be a strictly increasing sequence of positive integers with $t_0 = 0$ and $a_n \neq 0$ for $n = 1, 2, 3, \ldots$. The maximum modulus and maximum term of $f(z)$ respectively are $M(r, f) = \max_{|z|=r} |f(z)|$, $0 < r < 1$ and $\mu(r, f) = \max_{n \geq 0} (|a_n| r^n)$.

In 1968 Sons [4] defined the **order** ρ_f and **lower order** λ_f of f as

$$\rho_f = \limsup_{r \to 1^-} \frac{\log^+ \log^+ M(r, f)}{-\log(1 - r)}$$

and

$$\lambda_f = \liminf_{r \to 1^-} \frac{\log^+ \log^+ M(r, f)}{-\log(1 - r)},$$

where $\log^+ x = \max\{\log x, 0\}$, $0 \leq x \leq \infty$.

Then one can easily introduced the **hyper order** $\overline{\rho_f}$ and **hyper lower order** $\overline{\lambda_f}$ of f as

$$\overline{\rho_f} = \limsup_{r \to 1^-} \frac{\log^+ \log^+ \log^+ M(r, f)}{-\log(1 - r)}$$

and

$$\overline{\lambda_f} = \liminf_{r \to 1^-} \frac{\log^+ \log^+ \log^+ M(r, f)}{-\log(1 - r)}.$$

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Using the relation [3]

$$\mu(r,f) \leq M(r,f) \leq \frac{R}{R-r}\mu(R,f)$$

for $0 \leq r < R \leq 1$,we get the definition of ρ_f and λ_f in terms of $\mu(r,f)$ as

$$\rho_f = \limsup_{r\to 1^-} \frac{\log^+ \log^+ \mu(r,f)}{-\log(1-r)}$$

$$\lambda_f = \liminf_{r\to 1^-} \frac{\log^+ \log^+ \mu(r,f)}{-\log(1-r)}$$

and

$$\overline{\rho_f} = \limsup_{r\to 1^-} \frac{\log^+ \log^+ \log^+ \mu(r,f)}{-\log(1-r)}$$

$$\overline{\lambda_f} = \liminf_{r\to 1^-} \frac{\log^+ \log^+ \log^+ \mu(r,f)}{-\log(1-r)}.$$

For $0 < \rho_f < \infty$, the **type** σ_f and **lower type** τ_f of f are defined as

$$\sigma_f = \limsup_{r\to 1^-} \frac{\log^+ M(r,f)}{(1-r)^{-\rho_f}}$$

and

$$\tau_f = \liminf_{r\to 1^-} \frac{\log^+ M(r,f)}{(1-r)^{-\rho_f}}.$$

A number of results have been proved by several authors such as [1], [2], [4] etc. Let f and g be two analytic functions in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ and $|g(z)| < 1$. Then the composite function $f \circ g$ is defined by $(f \circ g)(z) = f(g(z)), \forall z \in D$.

The theory of distribution of values of entire function was studied by G. Valiron in (1949) [5].The function $N(r,f)$ is called enumerative function of f. It plays an important role in the theory of entire function. The ratio $\frac{N(r,f)}{N(r,g)}$ measures the comparative growth of f with respect to g in terms of enumerative function.

Now we study some growth properties of hyper exponent of converges of zeros of composite analytic function $f \circ g)$ in D. We introduce the defintion of hyper exponent of converges of zeros of an analytic function f in the unit disc D.

Let f be an analytic functions in the unit disc D, the **hyper-exponent of convergence** of zeros of f is denoted by $\rho_2(f)$ is defined by

$$\rho_2(f) = \limsup_{r\to 1^-} \frac{\log^+ \log^+ N(r,\frac{1}{f})}{-\log(1-r)},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x), k = 1, 2, ..., \log^{[0]} x = x$

In alternative notation

$$\rho_2(f) = \limsup_{r\to 1^-} \frac{\log^+ \log^+ N(r,0,f)}{-\log(1-r)}$$

Similarly one may define the hyper-exponent of convergence of distinct zeros of f denoted by

$$\overline{\rho_2}(f) = \limsup_{r\to 1^-} \frac{\log^+ \log^+ \overline{N}(r,0,f)}{-\log(1-r)}$$

Also replacing lim sup by lim inf we may define hyper lower exponent of convergence of zeros of f and hyper lower exponent of convergence of distinct zeros of f respectively by $\lambda_2(f)$ and $\overline{\lambda_2}(f)$.

