Journal of Fractional Calculus and Applications
Vol. 13(1) Jan. 2022, pp. 82-94.
ISSN: 2090-5858.
http://math-frac.org/Journals/JFCA/

A NOTE ON FRACTIONAL q-INTEGRALS

HONG-LI ZHOU, JIAN CAO* AND SAMA ARJIKA

Abstract. Since fractional q-integrals have been widely used in many fields, such as physics, engineering and economics, this paper considers fractional q-integrals. The main theorem is a fractional integral of k parameters. In this paper, fractional q-integrals are generated by the q-different equation. The purpose of this paper is to introduce fractional Askey-Wilson integral, Nassrallah-Rahman integral, Andrews-Askey integral and q-contour integral.

1. Introduction

The operators of fractional calculus provide very suitable tools in describing and solving a lot of problems in numerous areas of sciences and engineering (see, for details, [21] and [26]), such as physics, acoustics, electrochemistry and material science.Its theoretical and applied research has become a hot spot in the world. Their treatment from the viewpoint of the q-calculus can additionally open up new perspectives as it did, for example, in optimal control problems [9]. For further information about q-integrals and fractional q-integrals, see [1, 2, 3, 24, 34, 35, 36, 37, 38, 39, 41, 42, 40, 40].

The basic (or q-) hypergeometric function of the variable z and with $\mathfrak{r}$ numerator and $\mathfrak{s}$ denominator parameters is defined as follows (see, for details,[19]):

$$ {}_{\mathfrak{r}}\Phi_{\mathfrak{s}}\left[\begin{matrix} a_1, a_2, \cdots, a_{\mathfrak{r}}; \\ \\ b_1, b_2, \cdots, b_{\mathfrak{s}}; \end{matrix} q; z\right] := \sum_{n=0}^{\infty} \left[(-1)^n\, q^{\binom{n}{2}}\right]^{1+\mathfrak{s}-\mathfrak{r}} \frac{(a_1, a_2, \cdots, a_{\mathfrak{r}}; q)_n}{(b_1, b_2, \cdots, b_{\mathfrak{s}}; q)_n} \frac{z^n}{(q;q)_n}, $$

where $q \neq 0$ when $\mathfrak{r} > \mathfrak{s} + 1$. We also note that

$$ {}_{\mathfrak{r}+1}\Phi_{\mathfrak{r}}\left[\begin{matrix} a_1, a_2, \cdots, a_{\mathfrak{r}+1} \\ \\ b_1, b_2, \cdots, b_{\mathfrak{r}}; \end{matrix} q; z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_{\mathfrak{r}+1}; q)_n}{(b_1, b_2, \cdots, b_{\mathfrak{r}}; q)_n} \frac{z^n}{(q;q)_n}. $$

2010 *Mathematics Subject Classification.* 05A30, 11B65, 33D15, 33D45, 33D60, 39A13, 39B32.
Key words and phrases. Fractional q-integral, q-Difference equation, Fractional Askey–Wilson integral, Fractional Nassrallah–Rahman integral, Fractional Andrews–Askey integral, Fractional q-contour integral.
Submitted May 4, 2021.

The compact factorials of ${}_r\Phi_s$ are defined respectively by

$$(a;q)_0=1,\quad [a]_q:=\frac{1-q^a}{1-q},\quad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\quad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k) \tag{1}$$

and $(a_1,a_2,\ldots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$, where $m\in\mathbb{N}:=\{1,2,3,\cdots\}$ and $n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$.

The Thomae–Jackson q-integral is defined by [19, 20, 33]

$$\int_a^b f(x)\,\mathrm{d}_q\,x=(1-q)\sum_{n=0}^{\infty}\Big[bf(bq^n)-af(aq^n)\Big]q^n. \tag{2}$$

The Riemann–Liouville fractional q-integral operator is introduced in [1]

$$\Big(I_q^{\alpha}f\Big)(x)=\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x\big(qt/x;q\big)_{\alpha-1}f(t)\,\mathrm{d}_q\,t, \tag{3}$$

where the q-gamma function is defined by [19]

$$\Gamma_q(x)=\frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},\quad x\in\mathbb{R}\backslash\{0,-1,-2,\ldots\}. \tag{4}$$

