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CONVOLUTION PROPERTIES FOR CERTAIN SUBCLASSES OF MEROMORPHIC p-VALENT FUNCTIONS

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Abstract. In the present paper we introduce two subclasses $\mathcal{MS}^*_{p,q}(b;A,B)$ and $\mathcal{MK}_{p,q}(b;A,B)$ of meromorphic multivalent functions by using q-derivative operator defined in the punctured unit disc. Also, we derive several properties including convolution properties, the necessary and sufficient condition and coefficient estimates for these subclasses.

1. Introduction

Recently, the concept of q-calculus has magnitize a significant exertion of researchers due to its application in numerous branches of mathematics and physics. The q-calculus is an ordinary calculus without notion of limit point. Jackson [6, 7, 8] introduced and studied the q-derivative and q-integral. By making use of q-calculus various functions classes in Geometric Function Theory are introduced and investigated from different view points and perspectives (see [1], [11], [15], [16], [17], [19] and references therein). Purpose of this paper is to introduce and study two subclasses of p-valent meromorphic functions by applying q-derivative operators in conjuction with the principle of subordinations.

Let Σ_p denote the class of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \qquad (p \in \mathbb{N}), \tag{1}$$

which are analytic and p-valent in the punctured unit disc $\mathbb{U}^* = \mathbb{U}\backslash\{0\}$, where $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. Let g and f be two analytic functions in $\mathbb{U}$, then function g is said to be subordinate to f if there exists an analytic function w in the unit disk $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $g(z) = f(w(z))$ $(z \in \mathbb{U})$. We denote this subordination by $g \prec f$. In particular, if the function f is univalent in $\mathbb{U}$ the above subordination is equivalent to $g(0) = f(0)$ *and* $g(\mathbb{U}) \subseteq f(\mathbb{U})$.

For $0 < q < 1$, the q-derivative of a function f is defined by (see [5, 6, 7, 8])

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \qquad (z \in \mathbb{U}), \tag{2}$$

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provided that $f^{'}(0)$ exists.
From (2), it can be easily obtain that

$$D_q f(z) = \frac{-[p]_q}{q^p z^{p+1}} + \sum_{k=1}^{\infty} [k-p]_q a_k z^{k-p-1},$$

where

$$[k]_q = \frac{1-q^k}{1-q}.$$

As q$\to 1^-$, $[k]_q \to k$ and $\lim_{q\to 1^-} D_q f(z) = f^{'}(z)$. Also, we have

$$[k+p]_q = [k]_q + q^k [p]_q = q^p [k]_q + [p]_q,$$
$$[k-p]_q = q^{-p}[k]_q - q^{-p}[p]_q,$$
$$[0]_q = 0, [1]_q = 1.$$

For $f \in \Sigma_p$ given by (1) and $g \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \qquad (p \in \mathbb{N}),$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z).$$

Motivated essentially due to the work of Aouf et *al*. [3], Seoudy [13], Seoudy and Aouf [14] and Srivastava and Zayed [18], we define the following two subclasses of Σ_p by using the q-derivative operator D_q and the principle of subordination between analytic functions:

Definition 1 Let $0 < q < 1$, $-1 \le B < A \le 1$ and $b \in \mathbb{C}\backslash\{0\}$. A function f belonging to Σ_p is said to be in the class $\mathcal{MS}^*_{p,q}(b; A, B)$ if it satiesfies

$$1 - \frac{1}{b}\left[\frac{zD_q f(z)}{f(z)} + \frac{[p]_q}{q^p}\right] \prec \frac{1+Az}{1+Bz}. \tag{3}$$

Definition 2 Let $0 < q < 1$, $-1 \le B < A \le 1$ and $b \in \mathbb{C}\backslash\{0\}$. A function f belonging to Σ_p is said to be in the class $\mathcal{MK}_{p,q}(b; A, B)$ if it satiesfies

$$1 - \frac{1}{b}\left[\frac{D_q\big(zD_q f(z)\big)}{D_q f(z)} + \frac{[p]_q}{q^p}\right] \prec \frac{1+Az}{1+Bz}. \tag{4}$$

We also verify from both above definitions that

$$f \in \mathcal{MK}_{p,q}(b; A, B) \Leftrightarrow -\frac{q^p}{[p]_q} zD_q f \in \mathcal{MS}^*_{p,q}(b; A, B). \tag{5}$$

It may be pointed out here that, these classes generalizes several previously studied function classes. We deem it proper to demonstrate briefly the relevant connections with some of the well-known classes. Indeed, we have

(i)$\lim_{q\to 1^-} \mathcal{MS}^*_{1,q}(b; 1, -1) = \Sigma\mathcal{S}(b)$ and $\lim_{q\to 1^-} \mathcal{MK}_{1,q}(b; 1, -1) = \Sigma\mathcal{K}(b)$ (see [2]);