2. Main Results

Now we prove the following theorems.

Theorem 1 Let f and g be two analytic functions in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ and $|g(z)| < 1$. Also $0 < \lambda_{f \circ g} \leq \rho_{f \circ g} < \infty$ and $0 < \rho_g < \infty$. Then for any positive number A,

$$i) \liminf_{r \to 1^-} \frac{\log^+ \log^+ M(r, f \circ g)}{\log^+ \log^+ M(r^A, g)} \leq \frac{\rho_{f \circ g}}{A\rho_g} \leq \limsup_{r \to 1^-} \frac{\log^+ \log^+ M(r, f \circ g)}{\log^+ \log^+ M(r^A, g)}.$$

Further if $\lambda_g > 0$, then

$$ii) \frac{\lambda_{f \circ g}}{A\rho_g} \leq \liminf_{r \to 1^-} \frac{\log^+ \log^+ M(r, f \circ g)}{\log^+ \log^+ M(r^A, g)} \leq \frac{\lambda_{f \circ g}}{A\lambda_g} \leq \limsup_{r \to 1^-} \frac{\log^+ \log^+ M(r, f \circ g)}{\log^+ \log^+ M(r^A, g)} \leq \frac{\rho_{f \circ g}}{A\lambda_g}$$

and

$$\begin{aligned} iii) \liminf_{r \to 1^-} \frac{\log^+ \log^+ M(r, f \circ g)}{\log^+ \log^+ M(r^A, g)} &\leq \min\left\{\frac{\lambda_{f \circ g}}{A\lambda_g}, \frac{\rho_{f \circ g}}{A\rho_g}\right\} \leq \max\left\{\frac{\lambda_{f \circ g}}{A\lambda_g}, \frac{\rho_{f \circ g}}{A\rho_g}\right\} \\ &\leq \limsup_{r \to 1^-} \frac{\log^+ \log^+ M(r, f \circ g)}{\log^+ \log^+ M(r^A, g)}. \end{aligned}$$

Proof. $i)$ From the definition of order of $f \circ g$ and g, we have for $\varepsilon > 0$ and for all values of $r \to 1^-$,

$$\log^+ \log^+ M(r, f \circ g) \leq -(\rho_{f \circ g} + \varepsilon) \log(1 - r). \tag{1}$$

Also for a sequence of values of $r \to 1^-$,

$$\log^+ \log^+ M\left(r^A, g\right) \geq -A(\rho_g - \varepsilon) \log(1 - r). \tag{2}$$

Combining (1) and (2) for a sequence of values of $r \to 1^-$,

$$\frac{\log^+ \log^+ M(r, f \circ g)}{\log^+ \log^+ M(r^A, g)} \leq \frac{(\rho_{f \circ g} + \varepsilon)}{A(\rho_g - \varepsilon)}.$$

As $\varepsilon > 0$ is arbitrary, we obtain

$$\liminf_{r \to 1} \frac{\log^+ \log^+ M(r, f \circ g)}{\log^+ \log^+ M(r^A, g)} \leq \frac{\rho_{f \circ g}}{A\rho_g}. \tag{3}$$

Again, for a sequence of values of $r \to 1^-$,

$$\log^+ \log^+ M(r, f \circ g) \geq -(\rho_{f \circ g} - \varepsilon) \log(1 - r). \tag{4}$$

Also for all values of $r \to 1^-$,

$$\log^+ \log^+ M\left(r^A, g\right) \leq -A(\rho_g + \varepsilon) \log(1 - r). \tag{5}$$

Combining (4) and (5) for a sequence of values of $r \to 1^-$,

$$\frac{\log^+ \log^+ M(r, f \circ g)}{\log^+ \log^+ M(r^A, g)} \geq \frac{(\rho_{f \circ g} - \varepsilon)}{A(\rho_g + \varepsilon)}.$$

Since, $\varepsilon > 0$ is arbitrary, it follows from above

$$\limsup_{r \to 1^-} \frac{\log^+ \log^+ M(r, f \circ g)}{\log^+ \log^+ M(r^A, g)} \geq \frac{\rho_{f \circ g}}{A\rho_g}. \tag{6}$$

