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APPROXIMATE SOLUTIONS OF NONLINEAR NONLOCAL FRACTIONAL IMPULSIVE DIFFERENTIAL EQUATIONS VIA FAEDO-GALERKIN METHOD

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Abstract. In the present paper, we are concerned with a class of non linear fractional impulsive differential equations together with a non local condition in a reflexive Banach space. Convergence of approximate solutions is proved by using the properties of fractional power of a closed linear operator. Further, we used Faedo-Galerkin method to get unique solution. In the last, we demonstrated an application of results proved.

1. Introduction

Many mathematical phenomenon cannot be described by integer order differential equations, so we need non integer order differential equations to model such problems. Non local conditions give more better results than initial conditions. Fractional differential equations with a nonlocal condition have wide applicability in many fields of science and engineering. Literature shows that functional differential equations with nonlocal conditions have been studied by many authors [1, 2, 3, 4, 5, 6]. In recent few decades, researchers have shown great interest in fractional differential equations with nonlocal conditions due to its wide applicability in science and engineering. Later on, researchers have applied nonlocal conditions to non integer order differential equations. In [18], Zhou and Jiao have studied fractional neutral evolution equations with nonlocal conditions. Faedo-Galerkin method has been used to solve neutral fractional integro-differential equation with finite delay in a separable Hilbert space and demonstrated some convergence results. For the applications of Faedo-Galekin method to neutral differential equations, we refer to [13, 14] and references cited in these papers. In [17], Raheem and Bahuguna applied Rothe's method to find approximate solutions for non-integer order diffusion equations and proved some convergence results. Basic theories and lemmas

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on semigroup of bounded linear operators and on fractional derivatives have been discussed in books [15, 16].

On the other hand, the impulse conditions describe the dynamics of process in which discontinuous jumps occurs. The differential equations with impulsive conditions can be used in simulation of those discontinuous process in which impulses occur, that's why it becomes an important tool to handle the real process of mathematical models and phenomena such as in optimal control, electric circuit, biotechnology, population dynamics, fractals, neutral network, viscoelasticity, chemical technology. Papers [7, 9, 12, 20] deal with fractional differential equations with impulsive conditions. Chadha and Pandey [7] have established the existence and uniqueness of an impulsive fractional differential equation with a deviated arguments in a separable Hilbert space by using the Faedo-Galekin approximations.

In present work, we extended the applications of Faedo-Galerkin approximations to a class of impulsive fractional differential equations with a nonlocal condition. We used the probability density functions and fixed point theorems to prove the existence and uniqueness of approximate solutions.

Consider an impulsive fractional differential equation with a nonlocal condition in a reflexive Banach space $(\Upsilon, \|\cdot\|)$:

$$\begin{cases} {}^C D_t^\gamma \upsilon(t) + A\upsilon(t) = \Theta(t, \upsilon(t), \upsilon(e_1(t)), \upsilon(e_2(t)), \dots, \upsilon(e_p(t))), \\ \qquad\qquad\qquad\qquad t \in (0, T],\ t \neq t_l,\ l = 1, 2, \dots, q. \\ \Delta \upsilon(t_l) = \hat{J}_l(\upsilon(t_l)), \quad l = 1, 2, \dots, q, \\ \upsilon(t) = \chi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1.1}$$

where ${}^C D_t^\gamma$ is the Caputo fractional derivative of order $\gamma \in (0, 1)$, and $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}$ in Υ, the maps $\Theta : [0, T] \times \Upsilon^{m+1} \to \Upsilon$ and $\hat{J}_l : D(A^\alpha) \to D(A^\alpha)$ satisfy some suitable conditions, and the functions $e_i : [0, T] \to [0, T]$ are continuous.

The main aim of this paper is to obtain the Faedo-Galekin approximations and to prove their convergence to the unique solution of problem (1.1). An equivalent integral equation is obtained by using the concept of probability density functions introduced by El-Borai in [10]. By using Faedo-Galekin method, we obtained a sequence of approximate solutions and proved some convergence results. In the last section, we demonstrated an application of results proved.

2. Preliminaries and Assumptions

In this section, we give some definitions, assumptions and notations.

The fractional powerA^μ of A are well defined for all $0 \le \mu \le 1$ [16]. We may assume that, there exists $K > 0$ s.t. $\|S(t)\| \le K$ for all $t \ge 0$ and $0 \in \sigma(-A)$, $\sigma(-A)$ denotes the resolvent of $-A$ [16]. $D(A^\mu)$, the domain of A^μ is a Banach space with respect to the norm

$$\|\upsilon\|_\mu = \|A^\mu \upsilon\|, \quad \upsilon \in D(A^\mu).$$

Throughout the paper this Banach space is denoted by Υ_μ.

In the rest of this paper, we assume that $t_l \in (0, T]$ for all $l = 1, 2, \dots, q$ such that $t_1 < t_2 < \dots < t_q$, where $q \in \mathbb{N}$.

For $0 \le \mu \le 1$, we define $D_T^\mu = \{\upsilon : \upsilon : [-\tau, T] \to \Upsilon_\mu$ such that $\upsilon(t)$ is continuous except at $t = t_l$ but $\upsilon(t)$ is left continuous at $t = t_l$ and $\lim_{t \to t_l^+} \upsilon(t)$ exists for $l = 1, 2, \dots, q\}$.

It can be easily shown D^{μ}_{T} is a Banach space with respect to the norm

$$\|\upsilon\|_{T,\mu} = \sup_{s\in[-\tau,T]} \|\upsilon(s)\|, \quad \upsilon \in D^{\mu}_{T}.$$

We define

$$\tilde{\chi}(t) = \begin{cases} \chi(t), & t \in [-\tau, 0], \\ \chi(0), & t \in [0, T], \end{cases}$$

and, for $R > 0$

$$B_R(D^{\mu}_{T}, \tilde{\chi}) = \{\upsilon \in D^{\mu}_{T} \; : \; \|\upsilon - \tilde{\chi}\|_{T,\mu} \le R\}.$$

Consider the following assumptions:

(H1) Operator A satisfy the following conditions
- (i) A is closed
- (ii) A is positive definite
- (iii) A is self adjoint
- (iv) $D(A)$, the domain of A is dense in Υ
- (v) A has the pure point spectrum

$$0 < \alpha_0 \le \alpha_1 \le \cdots$$

and corresponding set of eigen functions is $\{\psi_i\}$, which may be assumed a complete orthonormal set, i.e.,

$$A\psi_i = \alpha_i \psi_i,$$

where

$$\langle \psi_i, \psi_j \rangle = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \ne j. \end{cases}$$

(H2) The function $\Theta(t)$ satisfies the following condition:

$$\|\Theta(t_1, \upsilon_1, \upsilon_2, \dots, \upsilon_{p+1}) - \Theta(t_2, \omega_1, \omega_2, \dots, \omega_{p+1})\| \le M_\Theta(r)\left[|t_1 - t_2| + \sum_{i=1}^{p+1} \|\upsilon_i - \omega_i\|_\mu\right]$$

for all $\upsilon_1, \upsilon_2, \dots, \upsilon_{p+1}, \omega_1, \omega_2, \dots, \omega_{p+1} \in B_R(D^{\mu}_{T}, \tilde{\chi})$, and $t_1, t_2 \in [0, T]$, where $M_\Theta : R^+ \to R^+$ is a nondecreasing function.

