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ON HARDY ROGERS TYPE CONTRACTIONS IN A-METRIC SPACES

I. YILDIRIM

Abstract. In this present work, we prove convergence to fixed point of Hardy Rogers type contractions of Mann iteration process in convex A-metric spaces. After, we also give a result based on our main result. Our results carry some well-known results from the literature to convex A-metric spaces.

1. Introduction

Banach fixed point theory is one of the cornerstones of mathematics and many other sciences. Various studies have been made using different generalizations of the contraction mappings in this theory. Some of this mappings are Kannan [7] contraction mapping, Chatterjea [2] contraction mapping, Reich [15] contraction mapping, Ciric [3] contraction mapping and Hardy and Rogers [6] contraction mapping. The mapping T has a fixed point in U if (U, d) is a complete metric space and $T : U \to U$ is a mapping that satisfies any of the above contractive conditions.

In addition to the generalizations of the contraction mappings in above, studies on fixed point theory have progressed in generalized metric spaces. Mustafa and Sims introduced a new class of generalized metric spaces called G-metric spaces (see [12], [13]) as a generalization of metric spaces (U, d). This was done to introduce and develop a new fixed point theory for a variety of mappings in this new setting. This helped to extend some known metric space results to this more general setting.

In 2015, Abbas et al. [1] introduced the concept of an A-metric space as follows:

Let U be nonempty set. Suppose a mapping $A : U^t \to \mathbb{R}$ satisfies the following conditions:

(A_1) $A(u_1, u_2, ..., u_{t-1}, u_t) \geq 0$,

(A_2) $A(u_1, u_2, ..., u_{t-1}, u_t) = 0$ if and only if $u_1 = u_2 = ... = u_{t-1} = u_t$,

(A_3) $A(u_1, u_2, ..., u_{t-1}, u_t) \leq A(u_1, u_1, ..., (u_1)_{t-1}, v) + A(u_2, u_2, ..., (u_2)_{t-1}, v)+$
$... + A(u_{t-1}, u_{t-1}, ..., (u_{t-1})_{t-1}, v) + A(u_t, u_t, ..., (u_t)_{t-1}, v)$

for any $u_i, v \in U$, $(i = 1, 2, ..., t)$ and $t \geq 2$. Then, (U, A) is said to be an A-metric space.

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It is clear that the an A-metric space for $t=2$ reduces to ordinary metric d. Also, an A-metric space is a generalization of the G-metric space.

[1] Let $U=\mathbb{R}$. Define a function $A:U^t\to\mathbb{R}$ by

$$\begin{aligned} A(u_1,u_2,...,u_{t-1},u_t) &= |u_1-u_2|+|u_1-u_3|+...+|u_1-u_t| \\ &\quad +|u_2-u_3|+|u_2-u_4|+...+|u_2-u_t| \\ &\quad \vdots \\ &\quad +|u_{t-2}-u_{t-1}|+|u_{t-2}-u_t|+|u_{t-1}-u_t| \\ &= \sum_{i=1}^{t}\sum_{i<j}|u_i-u_j|. \end{aligned}$$

Then (U,A) is an A-metric space.

[1] Let (U,A) be A-metric space. Then $A(u,u,\ldots,u,v)=A(v,v,\ldots,v,u)$ for all $u,v\in U$.

[1] Let (U,A) be A-metric space. Then for all for all $u,v,y\in U$ we have $A(u,u,\ldots,u,v)\le(t-1)A(u,u,\ldots,u,y)+A(v,v,\ldots,v,y)$ and $A(u,u,\ldots,u,v)\le(t-1)A(u,u,\ldots,u,y)+A(y,y,\ldots,y,v)$.

[1] Let (U,A) be A-metric space.

(i) A sequence $\{u_n\}$ in U is said to converge to a point $u\in U$ if $A(u_n,u_n,\ldots,u_n,u)\to 0$ as $n\to\infty$.