The generalized Riemann–Liouville fractional q-integral operator is given by [28]

$$\Big(I_{q,a}^{\alpha}f\Big)(x)=\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_a^x\big(qt/x;q\big)_{\alpha-1}f(t)\,\mathrm{d}_q\,t,\quad \alpha\in\mathbb{R}^+. \tag{5}$$

In fact, we rewrite fractional q-integral (5) equivalently as follows by (2)

$$\Big(I_{q,a}^{\alpha}f\Big)(x)=\frac{x^{\alpha-1}(1-q)}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty}\Big[x\big(q^{n+1};q\big)_{\alpha-1}f\big(xq^n\big)-a\big(aq^{n+1}/x;q\big)_{\alpha-1}f\big(aq^n\big)\Big]q^n.$$

Recently, Cao and Arjika [10], built the relations between the following fractional q-integrals and certain generating functions for q-polynomials.

Proposition 1. *For $\alpha\in\mathbb{R}^+$ and $0<a<x<1$, if $\max\{|as|,|az|\}<1$, we have*

$$I_{q,a}^{\alpha}\left\{\frac{(bxz,xt;q)_\infty}{(xs,xz;q)_\infty}\right\}=\frac{(1-q)^{\alpha}(abz,at;q)_\infty}{(as,az;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}\big(a/x;q\big)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\,{}_3\Phi_2\left[\begin{matrix}q^{-k},as,az\\ abz,at\end{matrix};q,q\right]. \tag{6}$$

Proposition 2 ([10, Theorem 2])**.** *For $\alpha\in\mathbb{R}^+$ and $0<a<x<1$, if $\max\{|as|,|az|,|au|\}<1$, we have*

$$\begin{aligned}&I_{q,a}^{\alpha}\left\{\frac{(bxz,xt,xru;q)_\infty}{(xs,xz,xu;q)_\infty}\right\}\\&=\frac{(1-q)^{\alpha}(abz,at,aru;q)_\infty}{(as,az,au;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}\big(a/x;q\big)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\,{}_4\Phi_3\left[\begin{matrix}q^{-k},as,az,au;\\ abz,at,aru;\end{matrix}\ q;q\right].\end{aligned} \tag{7}$$

In this paper, motivated by Jian and Arjika's results [10], we aim to establish more generalized relations for fractional q-integrals and derive: a generalization of Askey-Wilson interals, a generalization of reversal type Askey-Wilson integrals, a generalization of Ramanujan Askey-Wilson integrals, a generalization of Nassrallah-Rahman integrals and a generalization of Andrews-Askey integrals as applications of fractional q-integrals.

Theorem 3. *For* $\alpha \in \mathbb{R}^+, 0 < a < x < 1$, *if* $\max\{|at|, |az|, |ars_3|, |ars_4|, \cdots, |ars_k|\} < 1$,*we have*

$$I_{q,a}^{\alpha}\left\{\frac{(bxz, xt, xr_3u_3, \cdots, xr_ku_k; q)_\infty}{(xs, xz, xu_3, \cdots, xu_k; q)_\infty}\right\} = \frac{(1-q)^\alpha (abz, at, ar_3u_3, \cdots, ar_ku_k; q)_\infty}{(as, az, au_3, \cdots, au_k; q)_\infty} \times \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x; q)_{\alpha+k}}{a^k (q;q)_{\alpha+k}} {}_{k+1}\Phi_k \left[\begin{matrix} q^{-k}, as, az, au_3, \cdots, au_k; \\ abz, at, ar_3u_3, \cdots, ar_ku_k; \end{matrix} q; q\right]. \tag{8}$$

