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CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS DEFINED WITH GENERALIZED SÃLÃGEAN OPERATOR AND RELATED TO SIGMOID FUNCTION AND LEMNISCATE OF BERNOULLI

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Abstract. In this paper, the authors introduce new subclasses of multivalent functions defined with generalized Sãlãgean operator related to Sigmoid function and Lemniscate of Bernoulli. The initial coefficient bounds, Fekete-Szegö inequality and Hankel determinant problems are investigated for these classes. The results proved by various authors follow as special cases.

1. Introduction

The importance of theory of special functions can be gauged from the fact that it draws as much attention of scientists and engineers as that of the researchers working in the field of Physics and Computer science etc.

Out of the treasure of special functions, in this paper we shall focus on the sigmoid function given by

$$h(z) = \frac{1}{1+e^{-z}}, \tag{1}$$

whose working is analogous to the human brain.

The function $h(z)$ is a differentiable function possessing the following attributes:
(i) Its output ranges between 0 and 1.
(ii) Sufficiently large input domains are mapped onto a small output range.
(iii) Being a one-one function, no information is lost in the process.
(iv) The function is monotonically increasing.
The above mentioned properties make it quite useful in the Geometric function theory.

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Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{n=2}^{\infty} a_{n+p} z^{n+p}, \tag{2}$$

which are analytic and p-valent in the open unit disc $E = \{z : |z| < 1\}$.
Let U be the class of Schwarzian functions of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, \tag{3}$$

which are regular in the unit disc E and satisfying the conditions

$$w(0) = 0, |w(z)| < 1.$$

For the functions f and g in E, we say that f is subordinate to g in E, if a Schwarzian function $w(z) \in U$ can be found such that $f(z) = g(w(z))$, denoted by $f \prec g$. It follows from Schwarz lemma that $f(z) \prec g(z)$ implies that $f(0) = g(0)$ and $f(E) \subset g(E)$ (see detain in [11]).
Sokol and Thomas [22] introduced and studied the class S_L^* in the unit disc E, normalized by $f(0) = f'(0) - 1 = 0$ and satisfying the condition

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z} = q(z), z \in E,$$

where the branch of the square root is chosen to be $q(0) = 1$.
It is also noted that the set $q(E)$ lies in the region bounded by the right loop of the lemniscate of Bernoulli $\gamma_1 : (x^2 + y^2)^2 - 2(x^2 - y^2) = 0$.
For $f \in A_p$ and $\delta \geq 0$, Goyal et al. [8] introduced the following differential operator:

$$D_\delta^0 f(z) = f(z),$$

$$D_\delta^1 f(z) = (1-\delta) f(z) + \frac{\delta}{p} z f'(z) = D_\delta f(z),$$

and in general

$$D_\delta^t f(z) = D_\delta(D_\delta^{t-1} f(z)) = z^p + D_\delta f(z) + \sum_{k=p+1}^{\infty} \left[1 + \left(\frac{k}{p} - 1\right)\delta\right]^t a_k z^k, p \in N_0 = N \cup \{0\}$$

with $D_\delta^0 f(0) = 0$.
The operator $D_\delta^t f(z)$ is named as generalized Sãlãgean operator.
For $p = 1$, the above defined operator coincides with that introduced by Al-Aboudi [3].
For $p = 1, \delta = 1$, the operator $D_1^t f(z) \equiv D^t f(z)$, the well known Sãlãgean operator.
Let $\phi(z)$ be an analytic function with positive real part in E such that $\phi(0) = 1$ and $\phi'(0) > 0$ and maps E onto a region starlike with respect to 1 and symmetric with respect to the real axis.
In 1976, Noonan and Thomas [14] stated the qth Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q+1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has been investigated by several authors.
In the particular cases, $q = 2, n = p, a_1 = 1$ and $q = 2, n = p+1$, the Hankel determinant simplifies respectively to
$H_2(p) = |a_{p+2} - a_{p+1}^2|$ and $H_2(p+1) = |a_{p+1}a_{p+3} - a_{p+2}^2|$.
The functional $J_{n,m}(f) = a_n a_m - a_{m+n-1}$, $n, m \in N - \{1\}$, was investigated by Ma [10] and it is known as generalized Zalcman functional. The functional $J_{2,3}(f) = a_2 a_3 - a_4$ is a specific case of the generalized Zalcman functional. Various authors including Janowski [4, 5, 6, 21, 23] computed the upper bound for the functional $J_{2,3}(f)$ over different subclasses of analytic functions to obtain a bound for third Hankel determinant. For the functions in the class A_p, the Zalcman functional takes the form of $a_{p+1}a_{p+2} - a_{p+3}$.
For $q = 3$ and $n = p$ as

$$H_3(p) = \begin{vmatrix} a_p & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix},$$

which is known as Hankel determinant of order 3.
For $f \in A_p$ and $a_p = 1$, we have

$$H_3(p) = a_{p+2}(a_{p+1}a_{p+3} - a_{p+2}^2) - a_{p+3}(a_{p+3} - a_{p+1}a_{p+2}) + a_{p+4}(a_{p+2} - a_{p+1}^2),$$

and using the triangle inequality, it yields

$$|H_3(p)| \leq |a_{p+2}||a_{p+1}a_{p+3} - a_{p+2}^2| + |a_{p+3}||a_{p+3} - a_{p+1}a_{p+2}| + |a_{p+4}||a_{p+2} - a_{p+1}^2|. \tag{4}$$

A function $f(z) \in A_p$ is said to be in the class $S^*_{b,p}(\phi)$ if

$$1 + \frac{1}{b}\left[\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right] \prec \phi(z).$$

A function $f(z) \in A_p$ is said to be in the class $C_{b,p}(\phi)$ if

$$1 - \frac{1}{b} + \frac{1}{bp}\left[1 + \frac{zf''(z)}{f'(z)}\right] \prec \phi(z).$$

The classes $S^*_{b,p}(\phi)$ and $C_{b,p}(\phi)$, were studied in [1]. For $b = 1$, we have the classes $S^*_p(\phi)$ and $C_p(\phi)$(see [2]) and for $p = b = 1$ the classes reduce to $S^*(\phi)$ and $C(\phi)$ which were earlier introduced and investigated in [9]. These classes reduce to the classes of starlike and convex functions respectively when $\phi(z) = \dfrac{1+z}{1-z}$.
Also for $p = 1$ and $\phi(z) = \dfrac{1+z}{1-z}$, the classes $S^*_{b,p}(\phi)$ and $C_{b,p}(\phi)$ reduce to the classes $S^*(b)$ and $C(b)$ which were investigated in [13] and [24].
Motivated by above defined classes, we introduce the following subclasses of p-valent analytic functions of complex order related to sigmoid functions and Lemniscate of Bernoulli.
Definition 1.1 For $b \in C$, let the class $M_{p,\lambda}(\delta, t; b, \Phi_{m,n})$ denote the subclass of A_p consisting of functions of the form (2) and satisfying the following condition:

$$p + \frac{1}{b}\left[\frac{z(D^t_\delta f(z))' + \lambda z^2 (D^t_\delta f(z))''}{\lambda z(D^t_\delta f(z))' + (1-\lambda)(D^t_\delta f(z))} - p\right] > 0,$$

for $0 \leq \lambda \leq 1$ and $\Phi_{m,n}(z)$ is a simple logistic sigmoid activation function.
In particular, $M_{p,\lambda}(1, 0; b, \Phi_{m,n}) \equiv M_{p,\lambda}(b, \Phi_{m,n})$, the class studied by Singh and Singh [20].