Therefore from (3) and (6) we get

$$\liminf_{r\to 1^-}\frac{\log^+\log^+ M(r,f\circ g)}{\log^+\log^+ M(r^A,g)}\le\frac{\rho_{f\circ g}}{A\rho_g}\le\limsup_{r\to 1^-}\frac{\log^+\log^+ M(r,f\circ g)}{\log^+\log^+ M(r^A,g)}.$$

$ii)$ From the definition of lower order, we have for $\varepsilon>0$ and for all values of $r\to 1^-$,

$$\log^+\log^+ M(r,f\circ g)\ge -(\lambda_{f\circ g}-\varepsilon)\log(1-r). \tag{7}$$

Combining (5) and (7) for all values of $r\to 1^-$,

$$\frac{\log^+\log^+ M(r,f\circ g)}{\log^+\log^+ M(r^A,g)}\ge\frac{(\lambda_{f\circ g}-\varepsilon)}{A(\rho_g+\varepsilon)}.$$

As $\varepsilon>0$ is arbitrary, we obtain

$$\liminf_{r\to 1}\frac{\log^+\log^+ M(r,f\circ g)}{\log^+\log^+ M(r^A,g)}\ge\frac{\lambda_{f\circ g}}{A\rho_g}. \tag{8}$$

Again, for a sequence of values of $r\to 1^-$,

$$\log^+\log^+ M(r,f\circ g)\le -(\lambda_{f\circ g}+\varepsilon)\log(1-r). \tag{9}$$

Also for all values of $r\to 1^-$,

$$\log^+\log^+ M\left(r^A,g\right)\ge -A(\lambda_g-\varepsilon)\log(1-r). \tag{10}$$

Combining (9) and (10) for a sequence of values of $r\to 1^-$,

$$\frac{\log^+\log^+ M(r,f\circ g)}{\log^+\log^+ M(r^A,g)}\le\frac{(\lambda_{f\circ g}+\varepsilon)}{A(\lambda_g-\varepsilon)}.$$

Since, $\varepsilon>0$ is arbitrary, it follows from above

$$\liminf_{r\to 1^-}\frac{\log^+\log^+ M(r,f\circ g)}{\log^+\log^+ M(r^A,g)}\le\frac{\lambda_{f\circ g}}{A\lambda_g}. \tag{11}$$

Also, for a sequence of values of $r\to 1^-$,

$$\log^+\log^+ M\left(r^A,g\right)\le -A(\lambda_g+\varepsilon)\log(1-r). \tag{12}$$

Now from (7) and (12) we get for a sequence of values of $r\to 1^-$,

$$\frac{\log^+\log^+ M(r,f\circ g)}{\log^+\log^+ M(r^A,g)}\ge\frac{(\lambda_{f\circ g}+\varepsilon)}{A(\lambda_g-\varepsilon)}.$$

As $\varepsilon>0$ is arbitrary, we obtain from above that

$$\limsup_{r\to 1^-}\frac{\log^+\log^+ M(r,f\circ g)}{\log^+\log^+ M(r^A,g)}\ge\frac{\lambda_{f\circ g}}{A\lambda_g}. \tag{13}$$

Again, from (1) and (10), it follows for all values of $r\to 1^-$,

$$\frac{\log^+\log^+ M(r,f\circ g)}{\log^+\log^+ M(r^A,g)}\le\frac{(\rho_{f\circ g}+\varepsilon)}{A(\lambda_g-\varepsilon)}.$$

As $\varepsilon>0$ is arbitrary, it follows

$$\limsup_{r\to 1^-}\frac{\log^+\log^+ M(r,f\circ g)}{\log^+\log^+ M(r^A,g)}\le\frac{\rho_{f\circ g}}{A\lambda_g}. \tag{14}$$

Therefore from (8), (11) and (13) we get

$$\frac{\lambda_{f\circ g}}{A\rho_g}\le\liminf_{r\to 1^-}\frac{\log^+\log^+ M(r,f\circ g)}{\log^+\log^+ M(r^A,g)}\le\frac{\lambda_{f\circ g}}{A\lambda_g}\le\limsup_{r\to 1^-}\frac{\log^+\log^+ M(r,f\circ g)}{\log^+\log^+ M(r^A,g)}\le\frac{\rho_{f\circ g}}{A\lambda_g}.$$

iii) Using (3), (6), (11) and (13) we conclude that

$$\begin{aligned}\liminf_{r\to 1^-}\frac{\log^+\log^+ M(r,f\circ g)}{\log^+\log^+ M(r^A,g)} &\le \min\left\{\frac{\lambda_{f\circ g}}{A\lambda_g},\frac{\rho_{f\circ g}}{A\rho_g}\right\}\le \max\left\{\frac{\lambda_{f\circ g}}{A\lambda_g},\frac{\rho_{f\circ g}}{A\rho_g}\right\}\\ &\le \limsup_{r\to 1^-}\frac{\log^+\log^+ M(r,f\circ g)}{\log^+\log^+ M(r^A,g)}.\end{aligned}$$

This completes the proof.