Remark 4. *For* $u_3 = u_4 = \cdots = s_k$*=0 in Theorem 3, equation* (8) *reduces to* (6). *For* $u_4 = u_5 = \cdots = s_k$*=0, equation* (8) *reduces to* (7). *For* $z = u_3 = u_4 = \cdots = s_k$*=0 in Theorem 3 and making used of q-Chu-Vandermonde formula* [19, Eq. (II.6)]*) The q-Chu-Vandermonde formulas are given by*

$$ {}_2\Phi_1 \left[\begin{matrix} q^{-n}, a; \\ c; \end{matrix} q; q\right] = \frac{(c/a;q)_n}{(c;q)_n}\, a^n \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{9}$$

we get

$$I_{q,a}^{\alpha}\left\{\frac{(xt;q)_\infty}{(xs;q)_\infty}\right\} = \frac{(a/x;q)_\alpha (at;q)_\infty x^\alpha}{\Gamma_q(\alpha+1)(as;q)_\infty} \sum_{k=0}^{\infty} \frac{(aq^\alpha/x, t/s; q)_k}{(q^{1+\alpha}, at; q)_k} (xs)^k. \tag{10}$$

The rest of the paper is organized as follows: In Section 2, we give notations and lemmas to be used for the proof of Theorem 3. As applications of Theorem 3, we derive a generalization of Askey-Wilson interals, a generalization of reversal type Askey-Wilson integrals, a generalization of Ramanujan Askey-Wilson integrals, a generalization of Nassrallah-Rahman integrals and a generalization of Andrews-Askey integrals in Section 3.

2. Proof of Theorem 3

Before the proof of Theorem 3, we recall some notations and definitions to be used in sequel. The following usual q-difference operators are defined by [17, 31]

$$D_a\{f(a)\} := \frac{f(a) - f(qa)}{a}, \tag{11}$$

and their Leibniz rule is given by (see [30])

$$D_a^n\{f(a)g(a)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} D_a^k\{f(a)\} D_a^{n-k}\left\{g(q^k a)\right\}. \tag{12}$$

Here, and in what follows, D_a^0 is understood as the identity operator.

We also recall the definition of the Cauchy augmentation operator introduced by Chen and Gu [16]

$$\mathbb{T}(a, bD_c) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} (bD_c)^n. \tag{13}$$

Lemma 5 ([25, Proposition 1.2]). *Let $f(a,b,c)$ be a three-variable analytic function in a neighborhood of $(a,b,c)=(0,0,0)\in\mathbb{C}^3$. If $f(a,b,c)$ satisfies the q-difference equation*

$$(c-b)f(a,b,c)=abf(a,bq,cq)-bf(a,b,cq)+(c-ab)f(a,bq,c), \tag{14}$$

then we have

$$f(a,b,c)=\mathbb{T}(a,bD_c)\{f(a,0,c)\}. \tag{15}$$

Lemma 6 ([10, Lemma 5]). *For* $\max\{|as|,|az|,|au|,|ac|\}<1$, *we have*

$$(s-u)\frac{(abz,at,aru,ac\omega;q)_\infty}{(as,az,au,ac;q)_\infty}=ur\frac{(abz,at,aru,ac\omega q;q)_\infty}{(asq,az,au,acq;q)_\infty}$$
$$-u\frac{(abz,at,aru,ac\omega;q)_\infty}{(asq,az,au,ac;q)_\infty}+(s-ur)\frac{(abz,at,aru,ac\omega q;q)_\infty}{(as,az,au,acq;q)_\infty}. \tag{16}$$

Now, we are in position to prove Theorem 3.

Proof of Theorem 3. Denoting the RHS of the equation (8) by $f(r_k,s_k,s_1)$, and rewriting $f(r_k,s_k,s_1)$ equivalently by

$$f(r_k,u_k,s)=\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\sum_{j=0}^{k}\frac{(q^{-k})_jq^j}{(q;q)_j}\frac{(1-q)^\alpha(abzq^j,atq^j,ar_3u_3q^j,\cdots,ar_ku_kq^j;q)_\infty}{(asq^j,azq^j,au_3q^j,\cdots,au_kq^j;q)_\infty}, \tag{17}$$

we check that $f(r_k,u_k,s)$ satisfies the equation (14) of Lemma 5. Then, we have

