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A SURVEY ON FRACTIONAL CALCULUS IN GEOMETRIC FUNCTION THEORY

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Abstract. Recently considerable effort has been devoted to the study of fractional calculus in many branches of mathematics and physics. The main object of the present investigation is to provide a brief survey concerning fractional integral and derivative operators in Geometric Function Theory. The generalizations of these operators are also concerned along with numerous properties of these generalized operators. We also list some samples which reflect our recent investigations in the Geometric Function Theory.

1. Introduction

Geometric function theory is a highly developed branch of mathematics which suggests the significance of geometric ideas and problems in complex analysis. Recently, particular attention has been devoted to fractional integral and differential operators and its generalizations in Geometric Function Theory. The history of the theory goes back to seventeenth century, when in 1695 the derivative of order $\alpha = 1/2$ was investigated by Leibnitz in his letter to L'Hospital. Since then, the new theory turned out to be very attractive to mathematicians as well as physicists, biologists, engineers and economists. The fractional calculus operators have further been extensively used in describing and solving various problems in applied sciences and also in the Geometric Function Theory of Complex Analysis (see, for example

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[28] and [33]). There are several types of fractional integral and derivative operators, we refer the reader principally to [12, 13, 18, 19, 26, 27, 31] and the references therein for details. One of the important problem in Geometric Function Theory is univalent functions and how to construct a linear operators that preserves the class of the univalent functions and some of its subclasses. In [12], Biernacki conjectured that a certain integral operator maps the class of univalent into itself, but later Krzyz and Lewandowski provided a counterexample in [20] that the conjecture was not correct. While, Libera considered another linear integral operator in [21], which maps each of the subclasses of the starlike, convex and close-to-convex functions into itself. Among these operators in Geometric Function Theory, the operators which introduced and studied by Owa and Srivastava in [26, 27] as follows:

Definition 1.1. The fractional integral of order λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \ (\lambda > 0), \tag{1.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the complex $z-$plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 1.2. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \ (0 \leq \lambda < 1), \tag{1.2}$$

where the multiplicity of $(z-\zeta)^{-\delta}$ is removed as in Definition 1.1.

Definition 1.3. Under the hypotheses of Definition 1.2, the fractional derivative of order $k+\lambda$ is defined by

$$D_z^{k+\lambda} f(z) = \frac{d^k}{dz^k} D_z^{\lambda} f(z) \ (0 \leq \lambda < 1; \ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \ \mathbb{N} = \{1, 2, ...\}). \tag{1.3}$$

It is worthy of mention to recall here that a general form of the above operators, that is, the generalized fractional integral and generalized fractional derivative operators, was developed by Srivastava et al. [34] along the following lines:

Definition 1.4. For μ and η, the generalized fractional integral and derivative operators $I_{0,z}^{\lambda,\mu,\eta}$ for $\lambda > 0$ and $J_{0,z}^{\lambda,\mu,\eta} f(z)$ for $0 \leq \lambda < 1$ are defined by

$$I_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-\zeta)^{\lambda-1} f(\zeta) {}_2F_1\left(\mu+\lambda, -\eta; \lambda; 1-\frac{\zeta}{z}\right) d\zeta, \tag{1.4}$$

where $f(z)$ is an analytic function in a simply-connected region of the complex $z-$plane containing the origin with the order $f(z) = O(|z|^{\varepsilon}), \ z \to 0$ when $\varepsilon >$

$\max\{0,\mu-\eta\}-1$ and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$ and

$$J_{0,z}^{\lambda,\mu,\eta}f(z)=\begin{cases}\frac{d}{dz}\left\{\frac{z^{\lambda-\mu}\int_0^z(z-\zeta)^{-\lambda}f(\zeta)\,{}_2F_1\left(\mu-\lambda,1-\eta;1-\lambda;1-\frac{\zeta}{z}\right)d\zeta}{\Gamma(1-\lambda)}\right\},\\ \\ \frac{d^n}{dz^n}J_{0,z}^{\lambda-n,\mu,\eta}f(z)\ (n\le\lambda<n+1;\ n\in\mathbb{N}),\end{cases}\tag{1.5}$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as above and ${}_2F_1\,(a,b;c;z)$ is the Gaussian hypergeometric function defined by

$${}_2F_1\,(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n,$$

for $c\neq 0,-1,-2,...,\ z\in\mathbb{U}$ and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n=\begin{cases}1 & \text{if } n=0,\\ \lambda(\lambda+1)(\lambda+2)...(\lambda+n-1) & \text{if } n\in\mathbb{N}.\end{cases}$$

Noting that: $I_{0,z}^{\lambda,-\lambda,\eta}f(z)=D_z^{-\lambda}f(z)\ (\lambda>0)$ and $J_{0,z}^{\lambda,\lambda,\eta}f(z)=D_z^{\lambda}f(z)\ (0\le\lambda<1)$. It is significant to note that [32] and Liouville [22] defined the above operators associated with a real-valued function. An extension of the fractional calculus has been introduced by Kılıcman et al. in [17] as inserted below:

Definition 1.5. Let $f(z)$ be analytic in a simply connected region, for all $z\in\mathbb{U}$, containing the origin and $0<\alpha\le 1,\ 0<\beta\le 1$ such that $0\le\alpha-\beta<1$. Then the fractional integral operator $\mathcal{L}_z^{\alpha,\beta}$ is defined by

$$\mathcal{L}_z^{\alpha,\beta}f(z)=\frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}z^{1-\alpha}\int_0^z\frac{t^{\beta-1}\ f(\zeta)}{(z-\zeta)^{1-\alpha+\beta}}d\zeta,\tag{1.6}$$

where the multiplicity of $(z-\zeta)^{\alpha-\beta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$ and if $\alpha=\beta$, we have $\mathcal{L}_z^{\alpha,\alpha}f(z)=f(z)$.