Remark 1 The **Theorem 1** is also valid for maximum term of analytic functions in the unit disc $D=\{z\in\mathbb{C}:|z|<1\}$ in stead of maximum modulus.

The following theorem can also be deduced in the line of **Theorem 1** by using hyper order and hyper lower order.

Theorem 2 Let f and g be two analytic functions in the unit disc $D=\{z\in\mathbb{C}:|z|<1\}$ and $|g(z)|<1$. Also $0<\overline{\lambda}_{f\circ g}\le\overline{\rho}_{f\circ g}<\infty$ and $0<\overline{\rho}_g<\infty$. Then for any positive number A,

$$i)\liminf_{r\to 1^-}\frac{\log^+\log^+\log^+ M(r,f\circ g)}{\log^+\log^+\log^+ M(r^A,g)}\le\frac{\overline{\rho}_{f\circ g}}{A\overline{\rho}_g}\le\limsup_{r\to 1^-}\frac{\log^+\log^+\log^+ M(r,f\circ g)}{\log^+\log^+\log^+ M(r^A,g)}.$$

Further if $\overline{\lambda}_g>0$, then

$$\begin{aligned} ii)\frac{\overline{\lambda}_{f\circ g}}{A\overline{\rho}_g} &\le \liminf_{r\to 1^-}\frac{\log^+\log^+\log^+ M(r,f\circ g)}{\log^+\log^+\log^+ M(r^A,g)}\le\frac{\overline{\lambda}_{f\circ g}}{A\lambda_g}\\ &\le \limsup_{r\to 1^-}\frac{\log^+\log^+\log^+ M(r,f\circ g)}{\log^+\log^+\log^+ M(r^A,g)}\le\frac{\overline{\rho}_{f\circ g}}{A\overline{\lambda}_g}\end{aligned}$$

and

$$\begin{aligned} iii)\liminf_{r\to 1^-}\frac{\log^+\log^+\log^+ M(r,f\circ g)}{\log^+\log^+\log^+ M(r^A,g)} &\le \min\left\{\frac{\overline{\lambda}_{f\circ g}}{A\overline{\lambda}_g},\frac{\overline{\rho}_{f\circ g}}{A\overline{\rho}_g}\right\}\\ &\le \max\left\{\frac{\overline{\lambda}_{f\circ g}}{A\overline{\lambda}_g},\frac{\overline{\rho}_{f\circ g}}{A\overline{\rho}_g}\right\}\\ &\le \limsup_{r\to 1^-}\frac{\log^+\log^+\log^+ M(r,f\circ g)}{\log^+\log^+\log^+ M(r^A,g)}.\end{aligned}$$

Theorem 3 If f and g are two analytic functions in the unit disc $D=\{z\in\mathbb{C}:|z|<1\}$ and $|g(z)|<1$. Also $\rho_g<\infty$ and $\rho_{f\circ g}=\infty$. Then for every positive number A,

$$\limsup_{r\to 1^-}\frac{\log^+\log^+ M(r,f\circ g)}{\log^+\log^+ M(r^A,g)}=\infty.$$

Proof. Let us assume that the conclusion of the theorem does not hold. Then there exists a constant $B>0$ such that for all values of $r\to 1^-$,

$$\log^+\log^+ M(r,f\circ g)\le B\log^+\log^+ M\left(r^A,g\right). \tag{15}$$

Again from the definition of ρ_g, for all values of $r\to 1^-$, it follows that

$$\log^+\log^+ M\left(r^A,g\right)\le -A(\rho_g+\varepsilon)\log(1-r). \tag{16}$$

Combining (15) and (16) for a sequence of values of $r\to 1^-$,

$$\log^+\log^+ M(r,f\circ g)\le -AB(\rho_g+\varepsilon)\log(1-r). \tag{17}$$

From (17), it follows that $\rho_{f\circ g}<\infty$. So we arrive at a contradiction that $\rho_{f\circ g}=\infty$.