(H3) Maps $\hat{J}_l : \Upsilon_\mu \to \Upsilon_\mu$, $l = 1, 2, \dots, q$ satisfy the following two conditions
- (i) $\|\hat{J}_l(\upsilon)\|_\mu \le C_l$
- (ii) $\|\hat{J}_l(\upsilon_1) - \hat{J}_l(\upsilon_2)\|_\mu \le L_l\|\upsilon_1 - \upsilon_2\|_\mu$,

for all $\upsilon, \upsilon_1, \upsilon_2 \in B_R(D^{\mu}_{T}, \tilde{\chi})$, C_l and L_l, $l = 1, 2, \dots, q$, are positive constants.

(H4) Continuous functions $e_i : [0, T] \to [0, T]$ satisfy the condition that $0 \le e_i(t) \le t$ for all $t \in [0, T]$ and for all $i = 1, 2, \dots, p$, where $p \in \mathbb{N}$.

Next, we prove some lemmas which will be used for proving the main results.

Lemma 2.1. *Problem (1.1) is equivalent to the fractional integral equation:*

$$\upsilon(t) = \begin{cases} \tilde{\chi}(t), \qquad t \in [-\tau, 0], \\ \\ \tilde{\chi}(0) + \sum\limits_{l=1}^{q} \hat{J}_l(\upsilon(t_l)) - \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} A\upsilon(s) ds \\ \quad + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \Theta(s, \upsilon(s), \upsilon(e_1(s)), \ldots, \upsilon(e_p(s))) ds, \\ \hfill t \in [0, T], \end{cases} \tag{2.1}$$

Proof. If $t \in [0, t_1]$, then

$$\begin{aligned} {}^C D_t^\gamma \upsilon(t) &= -A\upsilon(t) + \Theta(t, \upsilon(t), \upsilon(e_1(t)), \upsilon(e_2(t)), \ldots, \upsilon(e_p(t))) \\ \upsilon(0) &= \chi(0). \end{aligned}$$

Applying fractional integral on both sides, we get

$$\begin{aligned} \upsilon(t) + c_1 &= -\frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} A\upsilon(s) ds \\ &\quad + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \Theta(s, \upsilon(s), \upsilon(e_1(s)), \ldots, \upsilon(e_p(s))) ds. \end{aligned}$$

Putting $t = 0$, we obtain

$$c_1 = -\chi(0).$$

Hence, we have

$$\begin{aligned} \upsilon(t) &= \chi(0) - \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} A\upsilon(s) ds \\ &\quad + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \Theta(s, \upsilon(s), \upsilon(e_1(s)), \ldots, \upsilon(e_p(s))) ds. \end{aligned}$$

If $t \in (t_1, t_2]$, then

$$\begin{aligned} D^\gamma \upsilon(t) &= -A\upsilon(t) + \Theta(t, \upsilon(t), \upsilon(e_1(t)), u(e_2(t)), \ldots, u(e_p(t))) \\ \upsilon(t_1^+) &= \upsilon(t_1^-) + \hat{J}_1(\upsilon(t_1)). \end{aligned}$$

Applying fractional integral on both sides, we get

$$\begin{aligned} \upsilon(t) + c_2 &= -\frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} A\upsilon(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\gamma-1} \Theta(s, \upsilon(s), \upsilon(e_1(s)), \ldots, \upsilon(e_p(s))) ds. \end{aligned}$$

Using condition $\upsilon(t_1^+) = \upsilon(t_1^-) + \hat{J}_1(\upsilon(t_1))$, we get

$$c_2 = -\chi(0) - \hat{J}_1(\upsilon(t_1)).$$

Thus,

$$\begin{aligned} \upsilon(t) &= \chi(0) + \hat{J}_1(\upsilon(t_1)) - \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} A\upsilon(s) ds \\ &\quad + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\alpha-1} \Theta(s, \upsilon(s), \upsilon(e_1(s)), \ldots, \upsilon(e_p(s))) ds. \end{aligned}$$

Similarly, if $t \in (t_l, t_{l+1}]$, we have

$$\begin{aligned} \upsilon(t) &= \chi(0) + \sum_{l=1}^{q} \hat{J}_l(\upsilon(t_l)) - \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} A\upsilon(s)ds \\ &\quad + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \Theta(s, \upsilon(s), \upsilon(e_1(s)), \ldots, \upsilon(e_p(s)))ds. \end{aligned}$$

This proves the lemma. □

Lemma 2.2. *If (2.1) holds, then we have*

$$\upsilon(t) = \begin{cases} \chi(t), \qquad t \in [-\tau, 0], \\ \\ S_\gamma(t)\chi(0) + \sum_{l=1}^{q} S_\gamma(t-t_l)\hat{J}_l(\upsilon(t_l)) \\ +\gamma \int_0^t (t-s)^{\gamma-1} T_\gamma(t-s)\Theta(s, \upsilon(s), \upsilon(e_1(s)), \ldots, \upsilon(e_p(s)))ds, \\ \hfill t \in [0, T]. \end{cases} \tag{2.2}$$

where

$$S_\gamma(t) = \int_0^\infty \psi_\gamma(\theta) S(t^\gamma \theta) d\theta, \quad T_\gamma(t) = \int_0^\infty \theta \psi_\gamma(\theta) S(t^\gamma \theta) d\theta,$$

ψ_γ *is a probability function to be defined latter.*

Proof. Applying Laplace transform to (2.2), we get

$$\omega(\lambda) = \frac{1}{\lambda}\chi(0) + \sum_{l=1}^{q} \frac{e^{-t_l\lambda}}{\lambda} \hat{J}_l(\upsilon(t_l)) - \frac{1}{\lambda^\gamma} A\omega(\lambda) + \frac{1}{\lambda^\gamma} H(\lambda),$$

where

$$\omega(\lambda) = L[\upsilon(t)], \quad H(\lambda) = L[\Theta(t, \upsilon(t), \upsilon(e_1(t)), \ldots, \upsilon(e_p(t)))].$$

$$\begin{aligned} \omega(\lambda) &= \lambda^{\gamma-1}(\lambda^\gamma I + A)^{-1}\chi(0) \\ &\quad + \lambda^{\gamma-1} \sum_{l=1}^{q} e^{-t_l\lambda} (\lambda^\gamma I + A)^{-1} \hat{J}_l(\upsilon(t_l)) \\ &\quad + (\lambda^\gamma I + A)^{-1} H(\lambda) \\ &= \lambda^{\gamma-1} \int_0^\infty e^{-\lambda^\gamma s} S(s)\chi(0) ds \\ &\quad + \lambda^{\gamma-1} \sum_{l=1}^{q} e^{-t_l\lambda} \int_0^\infty e^{-\lambda^\gamma s} S(s) \hat{J}_l(\upsilon(t_l)) ds \\ &\quad + \int_0^\infty e^{-\lambda^\gamma s} S(s) H(\lambda) ds. \end{aligned} \tag{2.3}$$