Definition 1.6. Let $f(z)$ be analytic in a simply connected region, for all $z\in\mathbb{U}$, containing the origin and $0<\alpha\le 1,\ 0<\beta\le 1$ such that $0\le\alpha-\beta<1$. Then the fractional integral operator $\mathcal{I}_z^{\alpha,\beta}$ is defined by

$$\mathcal{I}_z^{\alpha,\beta}f(z)=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)}\frac{d}{dz}\int_0^z\frac{t^{\alpha-1}\ f(\zeta)}{(z-\zeta)^{\alpha-\beta}}d\zeta,\tag{1.7}$$

where the multiplicity of $(z-\zeta)^{\beta-\alpha}$ is removed as in Definition 1.5 and if $\alpha=\beta$, we have $\mathcal{I}_z^{\alpha,\alpha}f(z)=f(z)$.

We are now ready to introduce the fractional q-derivative operator, we need the following notations.

For any complex number α, the q-shifted factorials are defined by

$$(\alpha;q)_0 = 1;\ (\alpha;q)_n = \prod_{k=0}^{n-1}\left(1-\alpha q^k\right),\ n \in \mathbb{N}. \tag{1.8}$$

In terms of the analogue of the gamma function

$$(q^{\alpha};q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}\ (n>0),$$

where the q-gamma function is defined by

$$\Gamma_q(x) = \frac{(q;q)_{\infty}(1-q)^{1-x}}{(q^x;q)_{\infty}}\ (0<q<1).$$

If $|q|<1$, the definition (1.8) remains meaningful for $n=\infty$ as a convergent infinite product

$$(\alpha;q)_{\infty} = \prod_{j=0}^{\infty}\left(1-\alpha q^j\right).$$

Also, the q-integral of a function $f(z)$ is defined by (see Gasper and Rahman [15])

$$\int\limits_0^z f(t)d_qt = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k).$$

Recalling the definition of fractional q-calculus operators of a complex-valued function $f(z)$, which were recently studied by Purohit and Raina in [29].

Definition 1.7. The fractional q-integral operator of order λ for a function $f(z)$ is defined by

$$I_{q,z}^{\lambda}f(z) = D_{q,z}^{-\lambda}f(z) = \frac{1}{\Gamma_q(\lambda)}\int_0^z (z-tq)_{\lambda-1}\, f(t)d_qt\ (\lambda>0),$$

where $f(z)$ is an analytic function in a simply-connected region of the complex z-plane containing the origin and the q-binomial function $(z-tq)_{\lambda-1}$ is defined by

$$(z-tq)_{\lambda-1} = z^{\lambda-1}\ {}_1\Phi_0\left[q^{1-\lambda};-;q,\frac{tq^{\lambda}}{z}\right]. \tag{1.9}$$

The series ${}_1\Phi_0\left(\lambda;-;q,z\right)$ is single-valued when $|\arg(-z)|<\pi$ and $|z|<1$ (see for details Gasper and Rahman [15, p. 104-106]) and therefore the function $(z-tq)_{\lambda-1}$ in (1.9) is single-valued when $\left|\arg\left(-\frac{tq^{\lambda}}{z}\right)\right|<\pi$, $\left|\frac{tq^{\lambda}}{z}\right|<1$ and $|\arg(z)|<\pi$.

Definition 1.8. The fractional q-derivative operator of order λ for a function $f(z)$ is defined by

$$D_{q,z}^{\lambda}f(z) = D_{q,z}^{\lambda}\ I_{q,z}^{1-\lambda}f(z) = \frac{1}{\Gamma_q(1-\lambda)}D_{q,z}\int_0^z (z-tq)_{-\lambda}\, f(t)d_qt\ (0\le\lambda<1),$$

where $f(z)$ is an analytic function in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-tq)_{-\lambda}$ is removed as in Definition 1.7.

Definition 1.9. Under the hypotheses of Definition 1.8, the fractional q-derivative for a function $f(z)$ of order λ is defined by

$$D_{q,z}^{\lambda} f(z) = D_{q,z}^{m} I_{q,z}^{m-\lambda} f(z),$$

where $m-1 \leq \lambda < 1,\ m \in \mathbb{N}_0$.

It is significant to note that Al-Salam [5, 6] and [2] (see also [1]) defined the aforementioned operators associated with a real-valued function.

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \tag{1.10}$$

which are analytic in the open unit disc $\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of functions which are univalent in $\mathbb{U}$.

A function $f(z) \in \mathcal{A}$ is said to be starlike and convex of order α, denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ $(0 \leq \alpha < 1,\ z \in \mathbb{U})$, respectively, if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \tag{1.11}$$

and

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha. \tag{1.12}$$

From (1.11) and (1.12), we have

$$f(z) \in \mathcal{C}(\alpha) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\alpha).$$

It is worth noting that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$. For example: the function $f(z) = z/(1-z)^2 \in \mathcal{S}^*$ because

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) = \mathrm{Re}\left(\frac{1+z}{1-z}\right) > 0,$$

while $f(z) = -\log(1-z) \in \mathcal{C}$ because

$$1 + \mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) = 1 + \mathrm{Re}\left(\frac{z}{1-z}\right) > \frac{1}{2}.$$

Let further $\mathcal{T}$ denote the subfamily of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \ (a_k \geq 0). \tag{1.13}$$

On the other hand, let $\mathcal{TS}^*(\alpha)$ and $\mathcal{TC}(\alpha)$ denote the subfamilies obtained by taking intersections, respectively of the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$, that is,

$$\mathcal{TS}^*(\alpha) = \mathcal{S}^*(\alpha) \cap \mathcal{T},\ \mathcal{TC}(\alpha) = \mathcal{C}(\alpha) \cap \mathcal{T}.$$

