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# *F*IXED POINT THEOREMS OF GENERALIZED $(\Psi,\Phi)_s-$RATIONAL TYPE CONTRACTIVE MAPPINGS IN QUASI $b-$METRIC SPACES

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Abstract. In this paper, we consider the setting of quasi $b-$metric spaces to establish results regarding the fixed point theorems with help of new notion $(\Psi,\Phi)_s-$rational type quasi $b-$metric spaces with application. An example is presented to support our results comparing with existing ones.

## 1. Introduction

The classical Banach contraction principle [1] is one of the most popular fundamental tool in fixed point theory. There are lots of generalizations and extensions in many directions by many researchers. Berinde in [2, 3] introduced the concept of almost contractions and proved several attracting results for a Ciric strong almost contraction. The concept of $b-$metric spaces was introduced by Czerwik (1993) [4]. Shah and Hussain [5] initiated the concept of quasi $b-$metric spaces and proved some fixed point results in quasi $b-$metric spaces. Also, in recent years several papers have been published in new results that are connected with the fixed point results of various classes of $b-$metric spaces (refer some similar works on, [6, 7, 8, 9, 10]). The aim of this paper is to prove common fixed point theorems for generalized $(\Psi,\Phi)_s-$rational type contraction mappings satisfying an ordered quasi $b-$metric spaces.

## 2. Preliminaries

**Definition 1** The function $\Phi:\mathbb{R}^+\to\mathbb{R}^+$ is said to be an altering distance function if the following conditions are satisfied:

(i) $\Phi$ is continuous and increasing
(ii) $\Phi(a)=0$ if and only if $a=0$

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**Definition 2** Let $\mathcal{B}$ be a nonvoid set and $s \geq 1$ be a given real number. A function $q_b : \mathcal{B} \times \mathcal{B} \to \mathbb{R}^+$ is called quasi $b-$metric space if for all $\eta, \lambda, \kappa \in X$, the following condition hold:

(i) $q_b(\eta, \lambda) = 0$ if and only if $\eta = \lambda$
(ii) $q_b(\eta, \lambda) = q_b(\lambda, \eta)$
(iii) $q_b(\eta, \lambda) \preceq s[q_b(\eta, \kappa) + q_b(\kappa, \lambda)]$.

In this case, the pair $(\mathcal{B}, q_b)$ is called a quasi $b-$metric space.
It should be noted that, the class of quasi $b-$metric spaces is effectively large than the class of metric space, since a quasi $b-$metric is a metric, when $s = 1$.
**Definition 3** Let $\mathcal{B}$ be a non void set. Then $(\mathcal{B}, q_b, \preceq)$ is called an ordered quasi $b-$metric space iff:

(i) $(\mathcal{B}, q_b)$ is a quasi $b-$metric space,
(ii) $(\mathcal{B}, \preceq)$ is partial ordered

**Definition 4** Let $(\mathcal{B}, q_b, \preceq)$ is partial ordered set. $\eta, \lambda \in \mathcal{B}$ are called comparable if $\eta \preceq \lambda$ or $\lambda \preceq \eta$ holds.
**Definition 5** Let $(\mathcal{B}, \preceq)$ partially ordered set. A mapping $T : \mathcal{B} \to \mathcal{B}$ is said to be strictly weakly increasing if $\eta_0 \preceq T\eta_0$, for all $\eta_0 \in \mathcal{B}$.
**Definition 6** Let $(\mathcal{B}, q_b)$ be a quasi $b-$metric space. Then a sequence $\{\eta_n\}$ is called:

(i) quasi $b-$convergent if and only if there exists $\eta \in \mathcal{B}$ such that $q_b(\eta_n, \eta) \to 0$, as $n \to \infty$. In this case, we write $\lim_{n\to\infty} \eta_n = \eta$.
(ii) quasi $b-$Cauchy if and only if $\lim_{n,m\to\infty} q_b(\eta_n, \eta_m) = 0$ as $m, n \to \infty$.

**Proposition 1** In a quasi $b-$metric space $(\mathcal{B}, q_b)$ the following assertions hold:

$(p_1)$ A quasi $b-$convergent sequence has a unique limit.
$(p_2)$ Each quasi $b-$convergent sequence is quasi $b-$Cauchy.
$(p_3)$ In general, a quasi $b-$metric is not continuous.