Consider the one sided stable probability density

$$\rho_\gamma(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-n\gamma-1} \frac{\Gamma(n\gamma+1)}{n!} \sin(n\pi\gamma), \quad \theta \in (0, \infty),$$

and its Laplace transform

$$\int_0^\infty e^{-\lambda\theta} \rho_\gamma(\theta) d\theta = e^{-\lambda^\gamma}.$$

Take the probability density function

$$\psi_\gamma(\theta) = \frac{1}{\gamma}\theta^{-1-\frac{1}{\gamma}}\rho_\gamma(\theta^{-\frac{1}{\gamma}}).$$

Consider

$$\begin{aligned}
&\lambda^{\gamma-1}\int_0^\infty e^{-\lambda^\gamma s}S(s)\chi(0)ds \\
&= \int_0^\infty \gamma(\lambda s)^{\gamma-1}e^{-(\lambda s)^\gamma}S(s^\gamma)\chi(0)ds \\
&= \int_0^\infty \left(-\frac{1}{\lambda}\right)\frac{d}{ds}\left[e^{-(\lambda s)^\gamma}\right]S(s^\gamma)\chi(0)ds \\
&= \int_0^\infty \int_0^\infty \theta e^{-\lambda s\theta}\rho_\gamma(\theta)d\theta S(s^\gamma)\chi(0)ds \\
&= \int_0^\infty e^{-\lambda s}\left\{\int_0^\infty \psi_\gamma(\theta)S(s^\gamma\theta)d\theta\right\}\chi(0)ds, \qquad (2.4)
\end{aligned}$$

$$\begin{aligned}
&\lambda^{\gamma-1}\sum_{l=1}^{q}e^{-t_l\lambda}\int_0^\infty e^{-\lambda^\gamma s}S(s)\hat{J}_l(\upsilon(t_l))ds \\
&= \sum_{l=1}^{q}e^{-t_l\lambda}\int_0^\infty \gamma(\lambda s)^{\gamma-1}e^{-(\lambda s)^\gamma}S(s^\gamma)\hat{J}_l(\upsilon(t_l))ds \\
&= \sum_{l=1}^{q}e^{-t_l\lambda}\int_0^\infty \left(-\frac{1}{\lambda}\right)\frac{d}{ds}\left[e^{-(\lambda s)^\gamma}\right]S(s^\gamma)\hat{J}_l(\upsilon(t_l))ds \\
&= \sum_{l=1}^{q}\int_0^\infty \left\{\int_0^\infty \theta e^{-\lambda(s\theta+t_l)}\rho_\gamma(\theta)d\theta\right\}S(s^\gamma)\hat{J}_l(\upsilon(t_l))ds \\
&= \int_0^\infty e^{-\lambda s}\left\{\sum_{l=1}^{q}\left(\int_0^\infty \psi_\gamma(\theta)S(s-t_l)^\gamma\theta)d\theta\right)\hat{J}_l(\upsilon(t_l)\right\}ds, \qquad (2.5)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty e^{-\lambda^\gamma s}S(s)H(\lambda)ds \\
= \quad & \int_0^\infty \gamma s^{\gamma-1}e^{-(\lambda s)^\gamma}S(s^\gamma) \\
& \qquad \int_0^\infty e^{-\lambda s_1}\Theta(s_1,\upsilon(s_1),\upsilon(e_1(s_1)),\dots,\upsilon(e_p(s_1)))ds_1 ds \\
= \quad & \int_0^\infty\int_0^\infty \gamma s^{\gamma-1}\left\{\int_0^\infty e^{-\lambda s\theta}\rho_\gamma(\theta)d\theta\right\}S(t^\gamma)e^{-\lambda s_1} \\
& \qquad \Theta(s_1,\upsilon(s_1),\upsilon(e_1(s_1)),\dots,\upsilon(e_p(s_1)))ds_1 ds \\
= \quad & \int_0^\infty\int_0^\infty\int_0^\infty \gamma s^{\gamma-1}e^{-\lambda(s_1+s\theta)}\rho_\gamma(\theta)S(s^\gamma) \\
& \qquad \Theta(s_1,\upsilon(s_1),\upsilon(e_1(s_1)),\dots,\upsilon(e_p(s_1)))d\theta ds_1 ds \\
= \quad & \int_0^\infty\int_0^{s_1}\int_0^\infty \gamma e^{-\lambda s}\rho_\gamma(\theta)S\left(\frac{(s-s_1)^\gamma}{\theta^\gamma}\right)\frac{(s-s_1)^{\gamma-1}}{\theta^\gamma}
\end{aligned}$$

$$\Theta(s_1, \upsilon(s_1), \upsilon(e_1(s_1)), \ldots, \upsilon(e_p(s_1)))d\theta ds ds_1$$

$$= \int_0^\infty e^{-\lambda s}\left[\gamma\int_0^s (s-s_1)^{\gamma-1}\left\{\int_0^\infty \theta\psi_\gamma(\theta)S((s-s_1)^\gamma\theta)d\theta\right\}\right.$$

$$\left.\Theta(s_1, \upsilon(s_1), \upsilon(e_1(s_1)), \ldots, \upsilon(e_p(s_1)))ds_1\right]ds. \tag{2.6}$$

Putting the values of (2.4), (2.5) and (2.6) in (2.3), and by taking inverse Laplace transform, we get

$$\begin{aligned}\upsilon(t) &= S_\gamma(t)\chi(0) + \sum_{l=1}^{q} S_\gamma(t-t_l)\hat{J}_l(\upsilon(t_l)) \\ &\quad +\gamma\int_0^t (t-s)^{\gamma-1}T_\gamma(t-s)\Theta(s, \upsilon(s), \upsilon(e_1(s)), \ldots, \upsilon(e_p(s)))ds \\ &\qquad t\in[0,T].\end{aligned}$$

Hence proved the lemma. □

Definition 2.3. *A continuous function $\upsilon : [-\tau, T] \to \Upsilon$ is said to be a mild solution of the problem (1.1) if $\upsilon(t)$ satisfies the integral equation (2.2).*

Lemma 2.4. *([7]) The operators $S_\gamma(t),\ t\geq 0$ and $T_\gamma(t),\ t\geq 0$ satisfy*

(i) $\|S_\gamma(t)\| \leq K$

(ii) $\|A^\mu T_\gamma(t)\| \leq \frac{\Gamma(2-\mu)c_\mu}{\Gamma(1+\gamma(1-\mu))t^{\gamma\mu}}$.