$$f(r_k,u_k,s)=\mathbb{T}(r_k,u_kD_s)f(r,0,s)$$
$$=\mathbb{T}(r_k,u_kD_s)\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\sum_{j=0}^{k}\frac{(q^{-k})_jq^j}{(q;q)_j}\frac{(1-q)^\alpha(abzq^j,atq^j,ar_3u_3q^j,\cdots,ar_{k-1}u_{k-1}q^j;q)_\infty}{(asq^j,azq^j,au_3q^j,\cdots,au_{k-1}q^j)_\infty}$$
$$=\mathbb{T}(r_k,u_kD_s)\left\{I_{q,a}^{\alpha}\left\{\frac{(xbz,xt,xr_3u_3,\cdots,xr_{k-1}u_{k-1};q)_\infty}{(xs,xz,xu_3,\cdots,xu_{k-1};q)_\infty}\right\}\right\}$$
$$=I_{q,a}^{\alpha}\left\{\mathbb{T}(r_k,u_kD_s)\left\{\frac{(xbz,xt,xr_3u_3,\cdots,xr_{k-1}u_{k-1};q)_\infty}{(xs,xz,xu_3,\cdots,xu_{k-1};q)_\infty}\right\}\right\}$$
$$=I_{q,a}^{\alpha}\left\{\frac{(xbz,xt,xr_3u_3,\cdots,xr_{k-1}u_{k-1};q)_\infty}{(xz,xu_3,\cdots,xu_{k-1};q)_\infty}.\mathbb{T}(r_k,u_kD_s)\left\{\frac{1}{(xs;q)_\infty}\right\}\right\}$$

which becomes the left-hand side of the equation (8) by making used of [16, Eq. (2.3)]

$$\mathbb{T}(r_k,u_kD_s)\left\{\frac{1}{(sx;q)_\infty}\right\}=\frac{(xr_ku_k;q)_\infty}{(xu_k,xst;q)_\infty},\ \max\{|xu_k|,|st|\}<1. \tag{18}$$

The proof is complete. □

We generalize fractional q-integrals and give applications of fractional q-integrals as follows in this paper.

3. Applications

In this section, we give and prove: a generalization of Askey-Wilson interals, a generalization of reversal type Askey-Wilson integrals, a generalization of Ramanujan Askey-Wilson integrals, a generalization of Nassrallah-Rahman integrals and a generalization of Andrews-Askey integrals as applications of Theorem 3.

3.1. **A generalization of Askey-Wilson interals.**

Proposition 7. [5, Theorem 2.1] *If max* $\{|a|,|b|,|c|,|d|\}<1$,*we have*

$$\int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;a,b,c,d)}d\theta=\frac{2\pi(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}, \tag{19}$$

where

$$h(\cos\theta;a)=(ae^{i\theta},ae^{-i\theta};q)_\infty,$$
$$h(\cos\theta;a_1,a_2,\cdots,a_m)=h(\cos\theta;a_1)h(\cos\theta;a_2)\cdots h(\cos\theta;a_m).$$

Theorem 8. *For* $\alpha\in\mathbb{R}^+$, *if* $\max\{|a|,|b|,|c|,|d|\}<1$, *we have*

$$\int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;a,b,c,d)}\sum_{k=0}^\infty \frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}{}_6\Phi_5\left[\begin{matrix}q^{-k},ae^{i\theta},ae^{-i\theta},abcd,au,av;\\ ab,ac,ad,auvy,avfu;\end{matrix}q;q\right]d\theta$$
$$=\frac{2\pi(a/x;q)_\alpha(abcd;q)_\infty}{(q;q)_\alpha(q,abcd,ad,bc,bd,cd;q)_\infty}\sum_{k=0}^\infty\frac{x^{k+\alpha}(aq^\alpha/x;q)_k}{a^k(q^{1+\alpha};q)_k}{}_3\Phi_2\left[\begin{matrix}q^{-k},au,av;\\ auvy,avfu;\end{matrix}q;q\right] \tag{20}$$

Remark 9. *For* $u=v=0$ *in Theorem 8, equation* (20) *reduces to* (19)

Proof of Theorem 8. The equation (19) can be rewrite equivalently by

$$\int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;b,c,d)}\frac{(xb,xc,ad,xuvy,xvfu;q)_\infty}{(xe^{i\theta},xe^{-i\theta},xbcd,xu,xv;q)_\infty}d\theta=\frac{2\pi}{(q,bc,bd,cd;q)_\infty}\frac{(xuvy,xvfu;q)_\infty}{(xu,xv;q)_\infty}. \tag{21}$$