(ii)$\lim_{q\to 1^-} \mathcal{MS}^*_{1,q}(b; A, B) = \Sigma\mathcal{S}^*_0(b; A, B)$ and $\lim_{q\to 1^-} \mathcal{MK}_{1,q}(b; A, B) = \Sigma\mathcal{K}_0(b; A, B)$ (see [3]);

(iii)$\lim_{q\to 1^-} \mathcal{MS}^*_{1,q}(b; A, B) = \Sigma\mathcal{S}^*(b; A, B)$ and $\lim_{q\to 1^-} \mathcal{MK}_{1,q}(b; A, B) = \Sigma\mathcal{K}(b; A, B)$ (see [4]);

(iv)$\lim_{q\to 1^-} \mathcal{MS}^*_{1,q}[(1-\alpha)e^{-\iota\mu} cos\mu; 1, -1] = \Sigma\mathcal{S}^\mu_1(\alpha)$ and $\lim_{q\to 1^-} \mathcal{MK}_{1,q}[(1-\alpha)e^{-\iota\mu} cos\mu; 1, -1]$

$=\Sigma\mathcal{K}_1^{\mu}(\alpha)(\mu\in\mathbb{R},\,|\mu|<\frac{\pi}{2},\,0\leq\alpha<1)$ (see [12]).

In the present investigations, we derive several properties including convolution properties, the necessary and sufficient condition and coefficient estimates for functions belonging to the subclasses $\mathcal{MS}^*_{p,q}(b;A,B)$ and $\mathcal{MK}_{p,q}(b;A,B)$. The inspiration of this paper is to renovate and generalize already known results.

2. Main Results

Unless otherwise mentioned, we assume throughout this section that $0<q<1$, $-1\leq B<A\leq 1$, $b\in\mathbb{C}\backslash\{0\}$ and $\theta\in[0,2\pi)$.

Theorem 1 If $f\in\Sigma_p$, then $f\in\mathcal{MS}^*_{p,q}(b;A,B)$ if and only if

$$z^p\left[f(z)*\frac{1+\big[M(\theta)-q\big]z}{z^p(1-z)(1-qz)}\right]\neq 0 \qquad (z\in\mathbb{U}^*), \tag{6}$$

where

$$M(\theta)=\frac{e^{-\iota\theta}+B}{(A-B)bq^p}. \tag{7}$$

Proof. It is easy to verify that for any function $f\in\Sigma_p$

$$f(z)*\frac{1}{z^p(1-z)}=f(z) \tag{8}$$

and

$$f(z)*\frac{1-\Big(q+\frac{1}{[p]_q}\Big)z}{z^p(1-z)(1-qz)}=-\frac{q^p}{[p]_q}zD_qf(z). \tag{9}$$

First, if $f\in\mathcal{MS}^*_{p,q}(b;A,B)$, in order to prove that (6) holds we will write (3) by using the definition of the subordination, that is

$$-\frac{q^p}{[p]_q}\frac{zD_qf(z)}{f(z)}=\frac{1+\Big[B+(A-B)b\frac{q^p}{[p]_q}\Big]w(z)}{1+Bw(z)} \qquad (z\in\mathbb{U}^*),$$

where w is a Schwarz function, hence

$$z^p\left[-q^p\big(1+Be^{\iota\theta}\big)zD_qf(z)-\Big\{[p]_q+\big(B[p]_q+(A-B)bq^p\big)e^{\iota\theta}\Big\}f(z)\right]\neq 0 \qquad (z\in\mathbb{U}^*). \tag{10}$$

Now from (8) and (9), we may write (10) as

$$z^p\Bigg[\big(1+Be^{\iota\theta}\big)\left(f(z)*\frac{\big\{1-\big(q+\frac{1}{[p]_q}\big)z\big\}[p]_q}{z^p(1-z)(1-qz)}\right)$$
$$-\{[p]_q+\big(B[p]_q+(A-B)bq^p\big)e^{\iota\theta}\}\left(f(z)*\frac{1}{z^p(1-z)}\right)\Bigg]\neq 0 \qquad (z\in\mathbb{U}^*),$$

which is equivalent to

$$z^p\left[f(z)*\frac{1+\Big(-q+\frac{1+Be^{\iota\theta}}{(A-B)bq^pe^{\iota\theta}}\Big)z}{z^p(1-z)(1-qz)}\big[-(A-B)bq^pe^{\iota\theta}\big]\right]\neq 0$$

or

$$z^p\left[f(z)*\frac{1+\Big(\frac{e^{-\iota\theta}+B}{(A-B)bq^p}-q\Big)z}{z^p(1-z)(1-qz)}\right]\neq 0 \qquad (z\in\mathbb{U}^*),$$