3. Approximate solutions and Convergence

Let Υ_n be the finite dimensional subspace of Υ which is spanned by $\{\phi_0, \phi_1, \ldots, \phi_n\}$ and $P^n : \Upsilon \to \Upsilon_n$ be the corresponding projection operator for $n = 0, 1, 2, \ldots$. We define $\Theta_n : [0,T]\times\Upsilon^{p+1}\to\Upsilon$ and $\hat{J}_{l,n} : \Upsilon\to\Upsilon$ by

$$\Theta_n(t, \upsilon(t), \upsilon(e_1(t)), \ldots, \upsilon(e_p(t))) = \Theta(t, P^n\upsilon(t), P^n\upsilon(e_1(t)), \ldots, P^n\upsilon(e_p(t)))$$

and

$$\hat{J}_{l,n}(\upsilon) = \hat{J}_l(P^n\upsilon), \quad \forall \upsilon\in\Upsilon, \quad n = 0, 1, 2, \ldots.$$

Using $(H2)$, we have

$$\|\Theta(t, \upsilon_1, \upsilon_2, \ldots, \upsilon_{p+1})\| \leq L_\Theta(R)[T + (p+1)R] + M,$$

where $M = \Theta(0, \chi(0), \chi(0), \ldots, \chi(0))$.

We assume that

$$\|(S_\gamma(t) - I)\chi(0)\|_\mu + K\sum_{l=1}^{q} C_l \leq \frac{R}{2}, \quad t\in[0,T], \tag{3.1}$$

$$\frac{\Gamma(1-\mu)c_\mu[L_\Theta(R)\{T+(p+1)R\}+M]T^{\gamma(1-\mu)}}{(1-\mu)\Gamma(1+\gamma(1-\mu))} \leq \frac{R}{2} \tag{3.2}$$

and

$$\left(\frac{\gamma C_\mu\Gamma(2-\mu)L_\Theta(R)(p+1)T^{\gamma(1-\mu)}}{\gamma(1-\mu)\Gamma(1+\gamma(1-\mu))} + K\sum_{l=1}^{q} L_l\right) < 1 \tag{3.3}$$

$$t\in[0,T].$$

We define a map F_n on $B_R(D_T^\mu, \tilde{\chi})$ by

$$(F_n\upsilon)(t) = \begin{cases} \tilde{\chi}(t), \qquad t \in [-\tau, 0], \\ \\ S_\gamma(t)\chi(0) + \sum_{l=1}^{q} S_\gamma(t-t_l)\hat{I}_{l,n}(\upsilon(t_l)) \\ +\gamma\int_0^t (t-s)^{\gamma-1} T_\gamma(t-s)\Theta_n(s, \upsilon(s), \upsilon(e_1(s)), \ldots, \upsilon(e_p(s)))ds, \\ \hfill t \in [0, T]. \end{cases}$$

Theorem 3.1. *Suppose that the assumptions $(H1)-(H4)$ are satisfied. Then there exists a unique $\upsilon_n \in B_R(D_T^\mu, \tilde{\chi})$ such that $F_n\upsilon_n = \upsilon_n$ for each $n = 0, 1, 2, 3, \ldots$, i.e., υ_n satisfies the approximate integral equation*

$$\upsilon_n(t) = \begin{cases} \tilde{\chi}(t), \qquad t \in [-\tau, 0], \\ \\ S_\gamma(t)\chi(0) + \sum_{l=1}^{q} S_\gamma(t-t_l)\hat{I}_{l,n}(\upsilon_n(t_l)) \\ +\gamma\int_0^t (t-s)^{\gamma-1} T_\gamma(t-s)\Theta(s, \upsilon_n(s), \upsilon_n(e_1(s)), \ldots, \upsilon_n(e_p(s)))ds, \\ \hfill t \in [0, T]. \end{cases} \tag{3.4}$$

Proof. To prove this theorem, first we need to show that $F_n : B_R(D_T^\mu, \tilde{\chi}) \to B_R(D_T^\mu, \tilde{\chi})$. Clearly $F_n : B_R(D_T^\mu, \tilde{\chi}) \to D_T^\mu$. For $t \in [-\tau, 0]$, we have

$$(F_n\upsilon)(t) - \tilde{\chi}(t) = 0.$$

If $t \in (0, t_1]$, then

$$\begin{aligned} &\|(F_n\upsilon)(t) - \tilde{\chi}(t)\|_\mu \\ &\leq \|(S_\gamma(t) - I)\chi(0)\|_\mu \\ &\quad +\gamma\int_0^t \|(t-s)^{\gamma-1}T_\gamma(t-s)\Theta_n(s, \upsilon(s), \upsilon(e_1(s)), \ldots, \upsilon(e_p(s)))\|_\mu ds \\ &\leq \|(S_\gamma(t) - I)\chi(0)\|_\mu \\ &\quad +\gamma\int_0^t (t-s)^{\gamma-1}\|A^\mu T_\gamma(t-s)\| \\ &\qquad \|\Theta_n(s, \upsilon(s), \upsilon(e_1(s)), \ldots, \upsilon(e_p(s)))\| ds \\ &\leq \|(S_\gamma(t) - I)\chi(0)\|_\mu \\ &\quad +\frac{\Gamma(1-\mu)c_\mu[L_\Theta(R)\{T + (p+1)R\} + M]T^{\gamma(1-\mu)}}{(1-\mu)\Gamma(1+\gamma(1-\mu))}. \end{aligned}$$

If $t \in (t_1, t_2]$, then

$$\begin{aligned} &\|(F_n\upsilon)(t) - \tilde{\chi}(t)\|_\mu \\ &\leq \|(S_\gamma(t) - I)\chi(0)\|_\mu \\ &\quad +\frac{\Gamma(1-\mu)c_\mu[L_\Theta(R)\{T + (p+1)R\} + M]T^{\gamma(1-\mu)}}{(1-\mu)\Gamma(1+\gamma(1-\mu))} \\ &\quad +K\|\hat{J}_{1,n}(\upsilon(t_1))\|_\mu \\ &\leq \|(S_\gamma(t) - I)\chi(0)\|_\mu \\ &\quad +\frac{\Gamma(1-\mu)c_\mu[L_\Theta(R)\{T + (p+1)R\} + M]T^{\gamma(1-\mu)}}{(1-\mu)\Gamma(1+\gamma(1-\nu))} + KC_1. \end{aligned}$$

Similarly, if $t \in (t_l, t_{l+1}]$, then

$$
\begin{aligned}
&\|(F_n\upsilon)(t)-\tilde{\chi}(t)\|_\mu \\
&\qquad \leq \quad \|(S_\gamma(t)-I)\chi(0)\|_\mu \\
&\qquad\qquad +\frac{\Gamma(1-\mu)c_\mu[L_\Theta(R)\{T+(p+1)R\}+M]T^{\gamma(1-\mu)}}{(1-\mu)\Gamma(1+\gamma(1-\mu))}+K\sum_{l=1}^{q}C_l.
\end{aligned}
$$

Using (3.1), (3.2) and taking supremum over $[-\tau, T]$, we get

$$\|(F_n\upsilon)(t)-\tilde{\chi}(t)\|_{T,\mu}\leq R.$$

Thus, $F_n : B_R(D^\mu_T, \tilde{\chi}) \to B_R(D^\mu_T, \tilde{\chi})$.