**Definition 7** The quasi $b-$metric space $(\mathcal{B}, q_b)$ is quasi $b-$complete if every quasi $b-$Cauchy sequence in $\mathcal{B}$ quasi $b-$converges.
Khan, Swaleh, and Sessa [11] introduced the concept of an altering distance functions and established some fixed point theorems in self-maps of a complete metric spaces. Several authors have studied fixed point results which are based on alerting distance functions. Here we introduce new notion of generalized $(\Psi, \Phi)_s-$ rational type quasi $b-$metric spaces where $\Psi$ and $\Phi$ are the altering distance functions.

## 3. Main theorems

In this section, we give our main results.
**Definition 8** Let $(\mathcal{B}, \preceq, q_b)$ be an ordered quasi $b-$metric space. Let $\Psi$ and $\Phi$ be altering distance functions. Then the mapping $T : \mathcal{B} \to \mathcal{B}$ is a generalized $(\Psi, \Phi)_s-$rational contraction mapping if there exists $C \geq 0$ and two altering distance functions $\Psi$ and $\Phi$ such that

$$\Psi(sq_b(T\eta, T\lambda)) \preceq \Psi(E_s(\eta, \lambda)) - \Phi(E_s(\eta, \lambda)) + C\Psi(F(\eta, \lambda)) \tag{1}$$

where

$$E_s(\eta, \lambda) = max\left\{q_b(\eta, \lambda), q_b(\lambda, T\lambda), \frac{q_b(\eta, T\lambda)q_b(\lambda, T\lambda)}{s(1 + q_b(\eta, \lambda) + q_b(\lambda, T\lambda))}\right\}$$

and

$$F(\eta, \lambda) = min\left\{\frac{q_b(\eta, T\eta)q_b(\lambda, T\eta)}{1 + q_b(\eta, \lambda)}, \frac{q_b(\eta, T\lambda)q_b(\lambda, T\eta)}{1 + q_b(\eta, \lambda)}\right\}$$

for all comparable $\eta, \lambda \in \mathcal{B}$.

**Lemma 1** Let $(\mathcal{B}, q_b)$ be a quasi $b-$metric space with $s \geq 1$, and suppose that $\{\eta_n\}$ and $\{\lambda_n\}$ quasi $b-$converge to $\eta, \lambda$ respectively. Then, we have

$$\frac{1}{s^2}q_b(\eta, \lambda) \leq \lim_{n\to\infty} \inf q_b(\eta_n, \lambda_n) \leq \lim_{n\to\infty} \sup q_b(\eta_n, \lambda_n) \leq s^2 q_b(\eta, \lambda).$$

In particular, if $\eta = \lambda$, then, $\lim\limits_{n,m\to\infty} q_b(\eta_n, \lambda_m) = 0$. Moreover, for each $\kappa \in X$ we have

$$\frac{1}{s}q_b(\eta, \kappa) \leq \lim_{n\to\infty} \inf q_b(\eta_n, \kappa) \leq \lim_{n\to\infty} \sup q_b(\eta_n, \kappa) \leq s q_b(\eta, \lambda).$$

**Proof** Suppose assume that $\{\eta_n\}$ and $\{\lambda_n\}$ are quasi $b-$convergent sequence, then using triangle inequality we may write

$$\begin{aligned} q_b(\eta, \lambda) &\leq s q_b(\eta, \eta_n) + s q_b(\eta_n, \lambda) \\ &= s q_b(\eta, \eta_n) + s^2 q_b(\eta_n, \lambda_n) + s^2 q_b(\lambda_n, \lambda) \end{aligned} \tag{2}$$

and

$$\begin{aligned} q_b(\eta_n, \lambda_n) &\leq s q_b(\eta_n, \eta) + s q_b(\eta, \lambda_n) \\ &= s q_b(\eta_n, \eta) + s^2 q_b(\eta, \lambda) + s^2 q_b(\lambda, \lambda_n) \end{aligned} \tag{3}$$

Letting the lower limit as $n$ approaches to infinity in (2) and upper limit as $n$ approaches to infinity in (3) we wish the first inequality. Similarly, using again the same argument the second inequality follows.

**Theorem 1** Let $(\mathcal{B}, \preceq)$ be a partially ordered set and suppose that there exists a quasi $b-$metric $q_b$ on $\mathcal{B}$ such that $(\mathcal{B}, q_b)$ is a quasi $b-$complete $b-$metric space. Let $T : \mathcal{B} \to \mathcal{B}$ be an increasing continuous mapping with respect to $\preceq$. Suppose that $T$ is an $(\Psi, \Phi)-$rational contractive mapping for all comparable $\eta, \lambda \in \mathcal{B}$, there exits $\eta_0 \in \mathcal{B}$ such that $\eta_0 \preceq T\eta_0$, then $T$ has a fixed point.