Hence the theorem follows.

Remark 2 If we take $\rho_f < \infty$ instead of $\rho_g < \infty$ in **Theorem 3** and the other condition remains the same then theorem also holds.

Remark 3 The **Theorem 3** is also valid for maximum term of analytic functions in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ in stead of maximum modulus.

Remark 4 The condition $\rho_g < \infty$ and $\rho_{f \circ g} = \infty$ are necessary in **Theorem 3**. Here we give two examples.

Example 1 Let us consider two analytic functions in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ as $f(z) = z$ and $g(z) = z^2$ then $f \circ g = z^2$.

Therefore, $\rho_g = \rho_{f \circ g} = \infty$.

We take $A = 1$, then we have

$$\limsup_{r \to 1^-} \frac{\log^+ \log^+ M(r, f \circ g)}{\log^+ \log^+ M(r, g)} = 1.$$

So we conclude that the condition $\rho_g < \infty$ is essential.

Example 2 Let us consider two analytic functions in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ as $f(z) = z$ and $g(z) = e^{e^z}$ then $f \circ g = e^{e^z}$.

Therefore, $\rho_g = \rho_{f \circ g} = 0$.

We take $A = 1$, then we have

$$\limsup_{r \to 1^-} \frac{\log^+ \log^+ e^{e^r}}{\log^+ \log^+ e^{e^r}} = 1.$$

So we conclude that the condition $\rho_{f \circ g} = \infty$ is essential.

Theorem 4 Let f and g be two analytic functions in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ and $|g(z)| < 1$. Also

i) $0 < \rho_g < \infty$

ii) $0 < \sigma_g < \infty$

iii) $\rho_{f \circ g} = \rho_g$

iv) $0 < \sigma_{f \circ g} < \infty$. Then

$$\liminf_{r \to 1^-} \frac{\log M(r, f \circ g)}{\log M(r, g)} \leq \frac{\sigma_{f \circ g}}{\sigma_g} \leq \limsup_{r \to 1^-} \frac{\log M(r, f \circ g)}{\log M(r, g)}.$$

Proof. From the definition of type of a composite function we have for arbitrary $\epsilon > 0$ and for all values of $r \to 1^-$.

$$\log^+ M(r, f \circ g) \leq (\sigma_{f \circ g} + \epsilon) (1 - r)^{-\rho_{f \circ g}} \tag{18}$$

Also for a sequence of values of r tending to 1^- ,

$$\log^+ M(r, g) \geq (\sigma_g - \epsilon) (1 - r)^{-\rho_g} \tag{19}$$

So combining (18) and (19) and using the condition (iii) it follows, for a sequence of values of $r \to 1^-$,

$$\frac{\log^+ M(r, f \circ g)}{\log^+ M(r, g)} \leq \frac{\sigma_{f \circ g} + \epsilon}{\sigma_g - \epsilon}$$

Since $\epsilon > 0$ is arbitrary, it follows from above that,

$$\liminf_{r \to 1^-} \frac{\log^+ M(r, f \circ g)}{\log^+ M(r, g)} \leq \frac{\sigma_{f \circ g}}{\sigma_g} \tag{20}$$

Again for a sequence of values $r \to 1^-$,

$$\log^+ M(r, f \circ g) \leq (\sigma_{f \circ g} - \epsilon) (1 - r)^{-\rho_{f \circ g}} \tag{21}$$

and also for all values of $r \to 1^-$,

$$\log^+ M(r, g) \geq (\sigma_g + \epsilon)(1 - r)^{-\rho_g} \tag{22}$$

So by condition (iii) , we obtain from (21) and (22) , for a sequence of values of $r \to 1^-$

$$\frac{\log^+ M(r, f \circ g)}{\log^+ M(r, g)} \leq \frac{\sigma_{f\circ g} - \epsilon}{\sigma_g + \epsilon}$$

Since $\epsilon > 0$ is arbitrary , we get from above

$$\limsup_{r \to 1^-} \frac{\log^+ M(r, f \circ g)}{\log^+ M(r, g)} \geq \frac{\sigma_{f\circ g}}{\sigma_g} \tag{23}$$

Thus the theorem follows from (20) and (23) .