Next for any $\upsilon, \omega \in B_R(C^\mu_T, \tilde{\chi})$ and $t \in [-\tau, 0]$, we have

$$(F_n\upsilon)(t)-(F_n\omega)(t)=0.$$

Let $\upsilon, \omega \in B_R(C^\mu_T, \tilde{\chi})$ and if $t \in (0, t_1]$, then

$$
\begin{aligned}
&\|F_n\upsilon(t)-F_n\omega(t)\|_\mu \\
&\qquad \leq \quad \gamma\int_0^t (t-s)^{\gamma-1}\|T_\gamma(t-s)\|_\mu L_\Theta(R)(p+1)\|\upsilon(s)-\omega(s)\|_\mu ds \\
&\qquad \leq \quad \frac{\gamma C_\mu\Gamma(2-\mu)L_\Theta(R)(p+1)T^{\gamma(1-\mu)}}{\gamma(1-\mu)\Gamma(1+\gamma(1-\mu))}\|u-v\|_{T,\mu}.
\end{aligned}
$$

If $t \in (t_1, t_2]$, then

$$
\begin{aligned}
&\|F_n\upsilon(t)-F_n\omega(t)\|_\mu \\
&\qquad \leq \quad K\|\hat{J}_{1,n}(\upsilon(t_1))-\hat{J}_{1,n}(\omega(t_1))\|_\mu \\
&\qquad\qquad +\frac{\gamma C_\mu\Gamma(2-\mu)L_\Theta(R)(p+1)T^{\gamma(1-\mu)}}{\gamma(1-\mu)\Gamma(1+\gamma(1-\mu))}\|\upsilon-\omega\|_{T,\mu} \\
&\qquad \leq \quad \left(KL_1+\frac{\gamma C_\mu\Gamma(2-\mu)L_\Theta(R)(p+1)T^{\gamma(1-\mu)}}{\gamma(1-\mu)\Gamma(1+\gamma(1-\mu))}\right)\|\upsilon-\omega\|_{T,\mu}.
\end{aligned}
$$

Similarly, if $t \in (t_l, t_{l+1}]$, then

$$
\begin{aligned}
&\|F_n\upsilon(t)-F_n\omega(t)\|_\mu \\
&\qquad \leq \quad \left(\frac{\gamma C_\mu\Gamma(2-\mu)L_\Theta(R)(m+1)T^{\gamma(1-\gamma)}}{\gamma(1-\mu)\Gamma(1+\gamma(1-\mu))}+K\sum_{l=1}^{q}L_l\right)\|\upsilon-\omega\|_{T,\mu}.
\end{aligned}
$$

Using the inequality (3.3) and taking the supremum over $[-\tau, T]$, we get

$$\|F_n\upsilon(t)-F_n\omega(t)\|_{T,\mu}<\|\upsilon-\omega\|_{T,\mu}.$$

It implies that F_n is a contraction mapping and therefore has a unique fixed point $\upsilon_n \in B_R(D^\mu_T, \tilde{\chi})$ i.e. $F_n\upsilon_n = \upsilon_n$ satisfying approximate integral equation (3.4). □

Lemma 3.2. *If $\chi(t) \in D(A)$ for all $t \in [-\tau, 0]$, then for any $t \in [-\tau, T]$, there exists a constant L_0, independent of n, such that*

$$\|A^\theta\upsilon_n(t)\|\leq L_0$$

for all $-\tau \leq t \leq T$ and $0 \leq \theta < 1$.

Proof. If $t \in [-\tau, 0]$, then

$$\|A^\theta\upsilon_n(t)\|=\|A^\theta\tilde{\chi}(t)\|\leq\|\chi\|_{0,\theta}.$$

If $t \in (0, t_1]$, then

$$\begin{aligned}
&\|A^\theta \upsilon_n(t)\| \\
&\leq \|A^\theta S_\gamma(t)\chi(0)\| + \gamma\int_0^t (t-s)^{\gamma-1}\|A^\theta T_\gamma(t-s)\| \\
&\qquad\qquad \|\Theta(s,\upsilon_n(s),\upsilon_n(e_1(s)),\ldots,\upsilon_n(e_p(s)))\|ds \\
&\leq K\|\chi(0)\|_\theta \\
&\quad + \frac{\gamma C\Gamma(2-\theta)}{\Gamma 1+\gamma(1-\theta)}\{L_\Theta(R)[T+(p+1)R]+M\}\int_0^t (t-s)^{\gamma-\gamma\theta-1}ds \\
&\leq K\|\chi(0)\|_\theta + \frac{\gamma C T^{\gamma(1-\theta)}\Gamma(2-\theta)}{\gamma(1-\theta)\Gamma(1+\gamma(1-\theta))}\{L_\Theta(R)[T+(p+1)R]+M\}.
\end{aligned}$$

If $t \in (t_1, t_2]$, then

$$\begin{aligned}
&\|A^\theta \upsilon_n(t)\| \\
&\leq K\|\chi(0)\|_\theta + \frac{\gamma C T^{\gamma(1-\theta)}\Gamma(2-\theta)}{\gamma(1-\theta)\Gamma(1+\gamma(1-\theta))}\{L_\Theta(R)[T+(p+1)R]+M\} \\
&\quad +KC_l.
\end{aligned}$$

Similarly, if $t \in (t_l, t_{l+1}]$, then

$$\begin{aligned}
\|A^\theta \upsilon_n(t)\| &\leq K\left(\|\chi(0)\|_\theta + \sum_{l=1}^{q} C_l\right) \\
&\quad + \frac{\gamma C T^{\gamma(1-\theta)}\Gamma(2-\theta)}{\gamma(1-\theta)\Gamma(1+\gamma(1-\theta))}\{L_\Theta(R)[T+(p+1)R]+M\}. \\
&= L_0.
\end{aligned}$$

This completes the proof of lemma. □

Theorem 3.3. *Suppose that the conditions (H1)-(H4) are satisfied and $\chi(t) \in D(A)$ for all $t \in [-\tau, 0]$. Then, $\{\upsilon_n\} \subset B_R(D_T^\gamma, \tilde{\chi})$ is a Cauchy sequence and converges to a unique solution $\upsilon \in B_R(D_T^\gamma, \tilde{\chi})$.*

Proof. If $t \in [-\tau, 0]$, then

$$\|\upsilon_{n_1}(t) - \upsilon_{n_2}(t)\|_\mu = 0.$$

If $t \in (0, t_1]$, then

$$\begin{aligned}
\|\upsilon_{n_1}(t) - \upsilon_{n_2}(t)\|_\mu &= \gamma\int_0^t (t-s)^{\gamma-1}\|A^\mu T_\gamma(t-s)\| \\
&\qquad \|\Theta_{n_1}(s,\upsilon_{n_1}(s),\upsilon_{n_1}(e_1(s)),\ldots,\upsilon_{n_1}(e_p(s))) \\
&\qquad\quad -\Theta_{n_2}(s,\upsilon_{n_2}(s),\upsilon_{n_2}(e_1(s)),\ldots,\upsilon_{n_2}(e_p(s)))\|ds.
\end{aligned}$$