**Proof** To prove that $T$ is a fixed point.

Let $\eta_0$ be an arbitrary point in $\mathcal{B}$. We define a sequence $\{\eta_n\}$ in $\mathcal{B}$ such that $\eta_{n+1} = T\eta_n$ for all $n \geq 0$. Since $\eta_0 \preceq T\eta_0 = \eta_1 = T\eta_0 \preceq \eta_2 = T\eta_1$ $T$, is an increasing sequence. Again, as $\eta_1 \preceq \eta_2$ and $T$ is an increasing, we have $\eta_2 = T\eta_1 \preceq \eta_3 = T\eta_2$. By induction, we have

$$\eta_0 \preceq \eta_1 \preceq ..... \preceq \eta_n \preceq \eta_{n+1} \preceq ....$$

If $\eta_n = \eta_{n+1}$ for some $n \in \mathbb{N}$, then $\eta_n = T\eta_n$ and hence $\eta_n$ is a fixed point of $T$. Consider $\eta_n \neq \eta_{n+1}$ for every $n \in \mathbb{N}$. By equation (1), we have

$$\begin{aligned} \Psi(q_b(\eta_n, \eta_{n+1})) &\preceq \Psi(s q_b(\eta_n, \eta_{n+1})) \\ &= \Psi(s q_b(T\eta_{n-1}, T\eta_n)) \\ &\preceq \Psi(E_s(\eta_{n-1}, \eta_n)) - \Phi(E_s(\eta_{n-1}, \eta_n)) + C\Psi(F(\eta_{n-1}, \eta_n)) \end{aligned} \tag{4}$$

where

$$
\begin{aligned}
&E_s(\eta_{n-1},\eta_n)\\
&= max\left\{q_b(\eta_{n-1},\eta_n), q_b(\eta_n,T\eta_n), \frac{q_b(\eta_{n-1},T\eta_n)q_b(\eta_n,T\eta_n)}{s(1+q_b(\eta_{n-1},\eta_n)+q_b(\eta_n,T\eta_n))}\right\}\\
&\preceq max\left\{q_b(\eta_{n-1},\eta_n), q_b(\eta_n,\eta_{n+1}), \frac{q_b(\eta_{n-1},\eta_{n+1})q_b(\eta_n,\eta_{n+1})}{s(1+q_b(\eta_{n-1},\eta_n)+q_b(\eta_n,\eta_{n+1}))}\right\}\\
&\preceq max\left\{q_b(\eta_{n-1},\eta_n), q_b(\eta_n,\eta_{n+1}), \frac{(sq_b(\eta_{n-1},\eta_n)+sq_b(\eta_n,\eta_{n+1}))q_b(\eta_n,\eta_{n+1})}{s(1+q_b(\eta_{n-1},\eta_n)+q_b(\eta_n,\eta_{n+1}))}\right\}\\
&= max\left\{q_b(\eta_{n-1},\eta_n), q_b(\eta_n,\eta_{n+1})\right\}
\end{aligned}
\tag{5}
$$

Since $|1+q_b(\eta_{n-1},\eta_n)+q_b(\eta_n,\eta_{n+1})| > |q_b(\eta_{n-1},\eta_n)+q_b(\eta_n,\eta_{n+1})|$.
Now, let us take,

$$
\begin{aligned}
&F(\eta_{n-1},\eta_n)\\
&\quad= min\left\{\frac{q_b(\eta_{n-1},T\eta_n)q_b(\eta_n,T\eta_{n-1})}{1+q_b(\eta_{n-1},\eta_n)}, \frac{q_b(\eta_{n-1},T\eta_n)q_b(\eta_n,T\eta_{n-1})}{1+q_b(\eta_{n-1},\eta_n)}\right\}\\
&\quad\preceq min\left\{\frac{q_b(\eta_{n-1},\eta_{n+1})q_b(\eta_n,\eta_n)}{1+q_b(\eta_{n-1},\eta_n)}, \frac{q_b(\eta_{n-1},\eta_{n+1})q_b(\eta_n,\eta_n)}{1+q_b(\eta_{n-1},\eta_n)}\right\} = 0.
\end{aligned}
\tag{6}
$$