Remark 5 The sign "$\leq$" in the above theorem can not be replaced by "$<$". This is shown in the following example.

Example 3 Let us consider two analytic functions $f(z)$ and $g(z)$ in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ as $f(z) = z$ and $g(z) = e^z$ then $f \circ g = e^z$. Now, $\rho_g = \limsup_{r \to 1^-} \frac{\log^+ \log^+ e^r}{-\log(1-r)} = \rho_{f\circ g}$ and $\sigma_g = \limsup_{r \to 1^-} \frac{\log^+ e^r}{(1-r)^{-\rho_g}} = \sigma_{f\circ g}$.Then we conclude that "$\leq$" can not be replaced by "$<$".

Theorem 5 Let f and g be two analytic functions in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ and $|g(z)| < 1$. Also $0 < \lambda_2(f \circ g) \leq \rho_2(f \circ g) < \infty$ and $0 < \lambda_2(g) \leq \rho_2(g) < \infty$. Then for any positive number A,

$$\begin{aligned} \frac{\lambda_2(f \circ g)}{A\rho_2(g)} &\leq \liminf_{r \to 1^-} \frac{\log^+ \log^+ N(r, 0, f \circ g)}{\log^+ \log^+ N(r^A, 0, g)} \leq \frac{\lambda_2(f \circ g)}{A\lambda_2(g)} \\ &\leq \limsup_{r \to 1^-} \frac{\log^+ \log^+ N(r, 0, f \circ g)}{\log^+ \log^+ N(r^A, 0, g)} \leq \frac{\rho_2(f \circ g)}{A\rho_2(g)}. \end{aligned}$$

Proof. From the definition of hyper exponent and hyper lower exponent of convergence of zeros of an analytic function f we have for arbitrary $\epsilon > 0$ and for all values of $r \to 1^-$,

$$\log^+ \log^+ N(r, 0, f \circ g) \geq -(\lambda_2(f \circ g) - \epsilon) \log(1 - r) \tag{24}$$

$$\log^+ \log^+ N(r, 0, g) \geq -A(\rho_2(g) + \epsilon) \log(1 - r) \tag{25}$$

Now from (24) and (25) it follows that for all values of $r \to 1^-$,

$$\frac{\log^+ \log^+ N(r, 0, f \circ g)}{\log^+ \log^+ N(r^A, 0, g)} \geq \frac{\lambda_2(f \circ g) - \epsilon}{A(\rho_2(g) + \epsilon)}$$

As $\epsilon > 0$ is arbitrary then,

$$\liminf_{r \to \infty} \frac{\log^+ \log^+ N(r, 0, f \circ g)}{\log^+ \log^+ N(r^A, 0, g)} \geq \frac{\lambda_2(f \circ g)}{A\rho_2(g)}. \tag{26}$$

Again for a sequence of values of $r \to 1^-$,

$$\log^+ \log^+ N(r, 0, f \circ g) \leq -(\lambda_2(f \circ g) - \epsilon) \log(1 - r) \tag{27}$$

and for values of $r \to 1^-$,

$$\log^+ \log^+ N(r^A, 0, g) \geq -A(\lambda_2(g) - \epsilon) \log(1 - r) \tag{28}$$

Now from (27) and (28) it follows, for a sequence of values of $r \to 1^-$, we get

$$\frac{\log^+ \log^+ N(r,0,f\circ g)}{\log^+ \log^+ N(r^A,0,g)} \le \frac{\lambda_2(f\circ g)}{A\lambda_2(g)}. \tag{29}$$

Also for a sequence of values of $r \to 1^-$, we may write

$$\log^+ \log^+ N(r^A,0,g) \le -A(\lambda_2(g)+\epsilon)\log(1-r). \tag{30}$$

Combining (24) and (30), it follows that for a sequence of values of $r \to 1^-$

$$\frac{\log^+ \log^+ N(r,0,f\circ g)}{\log^+ \log^+ N(r^A,0,g)} \le \frac{\lambda_2(f\circ g)-\epsilon}{A(\lambda_2(g)+\epsilon)}$$

Since $\epsilon > 0$ is arbitrary,

$$\limsup_{r\to 1^-} \frac{\log^+ \log^+ N(r,0,f\circ g)}{\log^+ \log^+ N(r^A,0,g)} \ge \frac{\lambda_2(f\circ g)}{A\lambda_2(g)}. \tag{31}$$