We have

$$\begin{aligned}
&\|\Theta_{n_1}(s,\upsilon_{n_1}(s),\upsilon_{n_1}(e_1(s)),\ldots,\upsilon_{n_1}(e_p(s))) \\
&\qquad\qquad -\Theta_{n_2}(s,\upsilon_{n_2}(s),\upsilon_{n_2}(e_1(s)),\ldots,\upsilon_{n_2}(e_p(s)))\| \\
\leq\ &\|\Theta_{n_1}(s,\upsilon_{n_1}(s),\upsilon_{n_1}(e_1(s)),\ldots,\upsilon_{n_1}(e_p(s))) \\
&\qquad\qquad -\Theta_{n_1}(s,\upsilon_{n_2}(s),\upsilon_{n_2}(e_1(s)),\ldots,\upsilon_{n_2}(e_p(s)))\| \\
&+\|\Theta_{n_1}(s,\upsilon_{n_2}(s),\upsilon_{n_2}(e_1(s)),\ldots,\upsilon_{n_2}(e_p(s))) \\
&\qquad\qquad -\Theta_{n_2}(s,\upsilon_{n_2}(s),\upsilon_{n_2}(e_1(s)),\ldots,\upsilon_{n_2}(e_p(s)))\| \\
\leq\ &L_\Theta(R)(p+1)\|\upsilon_{n_1}(t) - \upsilon_{n_2}(t)\|_\mu
\end{aligned}$$

$$
\begin{aligned}
&\qquad +L_{\Theta}(R)(p+1)\|A^{\mu-\theta}(P^{n_1}-P^{n_2})A^{\theta}\upsilon_{n_2}(t)\| \\
&\leq L_{\Theta}(R)(p+1)\|\upsilon_{n_1}(t)-\upsilon_{n_2}(t)\|_{\mu}+\frac{L_{\Theta}(R)(p+1)}{\lambda_{n_1}^{\theta-\mu}}\|A^{\theta}\upsilon_{n_2}(t)\| \\
&\leq L_{\Theta}(R)(p+1)\|\upsilon_{n_1}(t)-\upsilon_{n_2}(t)\|_{\mu}+\frac{L_{\Theta}(R)(p+1)}{\lambda_{n_1}^{\theta-\mu}}L_0. \qquad (3.5)
\end{aligned}
$$

Using inequality (3.5) and Lemma 2.4, we get

$$
\begin{aligned}
\|\upsilon_{n_1}(t)-\upsilon_{n_2}(t)\|_{\mu} &\leq \frac{\gamma\Gamma(2-\mu)c_{\mu}}{\Gamma(1+\gamma(1-\mu))} \\
&\quad\times\int_0^t (t-s)^{\gamma-\gamma\mu-1}\Big[L_{\Theta}(R)(p+1)\|\upsilon_{n_1}(t)-\upsilon_{n_2}(t)\|_{\mu} \\
&\qquad +\frac{L_{\Theta}(R)(p+1)}{\lambda_{n_1}^{\theta-\mu}}L_0\Big]ds \\
&\leq \frac{\gamma\Gamma(2-\mu)c_{\mu}L_{\Theta}(R)(p+1)L_0T^{\gamma-\gamma\mu}}{\lambda_{n_1}^{\theta-\mu}(\gamma-\gamma\mu)\Gamma(1+\gamma(1-\mu))} \\
&\quad +\frac{\gamma\Gamma(2-\mu)c_{\mu}L_{\Theta}(R)(p+1)T^{\gamma-\gamma\mu}}{\gamma(1-\mu)\Gamma(1+\gamma(1-\mu))}\|\upsilon_{n_1}-\upsilon_{n_2}\|_{T,\mu}.
\end{aligned}
$$

If $t\in(t_1,t_2]$, then

$$
\begin{aligned}
\|\upsilon_{n_1}(t)-\upsilon_{n_2}(t)\|_{\mu} &\leq K\|\hat{J}_{1,n_1}(\upsilon_{n_1}(t))-\hat{J}_{1,n_2}(\upsilon_{n_2}(t))\|_{\mu} \\
&\quad +\frac{\gamma\Gamma(2-\mu)c_{\mu}L_{\Theta}(R)(p+1)L_0T^{\gamma-\gamma\mu}}{\lambda_{n_1}^{\theta-\mu}(\gamma-\gamma\mu)\Gamma(1+\gamma(1-\mu))} \\
&\quad +\frac{\gamma\Gamma(2-\mu)c_{\mu}L_{\Theta}(R)(p+1)T^{\gamma-\gamma\mu}}{\gamma(1-\mu)\Gamma(1+\gamma(1-\mu))}\|\upsilon_{n_1}-\upsilon_{n_2}\|_{T,\mu} \\
&\leq KL_l\left[\|\upsilon_{n_1}(t)-\upsilon_{n_2}(t)\|_{\mu}+\frac{L_0}{\lambda_{n_1}^{\theta-\mu}}\right] \\
&\quad +\frac{\gamma\Gamma(2-\mu)c_{\mu}L_{\Theta}(R)(p+1)L_0T^{\gamma-\gamma\mu}}{\lambda_{n_1}^{\theta-\mu}(\gamma-\gamma\mu)\Gamma(1+\gamma(1-\mu))} \\
&\quad +\frac{\gamma\Gamma(2-\mu)c_{\mu}L_{\Theta}(R)(p+1)T^{\gamma-\gamma\mu}}{\gamma(1-\mu)\Gamma(1+\gamma(1-\mu))}\|\upsilon_{n_1}-\upsilon_{n_2}\|_{T,\mu} \\
&\leq \left[\frac{\gamma\Gamma(2-\mu)c_{\mu}L_{\Theta}(R)(p+1)T^{\gamma-\gamma\mu}}{\gamma(1-\mu)\Gamma(1+\gamma(1-\mu))}+KL_l\right] \\
&\quad \times\|\upsilon_{n_1}-\upsilon_{n_2}\|_{T,\mu} \\
&\quad +\left[\frac{\gamma\Gamma(2-\mu)c_{\mu}L_{\Theta}(R)(p+1)T^{\gamma(1-\mu)}}{\gamma(1-\mu)\Gamma(1+\gamma(1-\mu))}+L_l\right]\frac{L_0}{\lambda_{n_1}^{\theta-\mu}}.
\end{aligned}
$$

If $t\in(t_l,t_{l+1}]$, then

$$
\begin{aligned}
&\|\upsilon_{n_1}(t)-\upsilon_{n_2}(t)\|_{\mu} \\
&\quad\leq \left[\frac{\gamma\Gamma(2-\mu)c_{\mu}L_{\Theta}(R)(p+1)T^{\gamma(1-\mu)}}{(\gamma-\gamma\mu)\Gamma(1+\gamma(1-\mu))}+K\sum_{l=1}^{q}L_l\right]\|\upsilon_{n_1}-\upsilon_{n_2}\|_{T,\mu} \\
&\qquad +\left[\frac{\gamma\Gamma(2-\mu)c_{\mu}L_{\Theta}(R)(p+1)T^{\gamma(1-\mu)}}{(\gamma-\gamma\mu)\Gamma(1+\gamma(1-\mu))}+\sum_{l=1}^{q}L_l\right]\frac{L_0}{\lambda_{n_1}^{\theta-\mu}}.
\end{aligned}
$$