Next, apply the operator $I^\alpha_{q,a}$ with respect to the variable x, we get

$$\int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;b,c,d)}I^\alpha_{q,a}\left\{\frac{(xb,xc,ad,xuvy,xvfu;q)_\infty}{(xe^{i\theta},xe^{-i\theta},xbcd,xu,xv;q)_\infty}\right\}d\theta$$
$$=\frac{2\pi}{(q,bc,bd,cd;q)_\infty}I^\alpha_{q,a}\left\{\frac{(xuvy,xvfu;q)_\infty}{(xu,xv;q)_\infty}\right\}. \tag{22}$$

Taking $(s,z,u_3,u_4,u_5,b,t,r_3,r_4,r_5)=(e^{i\theta},e^{-i\theta},bcd,u,v,be^{i\theta},c,1/bc,vy,fu)$ and $u_6=\cdots=u_k=r_6=\cdots=r_k=0$ in Theorem 3, we have

$$I^\alpha_{q,a}\left\{\frac{(xb,xc,xd,xuvy,xvfu;q)_\infty}{(xe^{i\theta},xe^{-i\theta},xbcd,xu,xv;q)_\infty}\right\}$$
$$=\frac{(1-q^\alpha)(ab,ac,ad,auvy,avfu;q)_\infty}{(ae^{i\theta},ae^{-i\theta},abcd,au,av;q)_\infty}$$
$$\times\sum_{k=0}^\infty\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}{}_6\Phi_5\left[\begin{matrix}q^{-k},ae^{i\theta},ae^{-i\theta},abcd,au,av;\\ ab,ac,ad,auvy,avfu;\end{matrix}q;q\right]$$

and

$$I^\alpha_{q,a}\left\{\frac{(xuvy,xvfu;q)_\infty}{(xu,xv)_\infty}\right\}$$
$$=\frac{x^\alpha(a/x;q)_\alpha(auvy,afuv;q)_\infty}{\Gamma_q(\alpha+1)(au,av;q)_\infty}\sum_{k=0}^\infty\frac{x^k(aq^\alpha/x;q)_k}{a^k(q^{1+\alpha};q)_k}{}_3\phi_2\left[\begin{matrix}q^{-k},au,av;\\ auvy,avfu;\end{matrix}q;q\right]. \tag{23}$$

Combine the above two equations into (22), we achieve the proof of Theorem 8. $\square$

3.2. **A generalization of reversal type Askey-Wilson integrals.**

Proposition 10 ([6, Reversal Askey-Wilson integral])**.** *For* $|qabcd| < 1$*, there holds*

$$\int_{-\infty}^{\infty} \frac{h(i\sinh x; qa, qb, qc, qd)}{h(\cosh 2x; -q)} dx = \frac{(q, qab, qac, qad, qbc, qbd, qcd; q)_\infty}{(qabcd; q)_\infty} \log(q^{-1}), \tag{24}$$

where

$$h(i\sinh \alpha x; t) = \prod_{k=0}^{\infty}(1 - 2iq^k tsinh\alpha x + q^{2k}t^2) = (ite^{\alpha x}, -ite^{-\alpha x}; q)_\infty. \tag{25}$$

Theorem 11. *For* $\alpha \in \mathbb{R}^+$ *and* $|qabcd| < 1$*, we have*

$$\begin{aligned}\int_{-\infty}^{\infty} &\frac{h(i\sinh t; qa, qb, qc, qd)}{h(\cosh 2t; -q)} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x; q)_\infty}{a^k(q; q)_{\alpha+k}} \\ &\times {}_6\Phi_5\left[\begin{matrix} q^{-k}, qab, qac, qad, qau, qav; \\ iaqe^t, -iaqe^{-t}, qabcd, qauvy, qavfu; \end{matrix} q; q\right] dt \\ &= \frac{(a/x; q)_\alpha (q, qab, qac, qad, , qbc, qbd, qcd; q)_\infty}{(q; q)_\alpha (qabcd; q)_\infty} \\ &\times \sum_{k=0}^{\infty} \frac{x^{k+\alpha}(aq^\alpha/x; q)_k}{a^k(q^{1+\alpha}; q)_k} {}_3\Phi_2\left[\begin{matrix} q^{-k}, qau, qav; \\ qauvy, qavfu; \end{matrix} q; q\right] \log(q^{-1}).\end{aligned} \tag{26}$$