Taking (4) and (6) into account, (4) yields

$$
\begin{aligned}
\Psi(q_b(\eta_n,\eta_{n+1})) &\preceq \Psi\left(max\{q_b(\eta_{n-1},,\eta_n), sq_b(\eta_n,\eta_{n+1})\}\right)\\
&\quad - \Phi(max\{q_b(\eta_{n-1},\eta_n), sq_b(\eta_n,\eta_{n+1})\})\\
&\prec \Psi\left(max\{q_b(\eta_{n-1},\eta_n), sq_b(\eta_n,\eta_{n+1})\}\right).
\end{aligned}
\tag{7}
$$

Suppose

$$max\{q_b(\eta_{n-1},\eta_n), q_b(\eta_n,\eta_{n+1})\} = q_b(\eta_n,\eta_{n+1}),$$

then (7) becomes,

$$\Psi(q_b(\eta_n,\eta_{n+1})) \preceq \Psi\left(max\{q_b(\eta_{n-1},\eta_n), q_b(\eta_n,\eta_{n+1})\}\right) < \Psi q_b(\eta_n,\eta_{n+1}),$$

which gives a contradiction. Therefore,

$$max\{q_b(\eta_{n-1},\eta_n), q_b(\eta_n,\eta_{n+1})\} = q_b(\eta_{n-1},\eta_n).$$

Thus, inequality (7) becomes

$$\Psi(q_b(\eta_n,\eta_{n+1})) \preceq \Psi(q_b(\eta_{n-1},\eta_n)) - \Phi(q_b(\eta_{n-1},\eta_n)) \prec \Psi(q_b(\eta_{n-1},\eta_n)). \tag{8}$$

Since $\Psi$ is an increasing mapping, therefore $\{q_b(\eta_n,\eta_{n+1}) : n \in \mathbb{N}\cup\{0\}\}$ is an increasing sequence of positive numbers. So, there exists $\tau \geq 0$ such that

$$\lim_{n\to\infty} q_b(\eta_n,\eta_{n+1}) = \tau.$$

Letting $n$ approaches to infinity in (8), we get

$$\Psi(\tau) \leq \Psi(\tau) - \Phi(\tau) < \Psi(\tau).$$

Therefore $\Phi(\tau) = 0$, hence, $\tau = 0$. Thus, we have

$$\lim_{n\to\infty} q_b(\eta_n,\eta_{n+1}) = 0. \tag{9}$$

Now, we show that $\{\eta_n\}$ is a quasi $b-$Cauchy sequence in $\mathcal{B}$. Suppose conversely assume that $\{\eta_n\}$ is not a quasi $b-$Cauchy sequence. Then there exists $\gamma > 0$ for which we find that two subsequence's $\{\eta_{u_i}\}$ and $\{\eta_{v_i}\}$ of the sequence $\eta_n$ such that $v_i$ is the smallest index for which

$$v_i \succ u_i \succ i, \qquad q_b(\eta_{u_i},\eta_{v_i}) \succeq \gamma. \tag{10}$$

which shows that

$$q_b(\eta_{u_i},\eta_{v_i-1}) \prec \gamma. \tag{11}$$

From (10) and (11) implies that

$$\begin{aligned}\gamma &\preceq q_b(\eta_{u_i},\eta_{v_i})\\ &\preceq sq_b(\eta_{u_i},\eta_{u_i-1}) + sq_b(\eta_{u_i-1},\eta_{v_i})\\ &\preceq sq_b(\eta_{u_i},\eta_{u_i-1}) + s^2q_b(\eta_{u_i-1},\eta_{v_i-1}) + s^2q_b(\eta_{v_i-1},\eta_{v_i})\end{aligned}$$

Using (9) and taking the upper limit as $i$ approaches to infinity, we get

$$\frac{\gamma}{s^2} \preceq \lim_{i\to\infty} sup \ \ q_b(\eta_{u_i-1},\eta_{v_i-1}).$$

On the other hand, we have

$$q_b(\eta_{u_i-1},\eta_{v_i-1}) \preceq sq_b(\eta_{u_i-1},\eta_{v_i}) + sq_b(\eta_{u_i},\eta_{v_i-1}).$$

Using (9), (11) and taking the upper limit as $i$ tends to infinity, we get

$$\lim_{i\to\infty} q_b(\eta_{u_i-1},\eta_{v_i-1}) \preceq \gamma s.$$

So, we have

$$\frac{\gamma}{s^2} \preceq \lim_{i\to\infty} sup \ \ q_b(\eta_{u_i-1},\eta_{v_i-1}) \preceq \gamma s. \tag{12}$$