Taking supremum over $[-\tau, T]$ and after some simplifications, we get

$$\|\upsilon_{n_1}(t)-\upsilon_{n_2}(t)\|_\mu \leq \frac{L_0(\gamma-\gamma\mu)\Gamma(1+\gamma(1-\mu))\sum_{l=1}^{q} L_l + Q}{K\gamma(1-\mu)\Gamma(1+\gamma(1-\mu))\sum_{l=1}^{q} L_l + Q} \times \frac{L_0}{\lambda_{n_1}^{\theta-\mu}},$$

where $Q = \gamma\Gamma(2-\mu)c_\gamma L_\Theta(R)(p+1)T^{\gamma(1-\mu)}$. Since $\frac{1}{\lambda_{n_1}^{\theta-\mu}} \to 0$ as $n_1 \to \infty$. This proves $\{\upsilon_n\}$ is a Cauchy sequence and therefore converges to a unique $\upsilon \in B_R(D_T^\gamma, \tilde{\chi})$. This completes the proof of theorem.

With the help of Theorem 3.1 and Theorem 3.3, we have the following existence and uniqueness result.

Theorem 3.4. *Suppose that the conditions (H1)-(H4) are satisfied and $h(t) \in D(A)$ for all $t \in [-\tau, 0]$. Then, υ_n given by (3.4) converges in $B_R(D_T^\gamma, \tilde{\chi})$ to a unique solution $\upsilon \in B_R(D_T^\gamma, \tilde{\chi})$ of (2.2).*

4. Faedo-Galerkin Approximations

We know from the previous sections that there exists a unique $\upsilon \in B_R(D_T^\mu, \tilde{\chi})$, satisfying

$$\upsilon(t) = \begin{cases} \tilde{\chi}(t), \qquad t \in [-\tau, 0], \\ S_\gamma(t)\chi(0) + \sum_{l=1}^{q} S_\gamma(t-t_l)\hat{J}_l(\upsilon(t_l)) \\ +\gamma\int_0^t (t-s)^{\gamma-1}T_\gamma(t-s)\Theta(s,\upsilon(s),\upsilon(e_1(s)),\ldots,\upsilon(e_p(s)))ds, \\ \hfill t \in [0, T]. \end{cases} \tag{4.1}$$

Also, there is a unique solutions $\upsilon_n \in B_R(D_T^\mu, \tilde{\chi})$ of the approximation integral equations

$$\upsilon_n(t) = \begin{cases} \tilde{\chi}(t), \qquad t \in [-\tau, 0], \\ S_\gamma(t)\chi(0) + \sum_{l=1}^{q} S_\gamma(t-t_l)\hat{J}_{l,n}(\upsilon_n(t_l)) \\ +\gamma\int_0^t (t-s)^{\gamma-1}T_\gamma(t-s)\Theta(s,\upsilon_n(s),\upsilon_n(e_1(s)),\ldots,\upsilon_n(e_p(s)))ds, \\ \hfill t \in [0, T]. \end{cases}$$

The Faedo-Galerkin approximation of solution to the problem (1.1) is given by $\bar{\upsilon}_n = P^n\upsilon_n$ satisfying

$$\bar{\upsilon}_n(t) = \begin{cases} P^n\tilde{\chi}(t), \qquad t \in [-\tau, 0], \\ S_\gamma(t)P^n\chi(0) + \sum_{l=1}^{q} S_\gamma(t-t_l)P^n\hat{J}_{l,n}(\upsilon_n(t_l)) \\ +\gamma\int_0^t (t-s)^{\gamma-1}T_\gamma(t-s)P^n\Theta(s,\upsilon_n(s),\upsilon_n(e_1(s)),\ldots,\upsilon_n(e_p(s)))ds, \\ \hfill t \in [0, T]. \end{cases} \tag{4.2}$$

Solutions $\upsilon(t)$ and $\bar{\upsilon}_n(t)$ which are given by (4.1) and (4.2) respectively, have the following representation

$$\upsilon(t) = \sum_{i=0}^{\infty}\gamma_i(t)\phi_i, \quad \gamma_i(t) = \langle \upsilon(t), \phi_i\rangle \quad \text{for all} \quad i = 0, 1, 2, 3, \ldots,$$

and

$$\bar{\upsilon}_n(t) = \sum_{i=0}^{\infty} \gamma_i^n(t)\phi_i, \quad \gamma_i^n(t) = \langle \bar{\upsilon}_n(t), \phi_i \rangle \quad \text{for all} \quad i = 0, 1, 2, 3, \dots. \tag{4.3}$$

Using (4.3) in (4.2), we get

$$\begin{aligned} D^\gamma \gamma_i^n(t) + \lambda_i \gamma_i^n(t) &= F_i^n(t, \gamma_0^n(t), \cdots, \gamma_n^n(t), \gamma_0^n(e_1(t)), \dots, \gamma_n^n(e_1(t)), \\ &\qquad \cdots, \gamma_0^n(e_p(t)), \dots, \gamma_n^n(e_p(t))) \\ \Delta \gamma_i^n(t_l) &= \hat{J}_{li}^n(\gamma_i^n(t_l)), \\ \gamma_i^n(t) &= \psi_i(t), \end{aligned}$$

where

$$F_i^n = \left\langle \Theta\left(t, \sum_{i=0}^{n} \gamma_i^n(t)\upsilon_i, \sum_{i=0}^{n} \gamma_i^n(e_1(t))\upsilon_i, \dots, \sum_{i=0}^{n} \gamma_i^n(e_p(t))\upsilon_i\right), \phi_i \right\rangle,$$

$$\hat{J}_{li}^n(\gamma_i^n(t_l)) = \left\langle \hat{J}_l(\upsilon_n(t_l)), \phi_i \right\rangle, \quad \psi_i = \langle \tilde{\chi}(t), \phi_i \rangle.$$

As a consequence of Theorem 3.1 and Theorem 3.3, we have the following convergence result.

Theorem 4.1. *Suppose that the conditions (H1)-(H4) are satisfied and* $\chi(t) \in D(A)$ *for all* $t \in [-\tau, 0]$*. Then there exists a unique* $\bar{\upsilon}_n \in B_R(C_t^\gamma, \tilde{\chi})$ *satisfying (4.2) and* $\upsilon \in B_R(D_T^\gamma, \tilde{\chi})$ *satisfying (4.1) such that* $\bar{\upsilon}_n \to \upsilon$ *as* $n \to \infty$ *in* $B_R(D_T^\gamma, \tilde{\chi})$.