Remark 12. *For* $u = v = 0$ *in Theorem 11, equation* (26) *reduces to* (24)

Proof. The equation (24) can be rewrite equivalently by

$$\begin{aligned}\int_{-\infty}^{\infty} &\frac{h(i\sinh x; qb, qc, qd)}{h(\cosh 2x; -q)} I_{q,a}^{\alpha}\left\{\frac{(izqe^t, -izqe^{-t}, qzbcd, qzuvy, qzvfu; q)_\infty}{(qzb, qzc, qzd, qzu, qzv; q)_\infty}\right\} dx \\ &= (q, qbc, qbd, qcd; q)_\infty I_{q,a}^{\alpha}\left\{\frac{(qzuvy, qzvfu; q)_\infty}{(qzu, qzv; q)_\infty}\right\} \log(q^{-1}).\end{aligned} \tag{27}$$

Next, apply the operator $I_{q,a}^{\alpha}$ with respect to the variable z on the both sides of (27), we get

$$\begin{aligned}\int_{-\infty}^{\infty} &\frac{h(i\sinh x; qb, qc, qd)}{h(\cosh 2x; -q)} \frac{(izqe^t, -izqe^{-t}, qzbcd, qzuvy, qzvfu; q)_\infty}{(qzb, qzc, qzd, qzu, qzv; q)_\infty} dx \\ &= (q, qbc, qbd, qcd; q)_\infty \frac{(qzuvy, qzvfu; q)_\infty}{(qzu, qzv; q)_\infty} \log(q^{-1}).\end{aligned} \tag{28}$$

Now, taking $(s,z,u_3,u_4,u_5,b,t,r_3,r_4,r_5)=(qb,qc,qd,qu,qv,ie^t/c,-iqe^{-t},bc,vy,fu)$ and $u_6=\cdots=u_k=r_6=\cdots=r_k=0$ in Theorem 3, we have

$$I_{q,a}^{\alpha}\left\{\frac{(ixqe^t,-ixqe^{-t},qxbcd,qxuvy,qxvfu;q)_\infty}{(qxb,qxc,qxd,qxu,qxv;q)_\infty}\right\} = \frac{(1-q^\alpha)(iaqe^t,-iaqe^{-t},qabcd,qauvy,qavfu;q)_\infty}{(qab,qac,qad,qau,qav;q)_\infty} \times\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}{}_6\Phi_5\left[\begin{matrix}q^{-k},qab,qac,qad,qau,qav;\\ iaqe^t,-iaqe^{-t},qabcd,qauvy,qavfu;\end{matrix}\;q;q\right] \tag{29}$$

and

$$I_{q,a}^{\alpha}\left\{\frac{(qxuvy,qxvfu;q)_\infty}{(qxu,qxv)_\infty}\right\} = \frac{x^\alpha(a/x;q)_\alpha(qauvy,qafuv;q)_\infty}{\Gamma_q(\alpha+1)(qau,qav;q)_\infty}\sum_{k=0}^{\infty}\frac{x^k(aq^\alpha/x;q)_k}{a^k(q^{1+\alpha};q)_k}{}_3\Phi_2\left[\begin{matrix}q^{-k},qau,qav;\\ qauvy,qavfu;\end{matrix}\;q;q\right]. \tag{30}$$

Combine the above two equations into (28), we achieve the proof of Theorem 11. □

3.3. A generalization of Ramanujan Askey-Wilson integrals.

Proposition 13. [7, Atakishiyev integral] *If α is a real number and $q=e^{-2\alpha^2}$, then we have*

$$\int_{-\infty}^{\infty}h(i\sinh\alpha x;a,b,c,d)e^{-x^2}\cosh\alpha x\,dx=\sqrt{\pi}q^{-\frac{1}{8}}\frac{(ab/q,ac/q,ad/q,bc/q,bd/q,cd/q;q)_\infty}{(abcd/q^3;q)_\infty}. \tag{31}$$