Definition 1.2 For $b \in C$, let the class $G_{p,\lambda}(\delta, t; b, \Phi_{m,n})$ denote the subclass of A_p consisting of functions of the form (2) and satisfying the following condition:

$$p + \frac{1}{b}\left[\frac{z(D_\delta^t f(z))'}{D_\delta^t f(z)} + \lambda\frac{z^2(D_\delta^t f(z))''}{D_\delta^t f(z)} - p\right] > 0,$$

for $0 \leq \lambda \leq 1$ and $\Phi_{m,n}(z)$ is a simple logistic sigmoid activation function.
Particularly, $G_{p,\lambda}(1, 0; b, \Phi_{m,n}) \equiv G_{p,\lambda}(b, \Phi_{m,n})$, the class studied by Singh and Singh [20].
Definition 1.3 For $b \in C$, let the class $M_L(\delta, t; p, \lambda, b, \Phi_{m,n})$ denote the subclass of A_p consisting of functions of the form (2) and satisfying the following condition:

$$p + \frac{1}{b}\left[\frac{z(D_\delta^t f(z))' + \lambda z^2(D_\delta^t f(z))''}{\lambda z(D_\delta^t f(z))' + (1-\lambda)(D_\delta^t f(z))} - p\right] \prec p\sqrt{1+z},$$

where the branch of the square root is chosen to be $q(0) = 1, 0 \leq \lambda \leq 1$ and $\Phi_{m,n}(z)$ is a simple logistic sigmoid activation function.
Specifically, $M_L(1, 0; p, \lambda, b, \Phi_{m,n}) \equiv M_L(p, \lambda, b, \Phi_{m,n})$ the class studied by Olatunji and Dutta [18].
Definition 1.4 For $b \in C$, let the class $G_L(\delta, t; p, \lambda, b, \Phi_{m,n})$ denote the subclass of A_p consisting of functions of the form (2) and satisfying the following condition:

$$p + \frac{1}{b}\left[\frac{z(D_\delta^t f(z))'}{D_\delta^t f(z)} + \lambda\frac{z^2(D_\delta^t f(z))''}{D_\delta^t f(z)} - p\right] \prec \sqrt{1+z},$$

where the branch of the square root is chosen to be $q(0) = 1, 0 \leq \lambda \leq 1$ and $\Phi_{m,n}(z)$ is a simple logistic sigmoid activation function.
As a special case, $G_L(1, 0; p, \lambda, b, \Phi_{m,n}) \equiv G_L(p, \lambda, b, \Phi_{m,n})$ the class studied by Olatunji and Dutta [18].
Recently, various authors as Oladipo [15], Murugusundramoorthy et al. [12], Olatunji et al. [17], and Olatunji [16] have studied sigmoid function for different classes of analytic and univalent functions.
In the present work, we obtained initial coefficient bounds, Fekete-Szegö inequality, second and third Hankel determinants for the classes $M_{p,\lambda}(\delta, t; b, \Phi_{m,n})$, $G_{p,\lambda}(\delta, t; b, \Phi_{m,n})$, $M_L(\delta, t; p, \lambda, b, \Phi_{m,n})$ and $G_L(\delta, t; p, \lambda, b, \Phi_{m,n})$. The results proved by Singh and Singh [20] and Olatunji and Dutta [18] follows as special cases.
To prove our results, we shall make use of the following lemmas:
Lemma 1 [19] If a function $p \in P$ is given by $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$, then

$$|p_k| \leq 2, k \in N,$$

where P is the family of all functions analytic in E for which $p(0) = 1$ and $Re(p(z)) > 0$.
Lemma 2 [7] Let h be the sigmoid function defined in (1) and

$$\Phi_{m,n}(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right)^m,$$

then $\Phi(z) \in P$, $|z| < 1$, where $\Phi(z)$ is a modified sigmoid function.
Lemma 3 [7] Let

$$\Phi_{m,n}(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right)^m,$$

then $|\Phi_{m,n}(z)| < 2$.
Lemma 4 [7] If

$$\Phi(z) = 2h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

where $c_n = \dfrac{(-1)^{n+1}}{2n!}$, then $|c_n| \leq 2$, $n \in N$ and the result is sharp for each n.

2. The class $M_{p,\lambda}(\delta, t; b, \Phi_{m,n})$

2.1 Initial Coefficients

Theorem 2.1 If $f(z) \in A_p$, of the form (2) is belonging to $M_{p,\lambda}(\delta, t; b, \Phi_{m,n})$, then

$$|a_{p+1}| \leq \frac{p^{t+1}|b|[1+\lambda(p-1)]}{2[1+\lambda p](p+\delta)^t}, \tag{5}$$

$$|a_{p+2}| \leq \frac{p^{t+2}|b|^2[1+\lambda(p-1)]}{8[1+\lambda(p+1)](p+2\delta)^t} = h_1, \tag{6}$$

$$|a_{p+3}| \leq \frac{p^{t+1}|b|[1+\lambda(p-1)]}{24[1+\lambda(p+2)](p+3\delta)^t}\left|\frac{p^2b^2}{2} - \frac{1}{3}\right| = h_2 \tag{7}$$

and

$$|a_{p+4}| \leq \frac{p^{t+2}|b|^2[1+\lambda(p-1)]}{192[1+\lambda(p+3)](p+4\delta)^t}\left|\frac{p^2b^2}{2} - \frac{4}{3}\right| = h_3. \tag{8}$$

Proof. As $f \in M_{p,\lambda}(\delta, t; b, \Phi_{m,n})$, therefore

$$p + \frac{1}{b}\left[\frac{z(D^t_\delta f(z))' + \lambda z^2 (D^t_\delta f(z))''}{\lambda z(D^t_\delta f(z))' + (1-\lambda)(D^t_\delta f(z))} - p\right] = p\Phi_{m,n}(z), \tag{9}$$

where

$$\Phi_{m,n}(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + \frac{779}{20160}z^7 - ... \tag{10}$$