(ii) A sequence $\{u_n\}$ in U is called a Cauchy sequence if $A(u_n,u_n,\ldots,u_n,u_m)\to 0$ as $n,m\to\infty$.

(iii) The A-metric space (U,A) is said to be complete if every Cauchy sequence in U is convergent.

Recently, following the ideas of Hardy and Rogers [6], Yildirim [19] introduced the analogue of Hardy-Rogers type contraction mapping in A-metric space as follows:

Let (U,A) be A-metric space and $T:U\to U$ be a mapping. T is called a Hardy-Rogers type contraction mapping, if and only if, there exist $a,b,c\in\mathbb{R}^+$ with $a+2b+tc<1$ such that for all $u,v\in U$,

$$\begin{aligned} A(Tu,Tu,\ldots,Tu,Tv) &\le aA(u,u,\ldots,u,v)+b\left[A(Tu,Tu,\ldots,Tu,v)\right. \\ &\quad \left.+A(Tv,Tv,\ldots,Tv,v)\right]+c\left[A(Tu,Tu,\ldots,Tu,v)\right. \\ &\quad \left.+A(Tv,Tv,\ldots,Tv,u)\right]. \end{aligned} \tag{1}$$

It is clear that if we take $t=2$ in the Definition 1, we obtain the contractive definition of Hardy-Rogers in [6]. Yildirim [19] also proved the following fixed point theorem for above mapping (1).

Let (U,A) be complete A-metric space and $T:U\to U$ be a Hardy-Rogers type contraction mapping as Definition 1. Then T has a unique fixed point in U and Picard iteration process $\{u_n\}$ defined by $u_{n+1}=Tu_n$ converges to a fixed point of T.

The iterative approximation of a fixed point for certain classes of mappings is one of the main tools in the fixed point theory. As seen above, Picard iteration is generally used in metric spaces with there is no convex structure. Convex metric space structure is needed to use different iteration methods. After the convex metric space structure is defined, many authors ([4, 5, 8, 9, 10, 11, 14, 16, 17, 18]) discussed the existence of fixed points and convergence of different iterative processes for various mappings in such spaces.

Keeping the above in mind, our aim in this article is to establish the conditions necessary for the Mann iteration scheme to converge to the fixed point of this class of Hardy-Rogers type contraction mapping in convex complete A-metric space. The convex A-metric space we will use in these proofs is defined by Yildirim [20] as follows.

[20] Let (U, A) be a A-metric space and $I = [0, 1]$. A mapping $W : U^t \times I^t \to U$ is termed as a convex structure on U if

$$\begin{aligned} & A(u_1, u_2, ..., u_{t-1}, W(v_1, v_2, ..., v_{t-1}, v_t; \beta_1, \beta_2, ..., \beta_t)) \\ \leq \quad & \beta_1 A(u_1, u_2, ..., u_{t-1}, v_1) + \beta_2 A(u_1, u_2, ..., u_{t-1}, v_2) \\ & +... + \beta_t A(u_1, u_2, ..., u_{t-1}, v_t) \\ = \quad & \sum_{i=1}^{t} \beta_i A(u_1, u_2, ..., u_{t-1}, v_i) \end{aligned} \tag{2}$$

for real numbers $\beta_1, \beta_2, ..., \beta_t$ in $I = [0, 1]$ satisfying $\sum_{i=1}^{t} \beta_i = 1$ and u_i, $v_i \in X$ for all $i = 1, 2, ..., t$.

An A-metric space (U, A) with a convex structure W is called a convex A-metric space and denoted as (U, A, W).

A nonempty subset V of a convex A-metric space (U, A, W) is said to be convex if

$W(v_1, v_2, ..., v_{t-1}, v_t; \beta_1, \beta_2, ..., \beta_t) \in V$ for all $v_i \in V$ and $a_i \in I$, $i = 1, 2, ..., t$.