which leads to (6), which proves the necessary part of Theorem 1.
Reversely, suppose that $f \in \Sigma_p$ satisfy the condition (6). Since it was shown in the first part of the proof that assumption (6) is equivalent to (10), we obtain that

$$-\frac{q^p}{[p]_q}\frac{zD_qf(z)}{f(z)} \neq \frac{1+\left[B+(A-B)b\frac{q^p}{[p]_q}\right]e^{\iota\theta}}{1+Be^{\iota\theta}} \qquad (z \in \mathbb{U}^*), \qquad (11)$$

and let us assume that

$$\varphi(z) = -\frac{q^p}{[p]_q}\frac{zD_qf(z)}{f(z)} \quad and \quad \psi(z) = \frac{1+\left[B+(A-B)b\frac{q^p}{[p]_q}\right]z}{1+Bz}.$$

The relation (11) means that

$$\varphi(\mathbb{U}^*) \cap \psi(\partial\mathbb{U}^*) = \emptyset.$$

Thus, the simply connected domain is included in a connected component of $\mathbb{C}\backslash\psi(\partial\mathbb{U}^*)$. Therefore, using the fact that $\varphi(0) = \psi(0)$ and the univalence of the function ψ, it follows that $\varphi(z) \prec \psi(z)$, which implies that $f \in \mathcal{MS}^*_{p,q}(b; A, B)$. Thus, the proof of Theorem 1 is completed.

Theorem 2 If $f \in \Sigma_p$, then $f \in \mathcal{MK}_{p,q}(b; A, B)$ if and only if

$$z^p\left[f(z) * \frac{1-\left[\frac{1+M(\theta)}{[p]_q}-\left(M(\theta)-q-q^2\right)\right]z-\left(M(\theta)-q\right)\left(q+\frac{1}{[p]_q}\right)qz^2}{z^p(1-z)(1-qz)(1-q^2z)}\right] \neq 0 \qquad (z \in \mathbb{U}^*), \qquad (12)$$

where M(θ) is given by (7).

Proof. From (5) it follows that $f \in \mathcal{MK}_{p,q}(b; A, B)$ if and only if $-\frac{q^p}{[p]_q}zD_qf \in \mathcal{MS}^*_{p,q}(b; A, B)$. Then from Theorem 1, the function $-\frac{q^p}{[p]_q}zD_qf \in \mathcal{MS}^*_{p,q}(b; A, B)$ if and only if

$$z^p[-\frac{q^p}{[p]_q}zD_qf * g(z)] \neq 0 \qquad (z \in \mathbb{U}^*), \qquad (13)$$

where

$$g(z) = \frac{1+[M(\theta)-q]z}{z^p(1-z)(1-qz)}.$$

On a basic computation we note that

$$\begin{aligned} D_qg(z) &= \frac{g(qz)-g(z)}{(q-1)z} \\ &= \frac{-[p]_q+\left[1+M(\theta)-[p]_q\left(M(\theta)-q-q^2\right)\right]z+\left(M(\theta)-q\right)\left(q[p]_q+1\right)qz^2}{q^pz^{p+1}(1-z)(1-qz)(1-q^2z)} \end{aligned}$$

and therefore

$$-\frac{q^p}{[p]_q}zD_qg(z) = \frac{1-\left[\frac{1+M(\theta)}{[p]_q}-\left(M(\theta)-q-q^2\right)\right]z-\left(M(\theta)-q\right)\left(q+\frac{1}{[p]_q}\right)qz^2}{z^p(1-z)(1-qz)(1-q^2z)}.$$

Using the above relation and the identity

$$\left(-\frac{q^p}{[p]_q}zD_qf(z)\right) * g(z) = f(z) * \left(-\frac{q^p}{[p]_q}zD_qg(z)\right),$$

it is simple to check that (13) is identical to (12). Thus, the proof of Theorem 2 is completed.

Theorem 3 A necessary and sufficient condition for the function f defined by (1) to be in the class $\mathcal{MS}^*_{p,q}(b;A,B)$ is that

$$1+\sum_{k=1}^{\infty}\frac{\big(e^{-\iota\theta}+B\big)[k]_q+(A-B)bq^p}{(A-B)bq^p}a_kz^k\neq 0 \qquad (z\in\mathbb{U}^*). \tag{14}$$

Proof. From Theorem 1, we find that $f\in\mathcal{MS}^*_{p,q}(b;A,B)$ if and only if (6) holds. Since

$$\frac{1}{z^p(1-z)(1-qz)}=\frac{1}{z^p}+(1+q)z^{1-p}+(1+q+q^2)z^{2-p}+(1+q+q^2+q^3)z^{3-p}+\cdots, \quad (z\in\mathbb{U}^*),$$

hence

$$\frac{1+\big[M(\theta)-q\big]z}{z^p(1-z)(1-qz)}=\frac{1}{z^p}+\sum_{k=1}^{\infty}\Big(1+M(\theta)[k]_q\Big)z^{k-p},$$

where $M(\theta)$ is given by (7).