Again, using the triangular inequality, we have

$$\begin{gathered}q_b(\eta_{u_i-1},\eta_{v_i}) \preceq sq_b(\eta_{u_i-1},\eta_{v_i-1}) + sq_b(\eta_{v_i},\eta_{v_i}),\\ \gamma \preceq q_b(\eta_{u_i},\eta_{v_i}) \preceq sq_b(\eta_{u_i},\eta_{v_i-1}) + sq_b(\eta_{u_i-1},\eta_{v_i})\end{gathered}$$

and

$$\gamma \preceq q_b(\eta_{u_i},\eta_{v_i}) \preceq sq_b(\eta_{u_i},\eta_{v_i-1}) + sq_b(\eta_{v_i-1},\eta_{v_i}).$$

Taking the upper limits as $i$ approaches to infinity in the first and second inequalities above, and using (9) and (12) we get

$$\frac{\gamma}{s} \preceq \lim_{i\to\infty} sup \ \ q_b(\eta_{u_i-1},\eta_{v_i}) \preceq \gamma s^2. \tag{13}$$

Likewise, taking the upper limit as $i$ approaches to infinity in the third inequality above, and using (9) and (11) we get

$$\frac{\gamma}{s} \preceq \lim_{i\to\infty} q_b(\eta_{u_i},\eta_{v_i-1}) \preceq \gamma. \tag{14}$$

From (1), $(\Psi,\Phi)_s$−rational contraction mapping, we have

$$\begin{aligned}\Psi(sq_b(\eta_{u_i},\eta_{v_i})) &= \Psi(sq_b(T\eta_{u_i-1},T\eta_{v_i-1}))\\ &\le \Psi(E_s(\eta_{u_i-1},\eta_{v_i-1})) - \Phi(E_s(\eta_{u_i-1},\eta_{v_i-1}))\\ &\quad + C\Psi(F(\eta_{u_i-1},\eta_{v_i-1})),\end{aligned} \tag{15}$$

where

$$E_s(\eta_{u_i-1},\eta_{v_i-1})$$

$$\begin{aligned}
&= max\Big\{q_b(\eta_{u_i-1},\eta_{v_i-1}), q_b(\eta_{v_i-1},T\eta_{v_i-1}), \\
&\qquad \frac{q_b(\eta_{u_i-1},T\eta_{v_i-1})q_b(\eta_{v_i-1},T\eta_{v_i-1})}{s(1+q_b(\eta_{u_i-1},\eta_{v_i-1})+q_b(\eta_{v_i-1},T\eta_{v_i-1}))}\Big\} \\
&= max\left\{q_b(\eta_{u_i-1},\eta_{v_i-1}), q_b(\eta_{v_i-1},\eta_{v_i}), \frac{q_b(\eta_{u_i-1},\eta_{v_i})q_b(\eta_{v_i-1},\eta_{v_i})}{s(1+q_b(\eta_{u_i-1},\eta_{v_i-1})+q_b(\eta_{v_i-1},\eta_{v_i}))}\right\}
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
F(\eta_{u_i-1},\eta_{v_i-1}) &= min\Big\{\frac{q_b(\eta_{u_i-1},T\eta_{u_i-1})q_b(\eta_{v_i-1},T\eta_{u_i-1})}{1+q_b(\eta_{u_i-1},\eta_{v_i-1})}, \\
&\qquad \frac{q_b(\eta_{u_i-1},T\eta_{v_i-1})q_b(\eta_{v_i-1},T\eta_{u_i-1})}{1+q_b(\eta_{u_i-1},\eta_{v_i-1})}\Big\} \\
&= min\Big\{\frac{q_b(\eta_{u_i-1},\eta_{u_i})q_b(\eta_{v_i-1},\eta_{u_i})}{1+q_b(\eta_{u_i-1},\eta_{v_i-1})}, \\
&\qquad \frac{q_b(\eta_{u_i-1},\eta_{v_i})q_b(\eta_{v_i-1},\eta_{u_i})}{1+q_b(\eta_{u_i-1},\eta_{v_i-1})}\Big\}.
\end{aligned} \tag{17}$$

Taking the upper limit as $i$ approaches to infinity in (16) and (17) and using (9), (12), (13) and (14), we get

$$\begin{aligned}
\frac{\gamma}{s^2} &= min\left\{\frac{\gamma}{s^2}, 0\right\} \\
&\preceq \lim_{i\to\infty} sup\ E_s(q_b(\eta_{u_i-1},\eta_{v_i-1})) \\
&= max\left\{\lim_{i\to\infty} sup\ q_b(\eta_{u_i-1},\eta_{v_i-1}), 0, \right\} \\
&\preceq max\{\gamma s, 0\} \\
&= \gamma s.
\end{aligned}$$