The form of Mann iteration in a convex A-metric space is as follows:

[20] Let (U, A, W) be convex A-metric space with convex structure W and $T : U \to U$ be a mapping. Let $\{\beta_i^n\}$ be sequences in $[0, 1]$ for all $i = 1, 2, ..., t$ and $n \in \mathbb{N}$. Then for any given $u_0 \in U$, the iteration process defined by the sequence $\{u_n\}$ as

$$u_{n+1} = W\left(u_n, u_n, ..., u_n, Tu_n; \beta_1^n, \beta_2^n, ..., \beta_t^n\right), \tag{3}$$

is called Mann iteration process in the convex metric space (U, A, W).

2. **Main Results**

Now, we will prove the Mann iteration process converges to fixed point of Hardy-Rogers type contraction mapping in complete convex metric space (U, A, W).

Let (U, A, W) be a complete convex A-metric space with a convex structure W and, $T : U \to U$ be a Hardy-Rogers type contraction mapping as Definition 1 with $a + tb + tc < 1$ such that $a, b, c \in \mathbb{R}^+$. Let $\{u_n\}$ be defined iteratively by (3) and $u_0 \in U$, with $\{\beta_t^n\} \subset [0, 1],_{i=1}^{t} \beta_t^n = 1$ satisfying $_{n=0}^{\infty} \beta_t^n = \infty$ for all $n \in \mathbb{N}$ and $i = 1, 2, ..., t$. Then $\{u_n\}$ converges to a unique fixed point of T.

Proof. From the condition $a + tb + tc < 1$, we have that

$$a + 2b + tc < a + tb + tc < 1$$

for $t \geq 2$. Then from Theorem 1, we know that a Hardy-Rogers type contraction mapping has a unique fixed point in U. Let's call it u. That is $Tu = u$.

Let $\{u_n\}$ be the Mann iteration process (3), with $u_0 \in U$ arbitrary. Then

$$\begin{aligned} A(u,u,...,u,u_{n+1}) &= A(u,u,...,u,W\left(u_n,u_n,...,u_n,Tu_n;\beta_1^n,\beta_2^n,...,\beta_t^n\right)) \\ &\leq \beta_1^n A(u,u,...,u,u_n)+\beta_2^n A(u,u,...,u,u_n) \\ &\quad +...+\beta_t^n A(u,u,...,u,Tu_n) \\ &= \left(1-\beta_t^n\right) A(u,u,...,u,u_n)+\beta_t^n A(u,u,...,u,Tu_n). \end{aligned}$$

Using (1), we obtain that

$$\begin{aligned} A(u,u,...,u,Tu_n) &= A(Tu,Tu,...,Tu,Tu_n) \qquad (4) \\ &\leq aA(u,u,\ldots,u,u_n)+b\left[A(Tu,Tu,\ldots,Tu,u)\right. \\ &\quad \left.+A(Tu_n,Tu_n,\ldots,Tu_n,u_n)\right]+c\left[A(Tu,Tu,\ldots,Tu,u_n)\right. \\ &\quad \left.+A(Tu_n,Tu_n,\ldots,Tu_n,u)\right] \\ &= aA(u,u,\ldots,u,u_n)+bA(Tu_n,Tu_n,\ldots,Tu_n,u_n) \\ &\quad +cA(u,u,\ldots,u,u_n)+cA(Tu_n,Tu_n,\ldots,Tu_n,u) \\ &= (a+c)\,A(u,u,\ldots,u,u_n)+bA(Tu_n,Tu_n,\ldots,Tu_n,u_n) \\ &\quad +cA(u,u,...,u,Tu_n) \\ &\leq (a+c)\,A(u,u,\ldots,u,u_n)+b\,(t-1)\,A(Tu_n,Tu_n,\ldots,Tu_n,u) \\ &\quad +bA(u,u,\ldots,u,u_n)+cA(u,u,...,u,Tu_n) \\ &\leq (a+b+c)\,A(u,u,\ldots,u,u_n)+(b+c)\,(t-1)\,A(u,u,...,u,Tu_n) \end{aligned}$$

which implies that

$$\left[1-(b+c)\,(t-1)\right] A(u,u,...,u,Tu_n) \leq (a+b+c)\,A(u,u,\ldots,u,u_n).$$