Using (10), (9) can be expanded as

$$(1+\lambda p)\left(1+\frac{\delta}{p}\right)^t a_{p+1}z + 2(1+\lambda(p+1))\left(1+\frac{2\delta}{p}\right)^t a_{p+2}z^2$$
$$+3(1+\lambda(p+2))\left(1+\frac{3\delta}{p}\right)^t a_{p+3}z^3 + 4(1+\lambda(p+3))\left(1+\frac{4\delta}{p}\right)^t a_{p+4}z^4 + ...$$
$$= bp\left[\frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5\right]$$
$$.\Bigg[(1+\lambda(p-1)) + (1+\lambda p)\left(1+\frac{\delta}{p}\right)^t a_{p+1}z + (1+\lambda(p+1))\left(1+\frac{2\delta}{p}\right)^t a_{p+2}z^2$$
$$+ (1+\lambda(p+2))\left(1+\frac{3\delta}{p}\right)^t a_{p+3}z^3 + (1+\lambda(p+3))\left(1+\frac{4\delta}{p}\right)^t a_{p+4}z^4 + ...\Bigg]. \tag{11}$$

Equating the coefficients of z, z^2, z^3 and z^4 in (11), we obtain

$$a_{p+1} = \frac{p^{t+1}b[1+\lambda(p-1)]}{2[1+\lambda p](p+\delta)^t}, \tag{12}$$

$$a_{p+2} = \frac{p^{t+2}b^2[1+\lambda(p-1)]}{8[1+\lambda(p+1)](p+2\delta)^t}, \tag{13}$$

$$a_{p+3} = \frac{p^{t+1}b[1+\lambda(p-1)]}{24[1+\lambda(p+2)](p+3\delta)^t}\left[\frac{p^2b^2}{2}-\frac{1}{3}\right] \tag{14}$$

and

$$a_{p+4} = \frac{p^{t+2}b^2[1+\lambda(p-1)]}{192[1+\lambda(p+3)](p+4\delta)^t}\left[\frac{p^2b^2}{2}-\frac{4}{3}\right]. \tag{15}$$

Results (5), (6), (7) and (8) can be easily obtained from (12), (13), (14) and (15) respectively.

For $\delta = 1, t = 0$, Theorem 2.1 gives the following result due to Singh and Singh [20].

Corollary 2.1 If $f(z) \in A_p$, of the form (2) is belonging to $M_{p,\lambda}(b, \Phi_{m,n})$, then

$$|a_{p+1}| \leq \frac{p|b|[1+\lambda(p-1)]}{2[1+\lambda p]},$$

$$|a_{p+2}| \leq \frac{p^2|b|^2[1+\lambda(p-1)]}{8[1+\lambda(p+1)]},$$

$$|a_{p+3}| \leq \frac{p|b|[1+\lambda(p-1)]}{24[1+\lambda(p+2)]}\left|\frac{p^2b^2}{2}-\frac{1}{3}\right|$$

and

$$|a_{p+4}| \leq \frac{p^2|b|^2[1+\lambda(p-1)]}{192[1+\lambda(p+3)]}\left|\frac{p^2b^2}{2}-\frac{4}{3}\right|.$$

2.2 Fekete-Szegö Inequality

Theorem 2.2 If $f(z) \in A_p$, of the form (2) is belonging to $M_{p,\lambda}(\delta, t; b, \Phi_{m,n})$, then

$$|a_{p+2}-\mu a_{p+1}^2| \leq \frac{p^{2t+2}|b|^2[1+\lambda(p-1)]}{4[1+\lambda p]^2(p+\delta)^{2t}}\left|\frac{(1+\lambda p)^2(p+\delta)^{2t}}{2p^t[1+\lambda(p+1)](p+2\delta)^t}-\mu[1+\lambda(p-1)]\right|. \tag{16}$$

Proof. From (11) and (12), we have

$$a_{p+2}-\mu a_{p+1}^2 = \frac{p^{2t+2}|b|^2[1+\lambda(p-1)]}{4[1+\lambda p]^2(p+\delta)^{2t}}\left[\frac{(1+\lambda p)^2(p+\delta)^{2t}}{2p^t[1+\lambda(p+1)](p+2\delta)^t}-\mu[1+\lambda(p-1)]\right]. \tag{17}$$

Hence (16) can be easily obtained from (17).

In particular for $\mu = 1$,

$$|a_{p+2}-a_{p+1}^2| \leq \frac{p^{2t+2}|b|^2[1+\lambda(p-1)]}{4[1+\lambda p]^2(p+\delta)^{2t}}\left|\frac{(1+\lambda p)^2(p+\delta)^{2t}}{2p^t[1+\lambda(p+1)](p+2\delta)^t}-[1+\lambda(p-1)]\right| = h_4. \tag{18}$$

For $\delta = 1, t = 0$, Theorem 2.2 agrees with the following result due to Singh and Singh [20].

Corollary 2.2 If $f(z) \in A_p$, of the form (2) is belonging to $M_{p,\lambda}(b, \Phi_{m,n})$, then

$$|a_{p+2}-\mu a_{p+1}^2| \leq \frac{p^2|b|^2[1+\lambda(p-1)]}{4(1+\lambda p)^2}\left[\frac{(1+\lambda p)^2}{2[1+\lambda(p+1)]}-\mu[1+\lambda(p-1)]\right].$$

2.3 Second hankel determinant

Theorem 2.3 If $f(z) \in A_p$, of the form (2) is belonging to $M_{p,\lambda}(\delta, t; b, \Phi_{m,n})$, then

$$|a_{p+1}a_{p+3}-\mu a_{p+2}^2| \leq \frac{p^{2t+2}|b|^2[1+\lambda(p-1)]^2}{16}$$

$$\cdot\left|\frac{1}{3(1+\lambda p)[1+\lambda(p+2)](p+\delta)^t(p+3\delta)^t}\left[\frac{p^2b^2}{2}-\frac{1}{3}\right]-\mu\frac{p^2b^2}{4[1+\lambda(p+1)]^2(p+2\delta)^{2t}}\right|. \tag{19}$$

Proof. From (12), (13) and (14), we have

$$a_{p+1}a_{p+3}-\mu a_{p+2}^2=\frac{p^{2t+2}|b|^2[1+\lambda(p-1)]^2}{48(1+\lambda p)[1+\lambda(p+2)](p+\delta)^t(p+3\delta)^t}\left[\frac{p^2b^2}{2}-\frac{1}{3}\right] -\mu\frac{p^{2t+4}b^4[1+\lambda(p-1)]^2}{64[1+\lambda(p+1)]^2(p+2\delta)^{2t}}. \tag{20}$$