Moreover, a function $f(z) \in \mathcal{A}$ given by (1.10) and $0 < q < 1$, the q-derivative of a function $f(z)$ is defined by (see Gasper and Rahman [15])

$$D_q f(z) = \begin{cases} f'(0) & \text{if } z = 0, \\ \dfrac{f(qz) - f(z)}{(q-1)z} & \text{if } \ z \neq 0. \end{cases} \tag{1.14}$$

From (1.14), we deduce that $D_q f(z)$ for a function $f(z)$ of the form (1.10) is given by

$$D_q f(z) = 1 + \sum_{k=n+1}^{\infty} [k]_q \ a_k z^{k-1} \ (z \neq 0), \tag{1.15}$$

where

$$[i]_q = \frac{1-q^i}{1-q}.$$

As $q \to 1,\ [k]_q \to k$, we have

$$\lim_{q \to 1} \ D_q f(z) = f'(z).$$

Making use of the q-derivative D_q, we introduce the subclasses $\mathcal{S}_q^*(\alpha)$ and $\mathcal{C}_q(\alpha)$ as follows:

A function $f(z) \in \mathcal{A}$ is called q-starlike and q-convex of order α, denoted by $\mathcal{S}_q^*(\alpha)$ and $\mathcal{C}_q(\alpha)$ $(0 < q < 1,\ 0 \leq \alpha < 1,\ z \in \mathbb{U})$, respectively, if and only if

$$\left| \frac{1}{1-\alpha} \left(\frac{z D_q f(z)}{f(z)} - \alpha \right) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}$$

and

$$\left| \frac{1}{1-\alpha} \left(\frac{D_q(z D_q f(z))}{D_q f(z)} - \alpha \right) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}.$$

It is a considerable computational and conceptual advantage for the prescribed classes to be written as a closed disc with center $((1-\alpha q)/(1-q), 0)$ and radius $(1-\alpha)/(1-q)$ because it provides explicit and desirable properties one would like to have. In particular, the starlike functions have the property that the argument of f increases with z. The above classes turned out that the modulus of f increases with $|z|$. We refer the interested readers to [16] for more details concerning these classes.

By now, it is abundantly clear that the classes $\mathcal{S}_q^*(\alpha)$ and $\mathcal{C}_q(\alpha)$ satisfy the inclusion (see [16, 3, 4]):

$$\underset{0<q<1}{\cap}\mathcal{S}_q^*(\alpha)=\mathcal{S}^*(\alpha),\ \underset{0<q<1}{\cap}\mathcal{C}_q(\alpha)=\mathcal{C}(\alpha) \text{ and } f(z)\in\mathcal{C}_q(\alpha)\Leftrightarrow zD_qf(z)\in\mathcal{S}_q^*(\alpha).$$

Definition 1.10. Let $\mathcal{P}$ denote the class of analytic functions $p(z)$ in $\mathbb{U}$ such that $p(0)=1$ and $\operatorname{Re} p(z)>0,\ z\in\mathbb{U}$, this class is called the class of functions with positive real parts, also called Carathéodory class.

For example, the function $p(z)=(1+z)/(1-z)$ belongs to $\mathcal{P}$, this function gives maps conformally of $\mathbb{U}$ onto the right-half plane, and consequently it plays a fundamental role in the class $\mathcal{P}$. We also note that $\mathcal{P}$ is a convex set and a compact subset of $\mathcal{S}$.

For a detailed historical survey and an extended list of references on fractional and q-fractional calculus and their applications to the theory of univalent and multivalent functions, we refer, e.g., to [7, 8, 9, 10, 11, 23, 24, 25, 29, 30, 35, 36, 37, 38, 39, 40, 41] and elsewhere. Here, the authors obtained coefficient estimates, sharp bounds for the Fekete Szegö functional $\left|a_3-\mu a_2^2\right|$, sharp bounds for the second, third and fourth order Hankel determinant, differential subordination results within a generalized fractional calculus as well as some properties such as sufficient conditions, inclusion results and distortion theorems for functions belonging to families of univalent and multivlent functions.

2. Results involving the operators $D_z^{-\lambda}$ and D_z^{λ}

Theorem 2.1. [26] *There exists a univalent starlike functions of the form (1.10) in $\mathbb{U}$ such that $F(z)=\Gamma(2+\lambda)\,z^{-\lambda}D_z^{-\lambda}f(z)\in\mathcal{S}^*$.*

Example 2.1. [26] Let $f(z)=z+a_2z^2\in\mathcal{S}^*$, then, from (1.11) and Theorem 2.1, we have $2\left|a_2\right|<1$. Therefore,

$$\begin{aligned}\operatorname{Re}\left(\frac{zF'(z)}{F(z)}\right)&=\frac{(2+\lambda)^2+6\,(2+\lambda)\left|a_2\right|\left|z\right|\cos(\theta+\varphi)+8\left|a_2\right|^2\left|z\right|^2}{(2+\lambda)^2+4\,(2+\lambda)\left|a_2\right|\left|z\right|\cos(\theta+\varphi)+4\left|a_2\right|^2\left|z\right|^2}\\&>\frac{(2+\lambda-2\left|a_2\right|\left|z\right|)\,(2+\lambda-4\left|a_2\right|\left|z\right|)}{(2+\lambda+2\left|a_2\right|\left|z\right|)^2}\geq 0,\end{aligned}$$

for $|z|<1$ where $z=|z|\,e^{i\theta}$ and $a_2=|a_2|\,e^{i\varphi}$.

Remark 2.1. [26] If $f(z)=z+a_2z^2\in\mathcal{S}^*$, then $F(z)$ is an univalent convex function in $|z|<(2+\lambda)/4$ where λ is a positive real number.