Theorem 14. *For $\alpha\in R^+$ and $|abcdf/q^3|<1$, if α is a real number and $q=e^{-2\alpha^2}$, then we have*

$$\int_{-\infty}^{\infty}h(i\sinh\alpha t;a,b,c,d)e^{-x^2}\cosh\alpha t\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}} \times{}_6\Phi_5\left[\begin{matrix}q^{-k},ab/q,ac/q,ad/q,au/q,av/q;\\ iae^{\alpha t},-iae^{-\alpha t},abcd/q^3,auvy/q^3,avfu/q^3;\end{matrix}\;q;q\right]dt = \sqrt{\pi}q^{-\frac{1}{8}}\frac{(ab/q,ac/q,ad/q,bc/q,bd/q,cd/q;q)_\infty(a/x;q)_\alpha}{(q;q)_\alpha(abcd/q^3;q)_\infty} \times\sum_{k=0}^{\infty}\frac{x^{k+\alpha}(aq^\alpha/x;q)_k}{a^k(q^{1+\alpha};q)_k}{}_3\Phi_2\left[\begin{matrix}q^{-k},au/q,av/q;\\ auvy/q,avfu/q;\end{matrix}\;q;q\right]. \tag{32}$$

Remark 15. *For $u=v=0$ in Theorem 14,equation* (32) *reduces to* (31).

Proof of Theorem 14. The equation (32) can be rewritten as

$$\int_{-\infty}^{\infty} h(i\sinh\alpha t;b,c,d)e^{-x^2}\cosh\alpha t\frac{(ixe^{\alpha t},-ixe^{-\alpha t},xbcd/q^3,xuvy/q,xvfu/q;q)_\infty}{(xb/q,xc/q,xd/q,xu/q,xv/q;q)_\infty}dt$$
$$=\sqrt{\pi}q^{-\frac{1}{8}}(bc/q,bd/q,cd/q;q)_\infty\frac{(xuvy/q,xvfu/q;q)_\infty}{(xu/q,xv/q;q)_\infty}. \tag{33}$$

Next, apply the operator $I_{q,a}^{\alpha}$ with respect to the variable x on the both sides of (33), we get

$$\int_{-\infty}^{\infty} h(i\sinh\alpha t;b,c,d)e^{-x^2}\cosh\alpha t\;\; I_{q,a}^{\alpha}\left\{\frac{(ixe^{\alpha t},-ixe^{-\alpha t},xbcd/q^3,xuvy/q,xvfu/q;q)_\infty}{(xb/q,xc/q,xd/q,xu/q,xv/q;q)_\infty}\right\}dt$$
$$=\sqrt{\pi}q^{-\frac{1}{8}}(bc/q,bd/q,cd/q;q)_\infty I_{q,a}^{\alpha}\left\{\frac{(xuvy/q,xvfu/q;q)_\infty}{(xu/q,xv/q;q)_\infty}\right\}. \tag{34}$$

Taking $(s,z,u_3,u_4,u_5,b,t,r_3,r_4,r_5)=(b/q,c/q,d/q,u/q,v/q,iqe^{\alpha t}/c,-ie^{-\alpha t},bc/q^2,vy/q^2,fu/q^2)$ and $u_6=\cdots=u_k=r_6=\cdots=r_k=0$ in Theorem 3, we have

$$I_{q,a}^{\alpha}\left\{\frac{(ixe^{\alpha t},-ixe^{-\alpha t},xbcd/q^3,xuvy/q,xvfu/q;q)_\infty}{(xb/q,xc/q,xd/q,xu/q,xv/q;q)_\infty}\right\}$$
$$=\frac{(1-q^{\alpha})(iae^{\alpha t},-iae^{-\alpha t},abcd/q^3,auvy/q,avfu/q;q)_\infty}{(ab/q,ac/q,ad/q,au/q,av/q;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}$$
$$\times{}_6\Phi_5\left[\begin{matrix}q^{-k},ab/q,ac/q,ad/q,au/q,av/q;\\ iae^{\alpha t},-iae^{-\alpha t},abcd/q^3,auvy/q,avfu/q;\end{matrix}\;q;q\right] \tag{35}$$

and

$$I_{q,a}^{\alpha}\left\{\frac{(xuvy/q,xvfu/q;q)_\infty}{(xu/q,xv/q)_\infty}\right\}$$
$$=\frac{x^{\alpha}(a/x;q)_{\alpha}(auvy/q,afuv/q;q)_\infty}{\Gamma_q(\alpha+1)(au/q,av/q;q)_\infty}\sum_{k=0}^{\infty}\frac{x^k(aq^{\alpha}/x;q)_k}{a^k(q^{1+\alpha};q)_k}{}_3\Phi_2\left[\begin{matrix}q^{-k},au/q,av/q;\\ auvy/q,avfu/q;\end{matrix}\;q;q\right].$$