Hence (19) can be easily obtained from (20).
In particular for $\mu=1$,

$$|a_{p+1}a_{p+3}-a_{p+2}^2|\le\frac{p^{2t+2}|b|^2[1+\lambda(p-1)]^2}{16} \cdot\left|\frac{1}{3(1+\lambda p)[1+\lambda(p+2)](p+\delta)^t(p+3\delta)^t}\left[\frac{p^2b^2}{2}-\frac{1}{3}\right]-\frac{p^2b^2}{4[1+\lambda(p+1)]^2(p+2\delta)^{2t}}\right|=h_5. \tag{21}$$

For $\delta=1,t=0$, Theorem 2.3 coincides with the following result due to Singh and Singh [20].
Corollary 2.3 If $f(z)\in A_p$, of the form (2) is belonging to $M_{p,\lambda}(b,\Phi_{m,n})$, then

$$|a_{p+2}a_{p+3}-\mu a_{p+2}^2|\le\frac{p^2|b|^2[1+\lambda(p-1)]^2}{16} \cdot\left|\frac{1}{3(1+\lambda p)[1+\lambda(p+2)]}\left[\frac{p^2b^2}{2}-\frac{1}{3}\right]-\mu\frac{p^2b^2}{4[1+\lambda(p+1)]^2}\right|.$$

2.4 Zalcman Functional
Theorem 2.4 If $f(z)\in A_p$, of the form (2) is belonging to $M_{p,\lambda}(\delta,t;b,\Phi_{m,n})$, then

$$|a_{p+1}a_{p+2}-a_{p+3}|\le\frac{p^{t+1}|b|[1+\lambda(p-1)]}{8} \cdot\left|\frac{p^{t+2}b^2[1+\lambda(p-1)]}{2(1+\lambda p)[1+\lambda(p+1)](p+\delta)^t(p+2\delta)^t}-\frac{1}{3[1+\lambda(p+2)](p+3\delta)^t}\left[\frac{p^2b^2}{2}-\frac{1}{3}\right]\right|=h_6. \tag{22}$$

Proof. From (11), (12) and (13), the result (22) is obvious.
For $\delta=1,t=0$, Theorem 2.4 gives the following result due to Singh and Singh [20].
Corollary 2.4 If $f(z)\in A_p$, of the form (2) is belonging to $M_{p,\lambda}(b,\Phi_{m,n})$, then

$$|a_{p+1}a_{p+2}-a_{p+3}|\le\left|\frac{p^3b^3[1+\lambda(p-1)]^2}{16(1+\lambda p)[1+\lambda(p+1)]}-\frac{pb[1+\lambda(p-1)]}{24[1+\lambda(p+2)]}\left[\frac{p^2b^2}{2}-\frac{1}{3}\right]\right|.$$

2.5 Third Hankel determinant
Theorem 2.5 If $f(z)\in A_p$, of the form (2) is belonging to $M_{p,\lambda}(\delta,t;b,\Phi_{m,n})$, then

$$|H_3(p)|\le h_1h_5+h_2h_6+h_3h_4,$$

where h_1,h_2,h_3,h_4,h_5 and h_6 are given by (11), (12), (13), (14), (15), (16) respectively.
Proof. The proof of the result is obvious.

3. The class $G_{p,\lambda}(\delta,t;b,\Phi_{m,n})$

3.1 Initial Coefficients
Theorem 3.1 If $f(z)\in A_p$, of the form (2) is belonging to $G_{p,\lambda}(\delta,t;b,\Phi_{m,n})$, then

$$|a_{p+1}|\le\frac{p^{t+1}|b|}{2[1+\lambda p(p+1)](p+\delta)^t}, \tag{23}$$

$$|a_{p+2}| \leq \frac{p^{t+2}|b|^2}{4[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)](p+2\delta)^t} = k_1, \tag{24}$$

$$|a_{p+3}| \leq \frac{p^{t+1}|b|}{8[3+\lambda(p+2)(p+3)](p+3\delta)^t}\left|\frac{p^2b^2(p+2\delta)^t}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]} - \frac{1}{3}\right| = k_2 \tag{25}$$

and

$$|a_{p+4}| \leq \frac{p^{t+2}|b|^2}{16[4+\lambda(p+3)(p+4)](p+4\delta)^t}$$

$$\cdot\left|\frac{1}{[3+\lambda(p+2)(p+3)]}\left[\frac{p^2b^2(p+2\delta)^t}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]} - \frac{1}{3}\right] - \frac{1}{3[1+\lambda p(p+1)]}\right| = k_3. \tag{26}$$

Proof. As $f \in G_{p,\lambda}(\delta, t; b, \Phi_{m,n})$, therefore

$$p + \frac{1}{b}\left[\frac{z(D_\delta^t f(z))'}{D_\delta^t f(z)} + \lambda\frac{z^2(D_\delta^t f(z))''}{D_\delta^t f(z)} - p\right] = p\Phi_{m,n}(z), \tag{27}$$

where

$$\Phi_{m,n}(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + \frac{779}{20160}z^7 - ... \tag{28}$$

Using (28), (27) can be expanded as

$$\lambda p^{t+1}(p-1) + [1+\lambda p(p+1)](p+\delta)^t a_{p+1}z + [2+\lambda(p+1)(p+2)](p+2\delta)^t a_{p+2}z^2$$
$$+[3+\lambda(p+2)(p+3)](p+3\delta)^t a_{p+3}z^3 + [4+\lambda(p+3)(p+4)](p+4\delta)^t a_{p+4}z^4 + ...$$
$$= bp\left[\frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5\right]$$

$$\cdot\left[p^t + (p+\delta)^t a_{p+1}z + (p+2\delta)^t a_{p+2}z^2 + (p+3\delta)^t a_{p+3}z^3 + (p+4\delta)^t a_{p+4}z^4 + ...\right]. \tag{29}$$

Equating the coefficients of z, z^2, z^3 and z^4 in (29), we obtain

$$a_{p+1} = \frac{p^{t+1}b}{2[1+\lambda p(p+1)](p+\delta)^t}, \tag{30}$$

$$a_{p+2} = \frac{p^{t+2}b^2}{4[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)](p+2\delta)^t}, \tag{31}$$

$$a_{p+3} = \frac{p^{t+1}b}{8[3+\lambda(p+2)(p+3)](p+3\delta)^t}\left[\frac{p^2b^2(p+2\delta)^t}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]} - \frac{1}{3}\right] \tag{32}$$

and

$$a_{p+4} = \frac{p^{t+2}b^2}{16[4+\lambda(p+3)(p+4)](p+4\delta)^t}$$

$$\left[\frac{1}{[3+\lambda(p+2)(p+3)]}\left[\frac{p^2b^2(p+2\delta)^t}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]} - \frac{1}{3}\right] - \frac{1}{3[1+\lambda p(p+1)]}\right]. \tag{33}$$

Results (23), (24), (25) and (26) can be easily obtained from (30), (31), (32) and (33) respectively.