So, we have

$$\frac{\gamma}{s^2} \preceq \lim_{i\to\infty} sup\ E_s(q_b(\eta_{u_i-1},\eta_{v_i-1})) \preceq \gamma s. \tag{18}$$

and

$$\lim_{i\to\infty} F(\eta_{u_i-1},\eta_{v_i-1}) = 0. \tag{19}$$

Likewise, we can obtain

$$\frac{\gamma}{s^2} \preceq \lim_{i\to\infty} inf\ E_s(q_b(\eta_{u_i-1},\eta_{v_i-1})) \preceq \gamma s. \tag{20}$$

Now, taking the upper limit as $i$ approaches to infinity in (15) and (10), (18) and (19), we have

$$\begin{aligned}\Psi(\gamma s) &= \Psi(s \lim_{i\to\infty} sup\ q_b(T\eta_{u_i}, T\eta_{v_i}))\\ &\le \Psi(\lim_{i\to\infty} sup\ E_s(\eta_{u_i-1}, \eta_{v_i-1})) - \lim_{i\to\infty} inf\ \Phi(E_s(\eta_{u_i-1}, \eta_{v_i-1}))\\ &\le \Psi(\gamma s) - \Phi(\lim_{i\to\infty} inf\ E_s(\eta_{u_i-1}, \eta_{v_i-1})),\end{aligned}$$

which further implies that

$$\Phi(\lim_{i\to\infty} inf\ E_s(\eta_{u_i-1}, \eta_{v_i-1})) = 0,$$

So, $\lim_{i\to\infty} inf\ E_s(\eta_{u_i-1}, \eta_{v_i-1}) = 0$, which is contradiction to (20). Thus $\eta_{n+1} = T\eta_n$ is a quasi $b-$Cauchy sequence in $\mathcal{B}$. As $\mathcal{B}$ is a quasi $b-$complete space, there exists $v \in \mathcal{B}$ such that $\eta_n \to v$ as $n \to \infty$, and

$$\lim_{n\to\infty} \eta_{n+1} = \lim_{n\to\infty} T\eta_n = v.$$

Now, If $T$ is continuous. Using the triangular inequality, we get

$$q_b(v, Tv) \preceq sq_b(v, T\eta_n) + sq_b(T\eta_n, Tv).$$

Letting $n$ approaches to infinity, we get

$$q_b(v, Tv) \preceq s \lim_{n\to\infty} q_b(v, T\eta_n) + s \lim_{n\to\infty} q_b(T\eta_n, Tv) = 0.$$

So, we have $Tv = v$. Hence, $v$ is fixed point.

Without assuming the continuous the Theorem 1 we have the following fixed point theorem.

**Theorem 2** Let $(\mathcal{B}, \preceq)$ be a partially ordered quasi $b-$metric spaces such that the quasi $b-$metric is quasi $b-$complete. Let $T : \mathcal{B} \to \mathcal{B}$ be an increasing mapping with respect to $\preceq$. Suppose that $T$ is an $(\Psi, \Phi)_s-$rational contractive mapping for all comparable elements $\eta, \lambda \in \mathcal{B}$. Assume that whenever $\eta_n$ is an increasing sequence in $\mathcal{B}$ such that $\eta_n \to \eta \in \mathcal{B}$ implies $\eta_n \preceq \eta$ for all $n \in \mathbb{N}$, then $T$ has a fixed point.

**proof** The similar argument followed from the Theorem 1, we construct an increasing sequence $\{\eta_n\}$ in $\mathcal{B}$ such that $\eta_n \to v$ for some $v \in \mathcal{B}$. Using the assumption of $\mathcal{B}$, we have $\eta_n \preceq v$ for all $n \in \mathbb{N}$. Now, it is enough to show that $T$ has a fixed point. By $(\Psi, \Phi)_s-$rational contraction mapping, we have

$$\begin{aligned}\Psi(sq_b(\eta_{n+1}, Tv)) &= \Psi(sq_b(T\eta_n, Tv))\\ &\le \Psi(E_s(\eta_n, v)) - \Phi(E_s(\eta_n, v)) + C\Psi(F(\eta_n, v)),\end{aligned} \tag{21}$$

where

$$E_s(\eta_n, v) = max\left\{q_b(\eta_n, v), q_b(v, Tv), \frac{q_b(\eta_n, Tv)q_b(v, Tv)}{s(1 + q_b(\eta_n, v) + q_b(v, Tv))}\right\}, \tag{22}$$

and

$$\begin{aligned}F(\eta_n, v) &= min\left\{\frac{q_b(\eta_n, T\eta_n)q_b(v, T\eta_n)}{1 + q_b(\eta_n, v)}, \frac{q_b(\eta_n, Tv)q_b(v, T\eta_n)}{1 + q_b(\eta_n, v)}\right\}\\ &= min\left\{\frac{q_b(\eta_n, \eta_{n+1})q_b(v, \eta_{n+1})}{1 + q_b(\eta_n, v)}, \frac{q_b(\eta_n, Tv)q_b(v, \eta_{n+1})}{1 + q_b(\eta_n, v)}\right\}.\end{aligned} \tag{23}$$