From above inequality, we have

$$A(u,u,...,u,Tu_n) \leq \frac{a+b+c}{1-(b+c)\,(t-1)} A(u,u,\ldots,u,u_n). \qquad (5)$$

If we combine (4) and (5), we obtain that

$$\begin{aligned} A(u,u,...,u,u_{n+1}) &\leq \left(1-\beta_t^n\right) A(u,u,...,u,u_n)+\beta_t^n A(u,u,...,u,Tu_n) \\ &\leq \left(1-\beta_t^n\right) A(u,u,...,u,u_n)+\beta_t^n \frac{a+b+c}{1-(b+c)\,(t-1)} A(u,u,\ldots,u,u_n) \\ &= \left[1-\beta_t^n+\beta_t^n \frac{a+b+c}{1-(b+c)\,(t-1)}\right] A(u,u,\ldots,u,u_n) \qquad (6) \\ &= \left[1-\left(1-\frac{a+b+c}{1-(b+c)\,(t-1)}\right)\beta_t^n\right] A(u,u,\ldots,u,u_n) \\ &= \left[1-(1-\gamma)\,\beta_t^n\right] A(u,u,\ldots,u,u_n), \end{aligned}$$

where

$$\gamma = \frac{a+b+c}{1-(b+c)\,(t-1)} < 1 \text{ , as } a+tb+tc<1.$$

Inductively we obtain that

$$\begin{aligned} A(u,u,...,u,x_{n+1}) &\leq \left[1-(1-\gamma)\,\beta_t^n\right] A(u,u,\ldots,u,u_n) \qquad (7) \\ &\leq \left[1-(1-\gamma)\,\beta_t^n\right]\left[1-(1-\gamma)\,\beta_t^{n-1}\right] A(u,u,...,u,u_{n-1}) \\ &\quad \vdots \\ &\leq {}_{k=0}^{n}\left[1-(1-\gamma)\,\beta_t^k\right] A(u,u,...,u,u_0) \end{aligned}$$

Since $0 \leq \gamma < 1$, $\left\{\beta_t^k\right\} \subset [0,1]$ and ${}_{k=0}^{\infty}\beta_t^k = \infty$, we get

$$\lim_{n\to\infty} {}_{k=0}^{n} \left[1 - (1-\gamma)\,\beta_t^k\right] = 0,$$

which by (7) implies

$$\lim_{n\to\infty} A(u,u,...,u,u_{n+1}) = \lim_{n\to\infty} A(u_{n+1},u_{n+1},...,u_{n+1},u) = 0.$$

Therefore the sequence $\{u_n\}$ defined iteratively by (3) converges to the fixed point of T. □

As we said before, an A-metric space for $t = 2$ reduces to ordinary metric d. So, if we take $t = 3$ in above Theorem, we obtain the following result.

Let (U, d, W) be a complete convex metric space with a convex structure W and, $T : U \to U$ be a Hardy-Rogers type contraction mapping as follows:

$$d(Tu,Tv) \leq ad(u,v) + b\left[d(Tu,u) + d(Tv,v)\right] + c\left[d(Tu,v) + d(Tv,u)\right]$$

with $a + 2b + 2c < 1$ such that $a, b, c \in \mathbb{R}^+$. Let $\{u_n\}$ be defined iteratively by

$$u_{n+1} = W\left(u_n, Tu_n; \beta_1^n, \beta_2^n\right)$$

where $u_0 \in U$, with $\{\beta_t^n\} \subset [0,1], {}_{i=1}^{2}\,\beta_t^n = 1$ satisfying ${}_{n=0}^{\infty}\beta_2^n = \infty$ for all $n \in \mathbb{N}$ and $i = 1, 2$. Then $\{u_n\}$ converges to a unique fixed point of T.

References

[1] M. Abbas, B. Ali and Y.I. Suleiman, Generalized coupled common fixed point results in partially ordered A-metric spaces. Fixed Point Theory and Applications. 2015;2015:64. DOI: 10.1186/s13663-015-0309-2.