For $\delta = 1, t = 0$, Theorem 3.1 gives the following result due to Singh and Singh [20].

Corollary 3.1 If $f(z) \in A_p$, of the form (2) is belonging to $G_{p,\lambda}(b, \Phi_{m,n})$, then

$$|a_{p+1}| \leq \frac{p|b|}{2[1+\lambda p(p+1)]},$$

$$|a_{p+2}| \leq \frac{p^2|b|^2}{4[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]},$$

$$|a_{p+3}| \leq \frac{p|b|}{8[3+\lambda(p+2)(p+3)]}\left|\frac{p^2b^2}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]}-\frac{1}{3}\right|$$

and

$$|a_{p+4}| \leq \frac{p^2|b|^2}{16[4+\lambda(p+3)(p+4)]}$$

$$\cdot\left[\frac{1}{[3+\lambda(p+2)(p+3)]}\left[\frac{p^2b^2(p+2\delta)^t}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]}-\frac{1}{3}\right]-\frac{1}{3[1+\lambda p(p+1)]}\right].$$

3.2 Fekete-Szegö Inequality

Theorem 3.2 If $f(z) \in A_p$, of the form (2) is belonging to $G_{p,\lambda}(\delta, t; b, \Phi_{m,n})$, then

$$|a_{p+2}-\mu a_{p+1}^2| \leq \frac{p^{t+2}|b|^2}{4[1+\lambda p(p+1)]^2(p+\delta)^{2t}}\left|\frac{(p+\delta)^{2t}[1+\lambda p(p+1)]}{[2+\lambda(p+1)(p+2)](p+2\delta)^t}-\mu p^t\right|. \tag{34}$$

Proof. From (30) and (31), we have

$$a_{p+2}-\mu a_{p+1}^2 = \frac{p^{t+2}b^2}{4[1+\lambda p(p+1)]^2(p+\delta)^{2t}}\left[\frac{(p+\delta)^{2t}[1+\lambda p(p+1)]}{[2+\lambda(p+1)(p+2)](p+2\delta)^t}-\mu p^t\right]. \tag{35}$$

Hence (34) can be easily obtained from (35).

In particular for $\mu = 1$,

$$|a_{p+2}-a_{p+1}^2| \leq \frac{p^{t+2}|b|^2}{4[1+\lambda p(p+1)]^2(p+\delta)^{2t}}\left|\frac{(p+\delta)^{2t}[1+\lambda p(p+1)]}{[2+\lambda(p+1)(p+2)](p+2\delta)^t}-p^t\right| = k_4. \tag{36}$$

For $\delta = 1, t = 0$, Theorem 3.2 agrees with the following result due to Singh and Singh [20].

Corollary 3.2 If $f(z) \in A_p$, of the form (2) is belonging to $G_{p,\lambda}(b, \Phi_{m,n})$, then

$$|a_{p+2}-\mu a_{p+1}^2| \leq \frac{p^2|b|^2}{4[1+\lambda p(p+1)]^2}\left|\frac{[1+\lambda p(p+1)]}{[2+\lambda(p+1)(p+2)]}-\mu\right|.$$

3.3 Second hankel determinant

Theorem 3.3 If $f(z) \in A_p$, of the form (2) is belonging to $G_{p,\lambda}(\delta, t; b, \Phi_{m,n})$, then

$$|a_{p+1}a_{p+3}-\mu a_{p+2}^2| \leq \frac{p^{2t+2}|b|^2}{16[1+\lambda p(p+1)]^2}|\eta-\mu\sigma| \tag{37}$$

where

$$\eta = \frac{[1+\lambda p(p+1)]}{[3+\lambda(p+2)(p+3)](p+\delta)^t(p+3\delta)^t}\left[\frac{p^2b^2(p+2\delta)^t}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]}-\frac{1}{3}\right] \tag{38}$$

and

$$\sigma = \frac{p^2b^2}{[2+\lambda(p+1)(p+2)]^2(p+2\delta)^{2t}}. \tag{39}$$

Proof. From (30), (31) and (32), we have

$$a_{p+1}a_{p+3}-\mu a_{p+2}^2 = \frac{p^{2t+2}b^2}{16[1+\lambda p(p+1)][3+\lambda(p+2)(p+3)](p+\delta)^t(p+3\delta)^t}$$

$$\cdot\left[\frac{p^2b^2(p+2\delta)^t}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]}-\frac{1}{3}\right]$$

$$-\mu\frac{p^{2t+4}b^4}{16[2+\lambda(p+1)(p+2)]^2[1+\lambda p(p+1)]^2(p+2\delta)^{2t}}. \quad (40)$$

Hence (37) can be easily obtained from (40).
In particular for $\mu = 1$,

$$|a_{p+1}a_{p+3}-a_{p+2}^2| \le \frac{p^{2t+2}|b|^2}{16[1+\lambda p(p+1)]^2}|\eta-\sigma| = k_5. \quad (41)$$

For $\delta = 1, t = 0$, Theorem 3.3 coincides with the following result due to Singh and Singh [20].
Corollary 3.3 If $f(z) \in A_p$, of the form (2) is belonging to $G_{p,\lambda}(b, \Phi_{m,n})$, then

$$|a_{p+1}a_{p+3}-\mu a_{p+2}^2| \le \frac{p^2|b|^2}{16[1+\lambda p(p+1)]^2}|\eta-\mu\sigma| \quad (42)$$

where

$$\eta = \frac{[1+\lambda p(p+1)]}{[3+\lambda(p+2)(p+3)]}\left[\frac{p^2b^2}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]}-\frac{1}{3}\right] \quad (43)$$

and

$$\sigma = \frac{p^2b^2}{[2+\lambda(p+1)(p+2)]^2}. \quad (44)$$

3.4 Zalcman Functional

Theorem 3.4 If $f(z) \in A_p$, of the form (2) is belonging to $G_{p,\lambda}(\delta, t; b, \Phi_{m,n})$, then

$$|a_{p+1}a_{p+2}-a_{p+3}| \le \left|\frac{p^{2t+3}b^3}{8[1+\lambda p(p+1)]^2[2+\lambda(p+1)(p+2)](p+\delta)^t(p+2\delta)^t}\right.$$

$$\left.-\frac{p^{t+1}b}{8[3+\lambda(p+2)(p+3)](p+3\delta)^t}\left[\frac{p^2b^2(p+2\delta)^t}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]}-\frac{1}{3}\right]\right| = k_6. \quad (45)$$