Letting $n$ approaches to infinity in (22) and (23) and using Lemma 1, we get

$$\begin{aligned} 0 &= min\left\{0, q_b(v,Tv)\right\}. \\ &\preceq sup \lim_{n\to\infty} E_s(\eta_n, v) \\ &\preceq max\left\{0, q_b(v,Tv)\right\} \\ &= q_b(v,Tv). \end{aligned} \tag{24}$$

and

$$F(\eta_n, v) = 0.$$

Likewise, we can obtain

$$E_s(\eta_n, v) = \lim_{n\to\infty} inf\ E_s(\eta_n, v) \preceq q_b(v,Tv). \tag{25}$$

Again, taking the upper limit as $n$ approaches to infinity in (21) and using Lemma 1 and (24) we get

$$\begin{aligned} \Psi(q_b(v,Tv)) &\preceq \Psi(s\frac{1}{s}q_b(v,Tv)) \preceq \Psi(s \lim_{n\to\infty} sup\ q_b(\eta_{n+1}, Tv) \\ &\preceq \Psi(\lim_{n\to\infty} sup\ E_s(\eta, v) - \lim_{n\to\infty} inf\ \Phi(sq_b(\eta_{n+1}, Tv) \\ &\preceq \Psi(q_b(v,Tv)) - \Phi(\lim_{n\to\infty} inf\ E_s(\eta, v)). \end{aligned}$$

Therefore $\Phi(\lim_{n\to\infty} inf\ E_s(\eta_n, v)) \leq 0$, equally, $\lim_{n\to\infty} inf\ E_s(\eta_n, v)) = 0$. Thus, from (25) we get $v = Tv$ and hence $v$ is a fixed point of $T$.

**Corollary 1** Let $(\mathcal{B}, \preceq)$ be a partially ordered quasi $b-$metric spaces such that the quasi $b-$metric is $b-$complete. Let $T : \mathcal{B} \to \mathcal{B}$ be an increasing continuous mapping with respect to $\preceq$. Suppose that $r \in [0,1)$ and $C \geq 0$ such that

$$\begin{aligned} q_b(T\eta, T\lambda) \leq \frac{r}{s}\ max &\left\{q_b(\eta,\lambda), q_b(\lambda,T\lambda), \frac{q_b(\eta,T\lambda)q_b(\lambda,T\lambda)}{s(1+q_b(\eta,\lambda)+q_b(\lambda,T\lambda))}\right\} \\ &+ \frac{C}{s} min\left\{\frac{q_b(\eta,T\eta)q_b(\lambda,T\eta)}{1+q_b(\eta,\lambda)}, \frac{q_b(\eta,T\lambda)q_b(\lambda,T\eta)}{1+q_b(\eta,\lambda)}\right\} \end{aligned}$$

for all comparable elements $\eta, \lambda \in \mathcal{B}$. Suppose there exist $\eta_0 \in \mathcal{B}$ such that $\eta_0 \preceq T\eta_0$ then $T$ has a fixed point.

**Proof** It follows from the Theorem 1 let us consider $\Psi(t) = t$ and $\Phi(t) = (1-r)t$ for every $t \in \mathbb{R}^+$ and noticing that $T$ is generalized $(\Psi, \Phi)_s-$rational contraction mapping and hence it shows that $T$ has a fixed point.

Without assuming continuity of $T$ in the corollary 1.