Theorem 2.2. [26] *There exists a univalent convex functions of the form (1.10) in $\mathbb{U}$ such that $F(z)=\Gamma(2+\lambda)\,z^{-\lambda}D_z^{-\lambda}f(z)\in\mathcal{C}$.*

Example 2.2. [26] Let $f(z) = z + a_2 z^2 \in \mathcal{C}$, then, from (1.12) and Theorem 2.2, we have $4|a_2| < 1$ and

$$\operatorname{Re}\left(1 + \frac{zF''(z)}{F'(z)}\right) = \frac{(2+\lambda)^2 + 12(2+\lambda)|a_2||z|\cos(\theta+\varphi) + 32|a_2|^2|z|^2}{(2+\lambda)^2 + 8(2+\lambda)|a_2||z|\cos(\theta+\varphi) + 16a_2^2|z|^2}$$
$$> \frac{(2+\lambda - 4|a_2||z|)(2+\lambda - 8|a_2||z|)}{(2+\lambda+4|a_2||z|)^2} \geq 0,$$

for $|z| < 1$ where $z = |z| e^{i\theta}$ and $a_2 = |a_2| e^{i\varphi}$.

Theorem 2.3. [26] *There exists a univalent starlike functions of the form (1.10) in* $\mathbb{U}$ *such that* $G(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) \in \mathcal{S}^*$.

Example 2.3. [26] Let $f(z) = z + a_2 z^2 \in \mathcal{S}^*$, then

$$\operatorname{Re}\left(\frac{zG'(z)}{G(z)}\right) = \frac{(2-\lambda-2|a_2||z|)(2-\lambda-4|a_2||z|)}{(2-\lambda+2|a_2||z|)^2} > 0,$$

for $|z| < 1$ where $z = |z| e^{i\theta}$ and $a_2 = |a_2| e^{i\varphi}$.

Let $\mathcal{S}_F^*$ and C_F denote the classes of functions of the form (1.10) for which $F(z) \in \mathcal{S}^*$ and $F(z) \in \mathcal{C}$, respectively. Further, let $\mathcal{S}_F^*$ stands for the class of functions of the form (1.10) for which $G(z) \in \mathcal{S}^*$. The following theorems give the lower and upper bounds for $\left|D_z^{-\lambda} f(z)\right|$ and $\left|D_z^{\lambda} f(z)\right|$.

Theorem 2.4. [26] *If* $f(z) \in \mathcal{S}_F^*$, *then for* $|z| < 1$,

$$\frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)(1+|z|)^2} \leq \left|D_z^{-\lambda} f(z)\right| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)(1-|z|)^2},$$

where λ *is a positive real number. Equality holds for* $f(z) = z/(1-e^{i\theta}z)^2$.

Theorem 2.5. [26] *If* $f(z) \in \mathrm{C}_F$, *then for* $|z| < 1$,

$$\frac{|z|^{\lambda}}{\Gamma(2+\lambda)(1+|z|)} \leq \left|D_z^{-\lambda} f(z)\right| \leq \frac{|z|^{\lambda}}{\Gamma(2+\lambda)(1-|z|)},$$

where λ *is a positive real number. Equality holds for* $f(z) = z/(1-z)$.

Theorem 2.6. [26] *If* $f(z) \in \mathcal{S}_G^*$, *then for* $|z| < 1$ *and* $0 < \lambda < 1$,

$$\frac{|z|^{-\lambda}}{\Gamma(2-\lambda)(1+|z|)} \leq \left|D_z^{\lambda} f(z)\right| \leq \frac{|z|^{-\lambda}}{\Gamma(2-\lambda)(1-|z|)}.$$

Equality holds for $f(z) = z/(1-z)$.

3. Results involving the operators $I_{0,z}^{\lambda,\mu,\eta}$ and $J_{0,z}^{\lambda,\mu,\eta}$

Lemma 3.1. [34] *If $\lambda > 0$ and $\kappa > \mu - \eta - 1$, then*

$$I_{0,z}^{\lambda,\mu,\eta}(z^{\kappa}) = \frac{\Gamma(\kappa+1)\,\Gamma(\kappa-\mu+\eta+1)}{\Gamma(\kappa-\mu+1)\,\Gamma(\kappa+\lambda+\eta+1)} z^{\kappa-\mu}.$$

Theorem 3.1. [34] *Let $\lambda,\ \mu$ and η satisfy the inequalities*

$$\lambda > 0,\ \mu < 2,\ \lambda+\eta > -2 \text{ and } \mu - \eta < 2.$$

If $\mu(\lambda+\eta)/\lambda - 2 \le n$ and $f(z)$ of the form (1.13) is in $\mathcal{TS}^*(\alpha)$, then

$$\left|I_{0,z}^{\lambda,\mu,\eta} f(z)\right| \ge \frac{\Gamma(2-\mu+\eta)}{\Gamma(2-\mu)\Gamma(2+\lambda+\eta)} |z|^{1-\mu} \times \left\{1 - \frac{(1-\alpha)(-\mu+\eta+2)_n (n+1)!}{(n+1-\alpha)(-\mu+2)_n(\lambda+\eta+2)_n} |z|^n\right\}, \tag{3.1}$$

and

$$\left|I_{0,z}^{\lambda,\mu,\eta} f(z)\right| \le \frac{\Gamma(2-\mu+\eta)}{\Gamma(2-\mu)\Gamma(2+\lambda+\eta)} |z|^{1-\mu} \times \left\{1 + \frac{(1-\alpha)(-\mu+\eta+2)_n (n+1)!}{(n+1-\alpha)(-\mu+2)_n(\lambda+\eta+2)_n} |z|^n\right\}, \tag{3.2}$$

for $z \in \mathbb{U}$ if $\mu \le 1$ and $z \in \mathbb{U}-\{0\}$ if $\mu > 1$. Equalities in (3.1) and (3.2) are attained by the function

$$f(z) = z - \frac{1-\alpha}{n+1-\alpha} z^{n+1},$$

of certain values of z, where μ is assumed to be rational number for the case (3.2).