Proof. From (30), (31) and (32), the result (45) is obvious.
For $\delta = 1, t = 0$, Theorem 3.4 gives the following result due to Singh and Singh [20].
Corollary 3.4 If $f(z) \in A_p$, of the form (2) is belonging to $G_{p,\lambda}(b, \Phi_{m,n})$, then

$$|a_{p+1}a_{p+2}-a_{p+3}| \le \left|\frac{p^3b^3}{8[1+\lambda p(p+1)]^2[2+\lambda(p+1)(p+2)]}\right.$$

$$\left.-\frac{pb}{8[3+\lambda(p+2)(p+3)]}\left[\frac{p^2b^2}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]}-\frac{1}{3}\right]\right|.$$

3.5 Third Hankel determinant

Theorem 3.5 If $f(z) \in A_p$, of the form (2) is belonging to $G_{p,\lambda}(\delta, t; b, \Phi_{m,n})$, then

$$|H_3(p)| \le k_1k_5 + k_2k_6 + k_3k_4,$$

where k_1, k_2, k_3, k_4, k_5 and k_6 are given by (24), (25), (26), (36), (41), (45) respectively.
Proof. The proof of the result is obvious.

4. The class $M_L(\delta, t; p, \lambda; b, \Phi_{m,n})$

4.1 Initial Coefficients

Theorem 4.1 If $f(z) \in A_p$, of the form (2) is belonging to $M_L(\delta, t; p, \lambda; b, \Phi_{m,n})$, then

$$|a_{p+1}| \leq \frac{p^{t+1}|b|[1+\lambda(p-1)]}{8[1+\lambda p](p+\delta)^t}, \tag{46}$$

$$|a_{p+2}| \leq \frac{p^{t+1}|b|[1+\lambda(p-1)]|2bp-5|}{256[1+\lambda(p+1)](p+2\delta)^t} = l_1, \tag{47}$$

$$|a_{p+3}| \leq \frac{p^{t+1}|b|[1+\lambda(p-1)]|6p^2b^2-45pb+14|}{18432[1+\lambda(p+2)](p+3\delta)^t} = l_2 \tag{48}$$

and

$$|a_{p+4}| \leq \frac{p^{t+1}|b|[1+\lambda(p-1)]|12p^3b^3-180p^2b^2+337pb+651|}{1179648[1+\lambda(p+3)](p+4\delta)^t} = l_3. \tag{49}$$

Proof. As $f \in M_L(\delta, t; p, \lambda; b, \Phi_{m,n})$, therefore

$$p + \frac{1}{b}\left[\frac{z(D_\delta^t f(z))' + \lambda z^2 (D_\delta^t f(z))''}{\lambda z(D_\delta^t f(z))' + (1-\lambda)(D_\delta^t f(z))} - p\right] = p\sqrt{1+w(z)} = p\sqrt{1+\frac{\Phi_{m,n}(z)-1}{\Phi_{m,n}(z)+1}} \tag{50}$$

where

$$\Phi_{m,n}(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + \frac{779}{20160}z^7 - \ldots \tag{51}$$

Using (51), (50) can be expanded as

$$\begin{aligned}
&(1+\lambda p)\left(1+\frac{\delta}{p}\right)^t a_{p+1}z + 2(1+\lambda(p+1))\left(1+\frac{2\delta}{p}\right)^t a_{p+2}z^2 \\
&+3(1+\lambda(p+2))\left(1+\frac{3\delta}{p}\right)^t a_{p+3}z^3 + 4(1+\lambda(p+3))\left(1+\frac{4\delta}{p}\right)^t a_{p+4}z^4 + \ldots \\
&= bp\left[\frac{1}{8}z - \frac{5}{128}z^2 + \frac{7}{3072}z^3 + \frac{217}{98304}z^4 + \ldots\right] \\
&.\left[(1+\lambda(p-1)) + (1+\lambda p)\left(1+\frac{\delta}{p}\right)^t a_{p+1}z + (1+\lambda(p+1))\left(1+\frac{2\delta}{p}\right)^t a_{p+2}z^2\right. \\
&\left.+(1+\lambda(p+2))\left(1+\frac{3\delta}{p}\right)^t a_{p+3}z^3 + (1+\lambda(p+3))\left(1+\frac{4\delta}{p}\right)^t a_{p+4}z^4 + \ldots\right].
\end{aligned} \tag{52}$$

Equating the coefficients of z, z^2, z^3 and z^4 in (52), we obtain

$$a_{p+1} = \frac{p^{t+1}b[1+\lambda(p-1)]}{8[1+\lambda p](p+\delta)^t}, \tag{53}$$

$$a_{p+2} = \frac{p^{t+1}b[1+\lambda(p-1)](2bp-5)}{256[1+\lambda(p+1)](p+2\delta)^t}, \tag{54}$$

$$a_{p+3} = \frac{p^{t+1}b[1+\lambda(p-1)](6p^2b^2-45pb+14)}{18432[1+\lambda(p+2)](p+3\delta)^t} \tag{55}$$

and

$$a_{p+4} = \frac{p^{t+1}b[1+\lambda(p-1)](12p^3b^3-180p^2b^2+337pb+651)}{1179648[1+\lambda(p+3)](p+4\delta)^t}. \tag{56}$$

Results (46), (47), (48) and (49) can be easily obtained from (53), (54), (55) and (56) respectively.

For $\delta = 1, t = 0$, Theorem 4.1 gives the following result due to Olatunji and Dutta [18].

Corollary 4.1 If $f(z) \in A_p$, of the form (2) is belonging to $M_L(p, \lambda, b, \Phi_{m,n})$, then

$$|a_{p+1}| \leq \frac{p|b|[1+\lambda(p-1)]}{8[1+\lambda p]},$$

$$|a_{p+2}| \leq \frac{p|b|[1+\lambda(p-1)]|2bp-5|}{256[1+\lambda(p+1)]},$$

and

$$|a_{p+3}| \leq \frac{p|b|[1+\lambda(p-1)]|6p^2b^2-45pb+14|}{18432[1+\lambda(p+2)]}.$$

4.2 Fekete-Szegö Inequality

Theorem 4.2 If $f(z) \in A_p$, of the form (2) is belonging to $M_L(\delta, t; p, \lambda; b, \Phi_{m,n})$, then

$$|a_{p+2}-\mu a_{p+1}^2| \leq \frac{p^{t+1}|b|[1+\lambda(p-1)]}{64[1+\lambda p]^2(p+\delta)^{2t}} \left| \frac{(2pb-5)(1+\lambda p)^2(p+\delta)^{2t}}{4[1+\lambda(p+1)](p+2\delta)^t} - \mu p^{t+1} b[1+\lambda(p-1)] \right|. \tag{57}$$

Proof. From (53) and (54), we have

$$|a_{p+2}-\mu a_{p+1}^2| = \left| \frac{p^{t+1}b[1+\lambda(p-1)](2bp-5)}{256[1+\lambda(p+1)](p+2\delta)^t} - \mu \frac{p^{2(t+1)}b^2[1+\lambda(p-1)]^2}{64[1+\lambda p]^2(p+\delta)^{2t}} \right|. \tag{58}$$

Hence (57) can be easily obtained from (58).