Also for all values of $r \to 1^-$,

$$\log^+ \log^+ N(r,0,f\circ g) \le -(\rho_2(f\circ g)+\epsilon)\log(1-r) \tag{32}$$

Combining (28) and (32), we obtain for all values of $r \to 1^-$,

$$\frac{\log^+ \log^+ N(r,0,f\circ g)}{\log^+ \log^+ N(r^A,0,g)} \le \frac{\rho_2(f\circ g)+\epsilon}{A(\lambda_2(g)-\epsilon)}$$

Since $\epsilon > 0$ is arbitrary ,

$$\limsup_{r\to \infty} \frac{\log^+ \log^+ N(r,0,f\circ g)}{\log^+ \log^+ N(r^A,0,g)} \le \frac{\rho_2(f\circ g)}{A\lambda_2(g)}. \tag{33}$$

Thus the theorem follows from (26) , (29) , (31) and (33) .

Remark 6 The **Theorem 5** is still valid for the different zeros of f and g respectively with $\rho_2(f\circ g), \lambda_2(f\circ g), \lambda_2(g), \rho_2(f\circ g)$ etc is replaced by $\overline{\lambda_2}(f\circ g), \overline{\lambda_2}(g), \overline{\rho_2}(g), \overline{\rho}(f\circ g)$ respectively.

Theorem 6 Let f and g be two analytic functions in the unit disc $D = \{z\in\mathbb{C} : |z| < 1\}$ and $|g(z)| < 1$. Also $0 < \lambda_2(f\circ g) \le \rho_2(f\circ g) < \infty$ and $0 < \rho_2(g) < \infty$. Then for any positive no A,

$$\liminf_{r\to 1^-} \frac{\log^+ \log^+ N(r,0,f\circ g)}{\log^+ \log^+ N(r^A,0,g)} \le \frac{\rho_2(f\circ g)}{A\rho_2(g)} \le \limsup_{r\to 1^-} \frac{\log^+ \log^+ N(r,0,f\circ g)}{\log^+ \log^+ N(r^A,0,g)}$$

Proof. Let $\epsilon > 0$ be arbitrary, then from the definition we have for a sequence of values of $r \to 1^-$

$$\log^+ \log^+ N(r,0,f\circ g) \ge -(\rho(f\circ g)-\epsilon)\log(1-r). \tag{34}$$

$$\log^+ \log^+ N(r^A,0,g) \ge -A(\rho(g)-\epsilon)\log(1-r). \tag{35}$$

Again for all values of $r \to 1^-$,

$$\log^+ \log^+ N(r,0,f\circ g) \le -(\rho(f\circ g)+\epsilon)\log(1-r). \tag{36}$$

$$\log^+ \log^+ N(r^A,0,g) \le -A(\rho(g)+\epsilon)\log(1-r). \tag{37}$$

Again combining (36) and (35), it follows for sequence of values of $r \to 1^-$,

$$\frac{\log^+ \log^+ N(r,0,f\circ g)}{\log^+ \log^+ N(r^A,0,g)} \le \frac{\rho(f\circ g)+\epsilon}{A(\rho(g)-\epsilon)}$$

Since $\epsilon > 0$ is arbitrary,

$$\liminf_{r\to 1^-} \frac{\log^+ \log^+ N(r, 0, f\circ g)}{\log^+ \log^+ N(r^A, 0, g)} \leq \frac{\rho(f\circ g)}{A\rho(g)}. \tag{38}$$

Again combining (37) and (34), it follows for sequence of values of $r \to 1^-$,

$$\limsup_{r\to 1^-} \frac{\log^+ \log^+ N(r, 0, f\circ g)}{\log^+ \log^+ N(r^A, 0, g)} \leq \frac{\rho(f\circ g) - \epsilon}{A(\rho(g) + \epsilon)}$$

Since $\epsilon > 0$ arbitrary,

$$\limsup_{r\to 1^-} \frac{\log^+ \log^+ N(r, 0, f\circ g)}{\log^+ \log^+ N(r^A, 0, g)} \geq \frac{\rho(f\circ g)}{A\rho(g)}. \tag{39}$$

Thus from (38) and (39), the theorem follows.

Remark 7 The **Theorem 6** is also valid for the distinct zeros of f and g with the same replacement as in the above remark.

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