Theorem 3.2. [34] *Under the assumptions of Theorem 3.1, let the function $f(z)$ of the form (1.13) be in the $\mathcal{TC}(\alpha)$, then*

$$\left|I_{0,z}^{\lambda,\mu,\eta} f(z)\right| \ge \frac{\Gamma(2-\mu+\eta)}{\Gamma(2-\mu)\Gamma(2+\lambda+\eta)} |z|^{1-\mu} \times \left\{1 - \frac{(1-\alpha)(-\mu+\eta+2)_n n!}{(n+1-\alpha)(-\mu+2)_n(\lambda+\eta+2)_n} |z|^n\right\}, \tag{3.3}$$

and

$$\left|I_{0,z}^{\lambda,\mu,\eta} f(z)\right| \le \frac{\Gamma(2-\mu+\eta)}{\Gamma(2-\mu)\Gamma(2+\lambda+\eta)} |z|^{1-\mu} \times \left\{1 + \frac{(1-\alpha)(-\mu+\eta+2)_n n!}{(n+1-\alpha)(-\mu+2)_n(\lambda+\eta+2)_n} |z|^n\right\}, \tag{3.4}$$

for $z \in \mathbb{U}$ if $\mu \le 1$ and $z \in \mathbb{U}-\{0\}$ if $\mu > 1$. Equalities in (3.3) and (3.4) are attained by the function

$$f(z) = z - \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1},$$

of certain values of z, where μ is assumed to be rational number for the case (3.4).

4. Results involving the operators $\mathcal{L}_z^{\alpha,\beta}$ and $\mathcal{I}_z^{\alpha,\beta}$

Theorem 4.1. [17] *Let $f \in \mathcal{A}$ on $\mathbb{U}$. Then the operator $\mathcal{L}_z^{\alpha,\beta} : \mathfrak{U}^p \to \mathfrak{U}^p$ is a bounded operator and*

$$\left\|\mathcal{L}_z^{\alpha,\beta} f(z)\right\|_{\mathfrak{U}^p}^p \leq \|f(z)\|_{\mathfrak{U}^p}^p ,$$

for all $z \in \mathbb{U}$ where $\mathfrak{U}^p(\mathbb{U}),\ 0 < p < 1$ stands for the Bergman space for a function f analytic in $\mathbb{U}$ with the norm $\|f(z)\|_{\mathfrak{U}^p}^p < \infty$ defined by

$$\|f(z)\| = \frac{1}{\pi} \int\limits_{\mathbb{U}} |f(z)|^p \, d\mathfrak{U} < \infty,$$

where $d\mathfrak{U}$ is known as a Lebesgue measure over $\mathbb{U}$.

Theorem 4.2. [17] *(Compactness) Let $f(z) \in \mathcal{A}$ on $\mathbb{U}$. Then $\mathcal{L}_z^{\alpha,\beta} : \mathfrak{U}^p \to \mathfrak{U}^p$ is compact.*

Theorem 4.3. [17] *Let $f \in \mathcal{A}$, then*

$$\left|\mathcal{L}_z^{\alpha,\beta} f(z)\right| \leq r \left({}_2F_1(1,\beta,\alpha;r)\right)'.$$

Theorem 4.4. [17] *If $f \in \mathcal{C}$, then*

$$\left|\mathcal{L}_z^{\alpha,\beta} f(z)\right| \leq \frac{r\beta}{\alpha} \left({}_2F_1(1;\beta+1,\alpha+1;r)\right).$$

Theorem 4.5. [17] *If $f(z) \in \mathcal{C}$, then*

$$\mathcal{L}_z^{\alpha,\beta} f(z) \leq \frac{r\ \Gamma(\alpha)}{\Gamma(\beta)\ \Gamma(\alpha-\beta)} \int\limits_0^1 s^{\beta}(1-s)^{\alpha-\beta-1}(1-rs)^{-1} ds,$$

Theorem 4.6. [17] *If $f(z) \in \mathcal{A}$, then for $0 < \alpha \leq 1,\ 0 < \beta \leq 1$ and $0 \leq \alpha - \beta < 1$, we have*

$$\left|\frac{\alpha}{\beta}\left(\mathcal{L}_z^{\alpha,\beta} f(z)\right) - z\right| \leq \frac{(2-r)(\beta+1)}{(1-r)^2(\alpha+1)}.$$

5. Complementary Results

Definition 5.1. [38] Let $\mathcal{SP}_\lambda$ $(0 \leq \lambda \leq 1)$ be the class of functions $f \in \mathcal{A}_0$ satisfying the inequality

$$\mathrm{Re}\left\{\frac{z(\Omega^\lambda f)'}{(\Omega^\lambda f)}\right\} > \left|\frac{z(\Omega^\lambda f)'}{(\Omega^\lambda f)} - 1\right|,$$

where $\Omega^\lambda f = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z)$ $(z \in \mathbb{U})$.

For $c \neq 0, -1, -2, ...$ and $z \in \mathbb{U}$, let

$$\varphi(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} z^{n+1},$$

and

$$\mathcal{L}(a,c)f(z) = \varphi\,(a,c;z) * f(z),\ \ f \in \mathcal{A},$$

where $*$ stands for the convolution between two power series (see [14]).