[2] S. K. Chatterjea, Fixed-point theorems, Comptes Rendus de l'Acad ´emie Bulgare des Sciences, vol. 25, pp. 727–730, 1972.

[3] L. B. Ciri´c, Generalized contractions and fixed-point theorems, Publications de l'Institut Math´ematique, vol. 12, no. 26, pp. 19–26, 1971.

[4] K. Dogan, F. Gursoy and V. Karakaya, Some fixed point results in the generalized convex metric spaces, TWMS J. App. Eng. Math. V.10, N.1, 2020, pp.11-23.

[5] B. Gunduz and S. Akbulut, Strong convergence of an explicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces, Miskolc Mathematical Notes, Vol.14 (2013), No 3, pp. 905-913.

[6] G.E. Hardy and A.D. Rogers, A generalisation of fixed point theorem of Reich, Canad. Math. Bull., 16(1973), 201-208.

[7] R. Kannan, Some results on fixed points, Bulletin of the Calcutta Mathematical Society, vol. 60, pp. 71–76, 1968.

[8] A.R. Khan and M.A. Ahmed, Convergence of a general iterative scheme for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces and applications, Comput. Math. Appl. 59 (2010) 2990-2995.

[9] J. K. Kim, K. S. Kim and S. M. Kim, Convergence theorems of implicit iteration process for for finite family of asymptotically quasi-nonexpansive mappings in convex metric space, Nonlinear Analysis and Convex Analysis, 1484 (2006) 40-51.

[10] Q.Y. Liu, Z.B. Liu and N.J. Huang, Approximating the common fixed points of two sequences of uniformly quasi-Lipschitzian mappings in convex metric spaces, Appl. Math. Comp. 216 (2010) 883–889.

[11] G. Modi and B. Bhatt, Fixed point results for weakly compatible mappings in convex G-metric space, International Journal Of Mathematics And Statistics Invention, Volume 2, Issue 11 (2014), 34-38.

[12] Z. Mustafa, A New Structure For Generalized Metric Spaces – With Applications To Fixed Point Theory, PhD Thesis, the University of Newcastle, Australia, 2005.

[13] Z. Mustafa and B. Sims, A New Approach to Generalized Metric Spaces, Journal of Nonlinear and Convex Analysis, Volume 7, No. 2 (2006). 289–297.

[14] A. Rafik, Fixed Points of Ciric Quasi-contractive Operators in Generalized Convex Metric Spaces,General Mathematics Vol. 14, No. 3 (2006), 79–90.
[15] S. Reich, Some remarks concerning contraction mappings, Canadian Mathematical Bulletin, vol. 14, pp. 121–124, 1971.
[16] P. Thangavelu, S. Shyamala Malini and P. Jeyanthi, Convexity in D-Metric Spaces and its applications to fixed point theorems, International Journal of Statistika and Mathematika, Vol. 2, Issue 3, (2012), 05-12.
[17] I. Yildirim and S. H. Khan, Convergence theorems for common fixed points of asymptotically quasi-nonexpansive mappings in convex metric spaces, Applied Mathematics and Computation, Volume 218, Issue 9 (2012), 4860-4866.
[18] I. Yildirim, S. H. Khan and M. Ozdemir, Some fixed point results for uniformly quasi-Lipschitzian mappings in convex metric spaces, Journal of Nonlinear Analysis and Optimization: Theory & Applications, Vol. 4, No. 2, (2013), 143148.
[19] I. Yildirim, Fixed point results for some contraction type mappings in A-metric spaces, Submitted, 2021.
[20] I. Yildirim, On the stability and convergence of Mann iteration process in convex A-metric spaces, arXiv:2101.06042, 2021.

I. Yildirim, Department of Mathematics, Faculty of Science, Atatürk University, Erzurum, 25240, Turkey.
E-mail address: isayildirim@atauni.edu.tr