**Corollary 2** Let $(\mathcal{B}, \preceq)$ be a partially ordered quasi $b-$metric spaces such that the quasi $b-$metric is quasi $b-$complete. Let $T : \mathcal{B} \to \mathcal{B}$ be an increasing continuous mapping with respect to $\preceq$. Suppose that $r \in [0,1)$ and $C \geq 0$ such that

$$\begin{aligned} q_b(T\eta, T\lambda) \leq \frac{r}{s}\ max &\left\{q_b(\eta,\lambda), q_b(\lambda,T\lambda), \frac{q_b(\eta,T\lambda)q_b(\lambda,T\lambda)}{s(1+q_b(\eta,\lambda)+q_b(\lambda,T\lambda))}\right\} \\ &+ \frac{C}{s} min\left\{\frac{q_b(\eta,T\eta)q_b(\lambda,T\eta)}{1+q_b(\eta,\lambda)}, \frac{q_b(\eta,T\lambda)q_b(\lambda,T\eta)}{1+q_b(\eta,\lambda)}\right\} \end{aligned}$$

for all comparable elements $\eta, \lambda \in \mathcal{B}$, and assume that $\eta_n$ is increasing sequence in $\mathcal{B}$ such that $\eta_n \to \eta \in \mathcal{B}$ implies $\eta_n \preceq \eta$ for all $n \in \mathbb{N}$, then $T$ has a fixed point.

**Proof** It follows from the Theorem 2 let us consider $\Psi(t) = t$ and $\Phi(t) = (1-r)t$ for every $t \in \mathbb{R}^+$ hence it shows that $T$ has a fixed point.
**Example 1** Let $\mathcal{B} = \{0,1,2\}$ be furnish with the quasi $b-$metric. Define $q_b$ : $\mathcal{B} \times \mathcal{B} \to \mathbb{R}^+$ such that

$$q_b(\eta,\lambda) = \begin{cases} 0, & \eta = \lambda. \\ |2\eta - \frac{\lambda}{3}|^2, & otherwise. \end{cases}$$

Note that $q_b(\eta,\lambda) = 0$ for all $k,l \in \mathcal{B}$ and $q_b(\eta,\lambda) = 0$ iff $\eta = \lambda$. we also note it $q_b(\eta,\lambda) = q_b(\lambda,\eta)$ iff $\eta = \lambda$ so it is not symmetric.
If $\eta = 0, \lambda = 1, \kappa = 2$ then

$$q_b(\eta,\kappa) = \tfrac{4}{9},\ q_b(\eta,\lambda) = \tfrac{1}{9},\ q_b(\lambda,\kappa) = \tfrac{16}{9}.$$

So that the usual triangle inequality not satisfied. Therefore the metric is a quasi $b-$metric on $\mathcal{B}$. Thus $(\mathcal{B},\preceq,q_b)$ is a partially ordered quasi $b-$metric space with costant $s \geq 2$.

## 4. Applications

Let $\Omega$ be the set of all functions $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying in the following hypotheses:

(i) Every $\Phi \in \Omega$ is a Lebesgue integrable function on each compact subset of $\mathbb{R}^+$,
(ii) For all $\Phi \in \Omega$ and $\omega > 0$.

$$\int_0^{\omega} \Phi(t)dt > 0$$

Let the function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$\Psi(\tau) = \int_0^{\tau} \Phi(t)dt > 0$$

is an altering distance functions. Therefore, we consider the following result.
**Corollary 1** Let $(\mathcal{B},\preceq)$ be a partially ordered quasi $b-$metric spaces such that the quasi $b-$metric is $b-$complete. Let $T : \mathcal{B} \to \mathcal{B}$ be an increasing continuous mapping with respect to $\preceq$. Suppose that $r \in [0,1)$ and $C \geq 0$ such that

$$\int_0^{q_b(T\eta,T\lambda)} \Phi(t)dt \leq \frac{r}{s}\int_0^{max\left\{q_b(\eta,\lambda),q_b(\lambda,T\lambda),\frac{q_b(\eta,T\lambda)q_b(\lambda,T\lambda)}{s(1+q_b(\eta,\lambda)+q_b(\lambda,T\lambda))}\right\}} \Phi(t)dt$$
$$+\frac{C}{s}\int_0^{min\left\{\frac{q_b(\eta,T\eta)q_b(\lambda,T\eta)}{1+q_b(\eta,\lambda)},\frac{q_b(\eta,T\lambda)q_b(\lambda,T\eta)}{1+q_b(\eta,\lambda)}\right\}} \Phi(t)dt$$

for all comparable elements $\eta,\lambda \in \mathcal{B}$. Suppose there exists $\eta_0 \in \mathcal{B}$ such that $\eta_0 \preceq T\eta_0$, then $T$ has a fixed point.
**Proof** It follows from the Theorem 1 by taking

$$\Psi(\tau) = \int_0^{\tau} \Phi(t)dt.$$

and $\Phi(\tau) = (1-r)\tau$, for all $\tau \in \mathbb{R}^+$.

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