Further, it can verified that the Riemann map q of $\mathbb{U}$ onto the parabolic region

$$R = \left\{w : w = u + iv \ and\ v^2 < 2u - 1\right\},$$

satisfying $q(0) = 1$ and $q'(0) > 0$, is given by

$$\begin{aligned} q(z) &= 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \\ &= 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{2k+1}\right)z^n \\ &= \sum_{n=0}^{\infty} B_n z^n = 1 + \frac{8}{\pi^2}\left(z + \frac{2}{3}z^2 + \frac{23}{45}z^3 + \frac{44}{105}z^4 + ...\right),\ \ z \in \mathbb{U}, \end{aligned}$$

and the function $G(z)$ is given by

$$G(z) = \frac{1}{z}\left\{\mathcal{L}(2-\lambda, 2)\ z\left(\exp\int_0^z \frac{q(s)-1}{s}ds\right)\right\},\ \ z \in \mathbb{U}. \tag{5.1}$$

Theorem 5.1. [38] *Let* $0 \le \lambda < 1$ *and* $f \in \mathcal{SP}_\lambda$*, then*

$$G(-r) \le \left|\frac{f(z)}{z}\right| \le G(r),\ \ |z| = r, \tag{5.2}$$

and

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| \le \max_{\theta\in[0,2\pi]}\left\{\arg\left(G(re^{i\theta})\right)\right\}\ \ (z = re^{i\theta}), \tag{5.3}$$

where $G(z)$ *is given by (5.1). Equality holds true in 5.2 and 5.3 for some* $z \ne 0$ *if and only if* f *is a rotation of* $zG(z)$*.*

Corollary 5.1. [38] *If* $f \in \mathcal{SP}_\lambda$*, then*

$$\{w : |w| \le G(-1)\} \subseteq f\left(\mathbb{U}\right).$$

The result is sharp.

Theorem 5.2. [38] *Let* $f \in \mathcal{S}_\lambda\,(1/2)$ *and* $g \in \mathcal{SP}_\mu$ $(\lambda \le \mu)$*, then*

$$\Omega^\lambda f(z) * \Omega^\mu f(z) \in \mathcal{SP}_\mu.$$

In particular, if $f \in \mathcal{S}_\lambda\,(1/2)$ *and* $g \in \mathcal{SP}_\lambda$*, then*

$$\Omega^\lambda f(z) * \Omega^\lambda f(z) \in \mathcal{SP}_\lambda.$$

Theorem 5.3. [38] *Let* $f_j \in \mathcal{SP}_\lambda$ $(j = 1, 2, ..., n)$. *Also let*

$$\alpha_j > 0 \text{ and } \sum_{j=1}^{n} \alpha_j = 1,$$

Define a function g *by*

$$\Omega^\alpha g(z) = \prod_{j=1}^{n} (\Omega^\lambda f_j(z))^{\alpha_j},$$

then $g \in \mathcal{SP}_\lambda$.

Theorem 5.4. [38] *Let* $g \in \mathcal{P}$ *where*

$$g(z) = 1 + c_1 z + c_2 z^2 + ... = 1 + G(z),$$

then

$$|c_n| \leq 2 \ (n \in \mathbb{N}),$$

and

$$\left| c_2 - \frac{1}{2}\mu c_1^2 \right| \leq 2 + \frac{1}{2}(|\mu - 1| - 1) |c_1|^2 .$$

Furthermore, if we define the sequence $\{A_n\}_{n=1}^{\infty}$ *by*

$$\sum_{n=1}^{\infty} (-1)^{n-1} \gamma_{n-1} \{G(z)\}^n = \sum_{n=1}^{\infty} A_n z^n,$$

where $\gamma_0 = 1$,

$$\gamma_n = \frac{1}{2^n} \left[1 + \frac{1}{2} \sum_{j=1}^{n} \begin{pmatrix} n \\ j \end{pmatrix} B_n \right],$$

and the sequence $\{B_n\}_{n=1}^{\infty}$ *by*

$$h(z) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

then

$$|A_n| \leq 2 \ (n \in \mathbb{N}).$$

Theorem 5.5. [38] *Let the function* $f \in \mathcal{A}$ *be in the class* $\mathcal{SP}_\lambda$. *Then,*

$$\left| a_3 - \mu a_2^2 \right| \leq \begin{cases} \dfrac{4}{3\pi^2} (3-\lambda)(2-\lambda) \left(\frac{12(2-\lambda)\mu}{(3-\lambda)\pi^2} - \frac{4}{\pi^2} - \frac{1}{3} \right) & \text{if } \mu \geq \sigma_1, \\ \\ \dfrac{2}{3\pi^2} (3-\lambda)(2-\lambda) & \text{if } \sigma_2 \leq \mu \leq \sigma_1, \\ \\ \dfrac{4}{3\pi^2} (3-\lambda)(2-\lambda) \left(\frac{1}{3} + \frac{4}{\pi^2} - \frac{12(2-\lambda)\mu}{(3-\lambda)\pi^2} \right) & \text{if } \mu \leq \sigma_2, \end{cases} \tag{5.4}$$

where

$$\sigma_1 = \frac{3-\lambda}{2-\lambda} \left(\frac{1}{3} + \frac{5\pi^2}{72} \right) \text{ and } \sigma_2 = \frac{3-\lambda}{2-\lambda} \left(\frac{1}{3} - \frac{\pi^2}{72} \right).$$

Each of the estimates in (5.4) is sharp.

Definition 5.2. For $-\infty < \alpha < 2,\ 0 \leq \delta < 1,\ 0 \leq \beta < 1,\ 0 < q < 1$ and $z \in \mathbb{U}$, Purohit and Raina in [29] defined the families $\mathcal{F}_{q,\delta}^{\alpha}$ and $\mathcal{G}_{q,\delta}^{\alpha}$, respectively, by

$$\mathcal{F}_{q,\delta}^{\alpha} = \left\{ f \in \mathcal{T},\ \left| \frac{\Omega_{q,z}^{\alpha} f(z)}{\Omega_{q,z}^{\alpha} f(z) - 2\delta + 1} \right| < \beta \right\},$$

where

$$\begin{aligned} \Omega_{q,z}^{\alpha} f(z) &= \frac{\Gamma_q(2-\alpha)}{\Gamma_q(2)} z^{\alpha-1} D_{q,z}^{\alpha} f(z) \\ &= 1 + \sum_{k=2}^{\infty} \frac{\Gamma_q(2-\alpha)\Gamma_q(k+1)}{\Gamma_q(2)\Gamma_q(k+1-\alpha)} a_k z^{k-1}, \end{aligned}$$

and

$$\mathcal{G}_{q,\delta}^{\alpha} = \left\{ f \in \mathcal{T},\ \operatorname{Re}\left((1-\gamma)\Omega_{q,z}^{\alpha} f(z) + \alpha \frac{1-q^{1-\alpha}}{1-q} \Omega_{q,z}^{\alpha+1} f(z) \right) > \beta \right\}.$$