5. Application

Example 5.1. *Consider the following problem:*

$$\begin{cases} D^\alpha w(t,y) - \frac{\partial^2}{\partial y^2} w(t,y) = F(t, w(t,y), w(t-\tau_1, y), \dots, w(t-\tau_p, y)), \\ \qquad\qquad \tau_i > 0 \ (i = 1, 2, \dots, p), \\ w(t,0) = w(t,\pi) = 0, \quad t \in [0,T], \quad 0 < T < \infty, \\ \Delta w(t_l, y) = \frac{2w(t_l,y)}{2+w(t_l,y)}, \quad l = 1, 2, \dots, q, \\ \qquad y \in (0,\pi), \\ w(t,y) = g(t,y), \quad t \in [-\tau, 0], \quad y \in (0,\pi), \end{cases} \tag{5.1}$$

where F, I_l $l = 1, 2, \dots, q$, and g are real valued, sufficiently smooth functions. $\Delta w(t_l, y) = w(t_l^+, y) - w(t_l^-, y)$, where $w(t_l^+, y)$ and $w(t_l^-, y)$ are respectively the right and left hand limit of w at $(t,y) = (t_l, y)$. We assume that function F satisfies the following condition:

$$|F(t, \upsilon_1, \upsilon_2, \dots, \upsilon_{p+1}) - F(s, \omega_1, \omega_2, \dots, \omega_{p+1})| \leq L\left[|t-s| + \sum_{i=1}^{p+1} |\upsilon_i - \omega_i|\right].$$

Let $H = L^2(0,\pi)$. We define an operator A by

$$A\upsilon = -\upsilon''$$

with the domain

$$D(A) = \{\upsilon(\cdot) \in L^2(0,\pi) \ : \ \upsilon'' \in L^2(0,\pi), \upsilon(0) = \upsilon(\pi) = 0\}.$$

It can be easily proved that $-A$ is the infinitesimal generator of an analytic semigroup.

If we take $\gamma = \frac{1}{2}$, then $D(A^{\frac{1}{2}})$ which is denoted by $\Upsilon_{\frac{1}{2}}$ is the Banach space endowed with the norm

$$\|y\|_{\frac{1}{2}} = \|A^{\frac{1}{2}}y\|, \quad x \in D(A^{\frac{1}{2}}).$$

Also, for $t \in [0,T]$, we define $D_t^{\frac{1}{2}} = \{\upsilon : \upsilon$ is a map from $[-\tau, t]$ into $\Upsilon_{\frac{1}{2}}$ such that $\upsilon(t)$ is continuous at $t \neq t_l$, left continuous at $t = t_l$ and the right limit $\upsilon(t_l^+)$ exists for $l = 1, 2, \dots, q\}$.

The spectrum of A is given by $A\upsilon = -\upsilon'' = \alpha\upsilon$. The general solution υ of $A\upsilon = \alpha\upsilon$ is

$$\upsilon(y) = C\cos(\sqrt{\alpha}y) + D\sin(\sqrt{\alpha}y).$$

Using the boundary conditions $\upsilon(0) = \upsilon(\pi) = 0$, we obtain $C = 0$, $\alpha = \alpha_n = n^2$, $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$, the solution is given by $\upsilon_n(y) = D\sin ny$. If we take $D = \frac{\sqrt{2}}{\sqrt{\pi}}$, then $\langle \upsilon_n, \upsilon_m\rangle = 0$ for $n \neq m$, and $\langle \upsilon_n, \upsilon_m\rangle = 1$ for $n = m$. Thus A has pure point spectrum and the eigenvectors υ_n are orthonormal.

Here, $\hat{J}_l(w(t_l, y)) = \frac{2w(t_l,y)}{2+w(t_l,y)}$. If we define $\upsilon(t)(y) = w(t, y)$ and $\hat{J}_l(w(t_l, y)) = \hat{J}_l(\upsilon(t_l))(y)$, then $\hat{J}_l(\upsilon(t_l)) = \frac{2\upsilon(t_l)}{2+\upsilon(t_l)}$. For $\upsilon_1, \upsilon_2 \in D(A^{\frac{1}{2}})$, we have

$$\|\hat{J}_l(\upsilon_1) - \hat{J}_l(\upsilon_2)\|_{\frac{1}{2}} \leq \|\upsilon_1 - \upsilon_2\|_{\frac{1}{2}}$$

and

$$\|\hat{J}_l(\upsilon_1)\|_{\frac{1}{2}} \leq \|\upsilon_1\|_{\frac{1}{2}}.$$

If we define

$$\begin{gathered} \upsilon(t)(y) = w(t,y),\ e_i(t) = t - \tau_i, \quad i = 1, 2, \dots, p, \\ f(t, \upsilon(t), \upsilon(t-\tau_1), \dots, \upsilon(t-\tau_p))(y) \\ = F(t, w(t,y), w(t-\tau_1, y), \dots, w(t-\tau_1, y), \dots, w(t-\tau_p, y)). \\ \hat{J}_l(w(t_l, y)) = \hat{J}_l(\upsilon(t_l))(y), \text{ and } \chi(t)(y) = g(t,y), \end{gathered}$$

then problem (5.1) reduces to:

$$\begin{aligned} D^\gamma \upsilon(t) + A\upsilon(t) &= \Theta(t, \upsilon(t), \upsilon(e_1(t)), \upsilon(e_2(t)), \dots, \upsilon(e_p(t))), \\ & \qquad\qquad t \in [0,T] \\ \Delta\upsilon(t_l) &= \hat{J}_l(\upsilon(t_l)), \quad l = 1, 2, \dots, q \\ \upsilon(t) &= \chi(t), \quad t \in [-\tau, 0]. \end{aligned}$$

Next, we show that Θ satisfies the condition $(H2)$

$$\begin{aligned} & \|\Theta(t, \upsilon_1, \upsilon_2, \dots, \upsilon_{p+1}) - \Theta(t, \omega_1, \omega_2, \dots, \omega_{p+1})\|_{L^2} \\ \leq\ & L\Big[\int_0^\pi |\Theta(t, \upsilon_1(y,t), \upsilon_2(y,t), \dots, \upsilon_{p+1}(y,t) \\ & \qquad -\Theta(t, \omega_1(y,t), \omega_2(y,t), \dots, \omega_{p+1}(y,t))|^2 dy\Big]^{\frac{1}{2}} \\ \leq\ & L\Big[\int_0^\pi \left\{|t-s|^2 + |\upsilon_1(y,t) - \omega_1(y,t)|^2 + \dots + |\upsilon_{p+1}(y,t) - \omega_{p+1}|^2\right\} dy\Big]^{\frac{1}{2}} \\ \leq\ & 2L\Big[|t-s| + \|\upsilon_1 - \omega_1\|_{L^2} + \|\upsilon_2 - \omega_2\|_{L^2} + \dots + \|\upsilon_{p+1} - \omega_{p+1}\|_{L^2}\Big]. \end{aligned}$$

Hence $(H2)$ holds.

Thus, all the assumptions of Theorem 4.1 are satisfied. Therefore, Theorem 4.1 guarantees the existence of Faedo-Galerkin approximations and their convergence to the unique solution of (5.1).

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