In particular for $\mu = 1$,

$$|a_{p+2}-a_{p+1}^2| \leq \frac{p^{t+1}|b|[1+\lambda(p-1)]}{64[1+\lambda p]^2(p+\delta)^{2t}} \left| \frac{(2pb-5)(1+\lambda p)^2(p+\delta)^{2t}}{4[1+\lambda(p+1)](p+2\delta)^t} - p^{t+1} b[1+\lambda(p-1)] \right| = l_4. \tag{59}$$

For $\delta = 1, t = 0$, Theorem 4.2 agrees with the following result due to Olatunji and Dutta [18].

Corollary 4.2 If $f(z) \in A_p$, of the form (2) is belonging to $M_L(p, \lambda, b, \Phi_{m,n})$, then

$$|a_{p+2}-\mu a_{p+1}^2| \leq \frac{p|b|[1+\lambda(p-1)]}{64[1+\lambda p]^2} \left| \frac{(2pb-5)(1+\lambda p)^2}{4[1+\lambda(p+1)]} - \mu pb[1+\lambda(p-1)] \right|.$$

4.3 Second hankel determinant

Theorem 4.3 If $f(z) \in A_p$, of the form (2) is belonging to $M_L(\delta, t; p, \lambda; b, \Phi_{m,n})$, then

$$|a_{p+1}a_{p+3}-\mu a_{p+2}^2| \leq \frac{p^{2t+2}|b|^2[1+\lambda(p-1)]^2}{16384[1+\lambda(p+1)]^2}|\eta-\mu\sigma|, \tag{60}$$

where

$$\eta = \frac{[1+\lambda(p+1)]^2[6p^2b^2-45pb+14]}{9(1+\lambda p)[1+\lambda(p+2)](p+\delta)^t(p+3\delta)^t}$$

and

$$\sigma = \frac{(2pb-5)^2}{4(p+2\delta)^{2t}}.$$

Proof. Using 53), (54), (55) and following the procedure of Theorem 3.3, (60) can be easily obtained.

In particular for $\mu = 1$,

$$|a_{p+1}a_{p+3}-a_{p+2}^2| \leq \frac{p^{2t+2}|b|^2[1+\lambda(p-1)]^2}{16384[1+\lambda(p+1)]^2}|\eta-\sigma| = l_5. \tag{61}$$

For $\delta = 1, t = 0$, Theorem 4.3 coincides with the following result due to Olatunji and Dutta [18].

Corollary 4.3 If $f(z) \in A_p$, of the form (2) is belonging to $M_L(p, \lambda, b, \Phi_{m,n})$, then

$$|a_{p+1}a_{p+3} - \mu a_{p+2}^2| \leq \frac{p^2|b|^2[1+\lambda(p-1)]^2}{16384[1+\lambda(p+1)]^2}|\eta - \mu\sigma|,$$

where

$$\eta = \frac{[1+\lambda(p+1)]^2[6p^2b^2 - 45pb + 14]}{9(1+\lambda p)[1+\lambda(p+2)]}$$

and

$$\sigma = \frac{(2pb-5)^2}{4}.$$

4.4 Zalcman Functional

Theorem 4.4 If $f(z) \in A_p$, of the form (2) is belonging to $M_{p,\lambda}(\delta, t; b, \Phi_{m,n})$, then

$$|a_{p+1}a_{p+2} - a_{p+3}| \leq \left| \frac{p^{2t+2}b^2[1+\lambda(p-1)]^2(2pb-5)^2}{2048(1+\lambda p)[1+\lambda(p+1)](p+\delta)^t(p+2\delta)^t} - \frac{p^{t+1}b[1+\lambda(p-1)](6p^2b^2 - 45pb + 14)}{18432[1+\lambda(p+2)](p+3\delta)^t} \right| = l_6. \tag{62}$$

Proof. From (53), (54) and (55), the result (62) is obvious.

4.5 Third Hankel determinant

Theorem 4.5 If $f(z) \in A_p$, of the form (2) is belonging to $M_L(\delta, t; p, \lambda; b, \Phi_{m,n})$, then

$$|H_3(p)| \leq l_1 l_5 + l_2 l_6 + l_3 l_4,$$

where l_1, l_2, l_3, l_4, l_5 and l_6 are given by (47), (48), (49), (59), (61) and (62) respectively.

Proof. The proof of the result is obvious.

5. The class $G_L(\delta, t; p, \lambda; b, \Phi_{m,n})$

5.1 Initial Coefficients

Theorem 5.1 If $f(z) \in A_p$, of the form (2) is belonging to $G_L(\delta, t; p, \lambda; b, \Phi_{m,n})$, then

$$|a_{p+1}| \leq \frac{p^{t+1}|b|}{8[1+\lambda p(p+1)](p+\delta)^t}, \tag{63}$$

$$|a_{p+2}| \leq \frac{p^{t+1}|b||2bp - 5[1+\lambda p(p+1)]|}{128[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)](p+2\delta)^t} = d_1, \tag{64}$$

$$|a_{p+3}| \leq \frac{p^{t+1}|b|}{1024[3+\lambda(p+2)(p+3)](p+3\delta)^t} \cdot \left| \frac{pb[2pb - 5(1+\lambda p(p+1))] - 5pb[2+\lambda(p+1)(p+2)]}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]} + \frac{7}{3} \right| = d_2 \tag{65}$$

and

$$|a_{p+4}| \leq \frac{p^{t+1}|b|}{8192[4+\lambda(p+3)(p+4)](p+4\delta)^t}$$

$$\cdot\left|\frac{pb\{3[pb\{2pb-5(1+\lambda p(p+1))\}-5pb\{2+\lambda(p+1)(p+2)\}]+7[1+\lambda p(p+1)][2+\lambda(p+1)(p+2)]\}}{3[1+\lambda p(p+1)][2+\lambda(p+1)(p+2)][3+\lambda(p+2)(p+3)]}\right.$$

$$\left.+\frac{pb\{14[2+\lambda(p+1)(p+2)]-15[2pb-5(1+\lambda p(p+1))]\}}{6[1+\lambda p(p+1)][2+\lambda(p+1)(p+2)]}+\frac{217}{12}\right|=d_3. \quad (66)$$