Theorem 5.6. [29] *A function $f(z)$ of the form (1.13) belongs to $\mathcal{F}_{q,\delta}^{\alpha}$ if and only if*

$$\sum_{k=2}^{\infty} \frac{\Gamma_q(k+1)\Gamma_q(2-\alpha)}{\Gamma_q(2)\Gamma_q(k-\alpha+1)} (1+\beta) a_k \leq 2\beta(1-\delta).$$

The result is sharp for the function

$$f(z) = z - \frac{2\beta(1-\delta)\Gamma_q(2)\Gamma_q(n-\alpha+2)}{(1+\beta)\Gamma_q(n+2)\Gamma_q(2-\alpha)} z^{n+1}\ (n \in \mathbb{N}).$$

Theorem 5.7. [29] *A function $f(z)$ of the form (1.13) belongs to $\mathcal{G}_{q,\delta}^{\alpha}$ if and only if*

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{\Gamma_q(k+1)\Gamma_q(2-\alpha)}{\Gamma_q(2)\Gamma_q(k-\alpha+1)} a_k \left[(1-\gamma)(1-q) + \gamma\left(1-q^{k-\alpha}\right) \right] \\ &\leq (1-\beta-\gamma)(1-q) + \gamma\left(1-q^{1-\alpha}\right). \end{aligned}$$

The result is sharp for the function

$$f(z) = z - \frac{\left[(1-\beta-\gamma)(1-q) + \gamma\left(1-q^{1-\alpha}\right)\right]}{A_{n+1,q}(\alpha,\gamma)} z^{n+1}\ (n \in \mathbb{N}),$$

where

$$A_{n+1,q}(\alpha,\gamma) = \frac{\left[(1-\beta-\gamma)(1-q) + \gamma\left(1-q^{1-\alpha}\right)\right]\Gamma_q(k+1)\Gamma_q(2-\alpha)}{\Gamma_q(k-\alpha+1)\Gamma_q(2)}.$$

Theorem 5.8. [29] *Let $f(z)$ of the form (1.10) be in the $\mathcal{F}_{q,\delta}^{\alpha}$ $(-\infty < \alpha < 2,\ 0 \leq q < 1)$, then*

$$|z| - 2\beta\left(\frac{1-\delta}{1+\beta}\right) B(n,\alpha,q) |z|^{n+1} \leq |f(z)| \leq |z| + 2\beta\left(\frac{1-\delta}{1+\beta}\right) B(n,\alpha,q) |z|^{n+1},$$

where

$$B\left(n,\alpha,q\right)=\frac{\Gamma_{q}\left(2\right)\Gamma_{q}\left(n+2-\alpha\right)}{\Gamma_{q}\left(n+2\right)\Gamma_{q}\left(2-\alpha\right)}.$$

Further,

$$|z|-2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^{n+1}\leq\left|z\Omega_{q,z}^{\alpha}f(z)\right|\leq|z|+2\beta\left(\frac{1-\delta}{1+\beta}\right)|z|^{n+1}.$$

Theorem 5.9. [29] *Let $f(z)$ of the form (1.10) be in the $\mathcal{G}_{q,\delta}^{\alpha}$ $(-\infty<\alpha<2,\ 0\leq q<1)$, then*

$$|z|-B\left(n,\alpha,q\right)C\,|z|^{n+1}\leq|f(z)|\leq|z|+B\left(n,\alpha,q\right)C\,|z|^{n+1}.$$

Also,

$$|z|-CD\,|z|^{n+1}\leq\left|z\Omega_{q,z}^{\alpha}f(z)\right|\leq|z|+CD\,|z|^{n+1},$$

where

$$C=\frac{\left(1-\beta-\gamma\right)\left(1-q\right)+\gamma\left(1-q^{1-\alpha}\right)}{\left(1-\gamma\right)\left(1-q\right)+\gamma\left(1-q^{n+1-\alpha}\right)},$$

and

$$D=\frac{\Gamma_{q}\left(n-\alpha+2\right)\Gamma_{q}\left(2-\lambda\right)}{\Gamma_{q}\left(n-\lambda+2\right)\Gamma_{q}\left(2-\alpha\right)}.$$

References

[1] M. H. Annaby and Z. Mansour, q-fractional calculus and equation, Springer-Verlag Berlin Heidelberg, 2012.

[2] R.P. Agarwal, Certain fractional q-integrals and q-derivatives, Math. Proc. Cambridge Philos. Soc., 66 (2) (1969), 365 -370.

[3] S. Agrawal, Coefficient estimates for some classes of functions associated with q-function theory, Bull. Aust. Math. Soc., 95 (3) (2017), 446-456.

[4] S. Agrawal and S. K. Sahoo, A generalization of starlike functions of order alpha, Hokkaido Math. J. 46 (2017), 15-27.

[5] W.A. Al-Salam, Some fractional q-integrals and q-derivatives, Proc. Edinburgh Math. Soc., 2 (15) (1966/1967), 135-140.

[6] W.A. Al-Salam, q-analogues of Cauchy's formulas, Proc. Amer. Math. Soc., 17 (1966), 616-621.

[7] M.K. Aouf, Neighborhoods of a certain family of multivalent functions defined by using a fractional derivative operator, Bull. Belg. Math. Soc. Simon Stevin, 16 (2009), 31-40.