Now a simple computation shows that (6) is identical to (14). Thus, the proof of Theorem 3 is completed.

Theorem 4 A necessary and sufficient condition for the function f defined by (1) to be in the class $\mathcal{MK}_{p,q}(b;A,B)$ is that

$$1+\sum_{k=1}^{\infty}\frac{\big(e^{-\iota\theta}+B\big)[k]_q+(A-B)bq^p}{(A-B)bq^p}\left(1-\frac{[k]_q}{[p]_q}\right)a_kz^k\neq 0 \qquad (z\in\mathbb{U}^*). \tag{15}$$

Proof. From Theorem 2, we find that $f\in\mathcal{MK}_{p,q}(b;A,B)$ if and only if (12) holds.

Since

$$\frac{1}{z^p(1-z)(1-qz)(1-q^2z)}=\frac{1}{z^p}+(1+q+q^2)z^{1-p}+(1+q+2q^2+q^3+q^4)z^{2-p}$$
$$+(1+q+2q^2+2q^3+2q^4+q^5+q^6)z^{3-p}+\cdots, \qquad (z\in\mathbb{U}^*),$$

hence

$$\frac{1-\Big[\frac{1+M(\theta)}{[p]_q}-(M(\theta)-q-q^2)\Big]z-\Big(M(\theta)-q\Big)\Big(q+\frac{1}{[p]_q}\Big)qz^2}{z^p(1-z)(1-qz)(1-q^2z)}$$
$$=\frac{1}{z^p}+\sum_{k=1}^{\infty}\Big(1+M(\theta)[k]_q\Big)\left(1-\frac{[k]_q}{[p]_q}\right)z^{k-p} \qquad (z\in\mathbb{U}^*),$$

where $M(\theta)$ is given by (7).

Now a simple computation shows that (12) is identical to (15). Thus, the proof of Theorem 4 is completed.

Theorem 5 If $f\in\Sigma_p$ satisfies the inequality

$$\sum_{k=1}^{\infty}\Big[[k]_q\big(1+|B|\big)+(A-B)|b|q^p\Big]|a_k|<(A-B)|b|q^p \tag{16}$$

then $f\in\mathcal{MS}^*_{p,q}(b;A,B)$.

Proof. Since

$$\left|1+\sum_{k=1}^{\infty}\frac{\big(e^{-\iota\theta}+B\big)[k]_q+(A-B)bq^p}{(A-B)bq^p}a_kz^k\right|$$

$$\geq 1-\left|\sum_{k=1}^{\infty}\frac{\left(e^{-\iota\theta}+B\right)[k]_q+(A-B)bq^p}{(A-B)bq^p}a_kz^k\right|$$
$$\geq 1-\sum_{k=1}^{\infty}\frac{\left(1+|B|\right)[k]_q+(A-B)|b|q^p}{(A-B)|b|q^p}|a_k|>0.$$

Thus, the inequality (16) holds and our result follows from Theorem 3.
Using similar arguments to those in the proof of Theorem 5, we may also prove the next result.
Theorem 6 If $f\in\Sigma_p$ satisfies the inequality

$$\sum_{k=1}^{\infty}\Big[[k]_q\big(1+|B|\big)+(A-B)|b|q^p\Big]\Big(1-\frac{[k]_q}{[p]_q}\Big)|a_k|<(A-B)|b|q^p \tag{17}$$

then $f\in\mathcal{MK}_{p,q}(b;A,B)$.
Remarks Note that the results obtained in the present paper provide us a lot of interesting particular cases by assigning different values to the involved parameters, some illustration are given here :
(i) Taking $p=1$, $q\to 1^-$, $b=1$ and $e^{\iota\theta}=x$ in Theorem 1 and 2 we get the results of Ponnusamy [10].
(ii) Taking $p=1$, $q\to 1^-$, $b=(1-\alpha)e^{-\iota\mu}cos\mu$ $(\mu\in\mathbb{R},\ |\mu|<\frac{\pi}{2},\ 0\leq\alpha<1)$, A=1, B=-1 and $e^{\iota\theta}=x$ in Theorem 1 we get the result of Ravichandran et *al*. [12].
(iii) Taking $p=1$ in Theorem 1 and 2 our results matches with Mostafa et *al*. [9].
(iv) Taking $p=1$ and $q\to 1^-$ in Theorem 1 and 2, our results matches with Aouf [3] and Bulboacă et *al*. [4].

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