Proof. These results can be easily proved using the procedure of Theorem 4.1.
For $\delta=1, t=0$, Theorem 5.1 gives the following result due to Olatunji and Dutta [18].
Corollary 5.1 If $f(z)\in A_p$, of the form (2) is belonging to $G_L(p,\lambda,b,\Phi_{m,n})$, then

$$|a_{p+1}|\leq\frac{p|b|}{8[1+\lambda p(p+1)]},$$

$$|a_{p+2}|\leq\frac{p|b||2bp-5[1+\lambda p(p+1)]|}{128[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]},$$

and

$$|a_{p+3}|\leq\frac{p|b|}{1024[3+\lambda(p+2)(p+3)]}\left|\frac{pb[2pb-5(1+\lambda p(p+1))]-5pb[2+\lambda(p+1)(p+2)]}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]}+\frac{7}{3}\right|.$$

5.2 Fekete-Szegö Inequality
Theorem 5.2 If $f(z)\in A_p$, of the form (2) is belonging to $G_L(\delta,t;p,\lambda;b,\Phi_{m,n})$, then

$$|a_{p+2}-\mu a_{p+1}^2|\leq\frac{p^{t+1}|b|}{64[1+\lambda p(p+1)]^2(p+\delta)^{2t}} \quad (67)$$

$$\cdot\left|\frac{[2pb-5(1+\lambda p(p+1))][1+\lambda p(p+1)](p+\delta)^{2t}}{2[1+\lambda p(p+1)][2+\lambda(p+1)(p+2)](p+2\delta)^t}-\mu p^{t+1}b\right|.$$

Proof. The proof is similar to Theorem 4.2.
In particular for $\mu=1$,

$$|a_{p+2}-a_{p+1}^2|\leq\frac{p^{t+1}b}{64[1+\lambda p(p+1)]^2(p+\delta)^{2t}}$$

$$\cdot\left|\frac{[2pb-5(1+\lambda p(p+1))][1+\lambda p(p+1)](p+\delta)^{2t}}{2[1+\lambda p(p+1)][2+\lambda(p+1)(p+2)](p+2\delta)^t}-p^{t+1}b\right|=d_4. \quad (68)$$

For $\delta=1, t=0$, Theorem 5.2 agrees with the following result due to Olatunji and Dutta [18].
Corollary 5.2 If $f(z)\in A_p$, of the form (2) is belonging to $G_L(p,\lambda,b,\Phi_{m,n})$, then

$$|a_{p+2}-\mu a_{p+1}^2|\leq\frac{p|b|}{64[1+\lambda p(p+1)]^2}\left|\frac{[2pb-5(1+\lambda p(p+1))][1+\lambda p(p+1)]}{2[1+\lambda p(p+1)][2+\lambda(p+1)(p+2)]}-\mu pb\right|.$$

5.3 Second hankel determinant
Theorem 5.3 If $f(z)\in A_p$, of the form (2) is belonging to $G_L(\delta,t;p,\lambda;b,\Phi_{m,n})$, then

$$|a_{p+1}a_{p+3}-\mu a_{p+2}^2|\leq\frac{p^{2t+2}|b|^2}{8192[1+\lambda p(p+1)]^2}|\eta-\mu\sigma|, \quad (69)$$

where

$$\eta=\frac{[1+\lambda p(p+1)]}{[3+\lambda(p+2)(p+3)](p+\delta)^t(p+3\delta)^t}$$

$$\cdot\left[\frac{pb[2pb-5(1+\lambda p(p+1))]-5pb[2+\lambda(p+1)(p+2)]}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]}+\frac{7}{3}\right]$$

and

$$\sigma=\frac{p^{2t+2}b^2\{2bp-5[1+\lambda p(p+1)]\}^2}{2[2+\lambda(p+1)(p+2)]^2[1+\lambda p(p+1)]^2(p+2\delta)^{2t}}.$$

Proof. This theorem can be easily proved following the procedure of Theorem 4.3. In particular for $\mu=1$,

$$|a_{p+1}a_{p+3}-a_{p+2}^2|\le\frac{p^{2t+2}|b|^2}{8192[1+\lambda p(p+1)]^2}|\eta-\mu\sigma|=d_5. \tag{70}$$

For $\delta=1, t=0$, Theorem 5.3 coincides with the following result due to Olatunji and Dutta [18].

Corollary 5.3 If $f(z)\in A_p$, of the form (2) is belonging to $G_L(p,\lambda;b,\Phi_{m,n})$, then

$$|a_{p+1}a_{p+3}-\mu a_{p+2}^2|\le\frac{p^2|b|^2}{8192[1+\lambda p(p+1)]^2}|\eta-\mu\sigma|,$$

where

$$\eta=\frac{[1+\lambda p(p+1)]}{[3+\lambda(p+2)(p+3)]}\left[\frac{pb[2pb-5(1+\lambda p(p+1))]-5pb[2+\lambda(p+1)(p+2)]}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]}+\frac{7}{3}\right].$$

and

$$\sigma=\frac{p^2b^2\{2bp-5[1+\lambda p(p+1)]\}^2}{2[2+\lambda(p+1)(p+2)]^2[1+\lambda p(p+1)]^2}.$$

5.4 Zalcman Functional

Theorem 5.4 If $f(z)\in A_p$, of the form (2) is belonging to $G_L(\delta,t;b,\lambda,\Phi_{m,n})$, then

$$|a_{p+1}a_{p+2}-a_{p+3}|\le\left|\frac{p^{2t+2}b^2\{2pb-5[1+\lambda p(p+1)]\}}{1024[1+\lambda p(p+1)]^2[2+\lambda(p+1)(p+2)](p+\delta)^t(p+2\delta)^t}\right.$$
$$-\frac{p^{t+1}b}{1024[3+\lambda(p+2)(p+3)](p+3\delta)^t}$$
$$\left.\cdot\left[\frac{pb\{2pb-5[1+\lambda p(p+1)]\}-5pb[2+\lambda(p+1)(p+2)]}{[2+\lambda(p+1)(p+2)][1+\lambda p(p+1)]}+\frac{7}{3}\right]\right|=d_6. \tag{71}$$

Proof. The proof is similar to that of the Theorem 4.4.

5.5 Third Hankel determinant

Theorem 5.5 If $f(z)\in A_p$, of the form (2) is belonging to $G_L(\delta,t;p,\lambda;b,\Phi_{m,n})$, then

$$|H_3(p)|\le d_1d_5+d_2d_6+d_3d_4,$$

where d_1,d_2,d_3,d_4,d_5 and d_6 are given by (64), (65), (66), (68), (70) and (71) respectively.

Proof. The proof of the result is obvious.

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