[8] M.K. Aouf and J. Dziok, Distortion and convolutional theorems for operators of generalized fractional calculus involving Wright function, J. Appl. Anal., 14 (2008), 183-192.

[9] M.K. Aouf, A.O. Mostafa and H.M. Zayed, Some characterizations of integral operators associated with certain classes of p-valent functions defined by the Srivastava-Saigo-Owa fractional differintegral operator, Complex Anal. Oper. Theory, 10 (6) (2016), 1267-1275.

[10] M.K. Aouf, A.O. Mostafa and H.M. Zayed, Subordination and superordination properties of p-valent functions defined by a generalized fractional differintegral operator, Quaest. Math., 39 (4) (2016), 545-560.

[11] M.K. Aouf, A.O. Mostafa and H.M. Zayed, On certain subclasses of multivalent functions defined by a generalized fractional differintegral operator, Afr. Mat., 28 (1-2) (2017), 99-107.

[12] M. Biernacki, Sur lintegrale des fonctions univalentes, Bull. Acad. Polon. Sci, 8, 29-34 (1960).

[13] B.C. Carlson and D.B. Shaffer, Starlike and prestarlike hypergeometric functions. SIAM J. Math. Anal., 15 (1984), 737-745.

[14] P.L. Duren, Univalent Functions, Springer-Verlag, New York, 1983.

[15] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.

[16] M. E. H. Ismail, E. Merkes and D. Styer, A gerneralization of starlike functions, Complex Variables Theory Appl., 14 (1-4) (1990), 77-84.

[17] A. Kılıcman, R.W. Ibrahim and Z.E. Abdulnaby, On a generalized fractional integral operator in a complex domain, Appl. Math. Inf. Sci., 10 (3) (2016), 1053-1059.

[18] V.S. Kiryakova, The operators of generalized fractional calculus and their action in classes of univalent functions, In: Proc. Inter. Symp. on Geometric Function Theory and Applications 2010, Sofia, August 27-31 (2010).

[19] V.S. Kiryakova, M. Saigo and H.M. Srivastava, Some criteria for univalence of analytic functions involving generalized fractional calculus operators, Fract. Calc. Appl. Anal., 1 (1998), 79–104.

[20] J. Krzyz and Z. Lewandowski, On the integral of univalent functions, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom, 11 (1963), 447-448.

[21] R.J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc., 16 (4) (1965), 755–758.

[22] J. LiouvilIe, (1832), M emoire sur le Calcul des di erentielles a indices quelconques, J. Ecole polytech., Vol. 13, pp.71-162.

[23] A.K. Mishra and P. Gochhayat, The Fekete-Szego problem for k-uniformly convex functions and for a class defined by the Owa-Srivatava operator, J. Math. Anal. Appl., 347 (2008), 563-572.

[24] A.K. Mishra and P. Gochhayat, Second Hankel determinant for a class of analytic functions defined by fractional derivative, Inter. J. Math. Math. Sci., 2008 (2008), 10pp.

[25] A.O. Mostafa, M.K. Aouf, H.M. Zayed and T. Bulboacă, Multivalent functions associated with Srivastava-Saigo-Owa fractional differintegral operator, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math., 112 (4) (2018), 1409-1429.

[26] S. Owa, On the distortion theorems. I, Kyungpook Math. J., 18 (1978), 53-59.

[27] S. Owa and H.M. Srivastava, Univalent and starlike generalized hypergeometric functions, Canad. J. Math., 39, 1057-1077 (1987).

[28] I. Podlubny, Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Math. Sciences Engin. 198, Academic Press, San Diego 1999.

[29] S.D. Purohit and R.K. Raina, Certain subclasses of analytic functions associated with fractional q-calculus operators, Math. Scand., 109 (1) (2011), 55-70.

[30] S.D. Purohit and R.K. Raina, Fractional q-calculus and certain subclasses of univalent analytic functions, Mathematica (Cluj), 55 (78) (2013), 62-74.

[31] S. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc., 49 (1) (1975), 109–115.

[32] M. Saigo, A remark on integral operators involving the Gauss hypergeometric functions, Math. Rep. College General Ed. Kyushu Univ., 11 (1978), 135-143.

[33] H.M. Srivastava and S. Owa (editors), Current Topics in Analytic Function Theory,World Scientific, Singapore 1992.

[34] H.M. Srivastava, M. Saigo and S. Owa, A class of distortion theorems involving certain operators of fractional calculus, J. Math. Anal. Appl., 131 (1988), 412-420.

[35] H.M. Srivastava and M.K. Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients, J. Math. Anal. Appl., 171 (1992), 1-13.

[36] H.M. Srivastava and M.K. Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. II, J. Math. Anal. Appl., 171 (1995), 673-688.

[37] H.M. Srivastava and M.K. Aouf, Some applications of fractional calculus operators to certain subclasses of prestarlike functions of negative coefficients, Comput Math Appl., 30 (1) (1995), 53-61.

[38] H.M. Srivastava and A.K. Mishra, Applications of fractional calculus to parabolic starlike and uniformly convex functions, Comput. Math. Appl., 39 (2000), 57-69.

[39] B. Wongsaijai and N. Sukantamala, Applications of fractional q-calculus to certain subclass of analytic p-valent functions with negative coefficients, Abstr. Appl. Anal., 2015 (2015), 12pp.

[40] H.M. Zayed, S.A. Mohammadein and M.K. Aouf, Sandwich results of p-valent functions defined by a generalized fractional derivative operator with application to vortex motion, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math., 113 (2019), 1499-1514.

[41] H.M. Zayed and T. Bulboacă, Applications of differential subordinations involving a generalized fractional differintegral operator, J. Inequal. Appl., 2019 (2019), Article 242, 13 pp.

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