Combine the above two equations into (34), we get the desired results. □

3.4. **A generalization of Nassrallah-Rahman integrals.**

Proposition 16 (Nassrallah-Rahman Integral)**.** *For* $\max\{|a|,|b|,|c|,|d|,|u|\}<1$, *we have*

$$\int_0^{\pi}\frac{h(\cos 2\theta;abcdu)}{h(\cos\theta;a,b,c,d,u)}d\theta=\frac{2\pi(abcu,abcd,abdu,acdu,bcdu;q)_\infty}{(q,ab,ac,ad,au,bc,bd,buu,cd,cu,du;q)_\infty}. \tag{36}$$

We have the following extension

Theorem 17. *For $\alpha \in \mathbb{R}^+$ and if* $\max\{|a|,|b|,|c|,|d|,|u|\} < 1$ *we have*

$$\int_0^\pi \frac{h(\cos 2\theta; abcdu)}{h(\cos\theta; a,b,c,d,u)} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}} \times {}_8\Phi_7\left[\begin{matrix} q^{-k}, ae^{i\theta}, ae^{-i\theta}, abcu, abcd, abdu, acdu, ag; \\ abcdue^{i\theta}, abcdue^{-i\theta}, ab, ac, ad, au, af; \end{matrix} q;q\right] d\theta$$
$$= \frac{2\pi x^\alpha (a/x;q)_\alpha (abcu, abcd, abdu, acdu, bcdu; q)_\infty}{(q;q)_\alpha (q, ab, ac, ad, au, bc, bd, bu, cd, cu, du; q)_\infty} \sum_{k=0}^{\infty} \frac{(aq^\alpha/x, f/g; q)_k (xg)^k}{(q^{1+\alpha}, af; q)_k}. \tag{37}$$

Remark 18. *For $f = g = 0$ in Theorem 17, equation* (37) *reduces to* (36).

Proof of Theorem 17. The equation (36) can be rewrite equivalently by

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; b,c,d,u)} \frac{(xbcdue^{i\theta}, xbcdue^{i\theta}, xb, xc, xd, xu, xf; q)_\infty}{(xe^{i\theta}, xe^{i\theta}, xbcu, xbcd, xbdu, xcdu, xg; q)_\infty}$$
$$= \frac{2\pi (bcdu;q)_\infty}{(q, bc, bd, bu, cd, cu, du; q)_\infty} \frac{(xf;q)_\infty}{(xg;q)_\infty}. \tag{38}$$

Next, apply the operator $I_{q,a}^{\alpha}$ with respect to the variable x on the both sides of (38), we get

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; b,c,d,u)} I_{q,a}^{\alpha}\left\{\frac{(xbcdue^{i\theta}, xbcdue^{-i\theta}, xb, xc, xd, xu, xf; q)_\infty}{(xe^{i\theta}, xe^{-i\theta}, xbcu, xbcd, xbdu, xcdu, xg; q)_\infty}\right\}$$
$$= \frac{2\pi (bcdu;q)_\infty}{(q, bc, bd, bu, cd, cu, du; q)_\infty} I_{q,a}^{\alpha}\left\{\frac{(xf;q)_\infty}{(xg;q)_\infty}\right\}. \tag{39}$$

Taking $(s, z, u_3, u_4, u_5, u_6, u_7, b, t, r_3, r_4, r_5, r_6, r_7) = (e^{i\theta}, e^{-i\theta}, bcu, bcd, bdu, cdu, g, bcdu, bcdue^{i\theta}, 1/cu, 1/bd, 1/bu, 1/cd, f/g)$ and $u_8 = \cdots = u_k = r_8 = \cdots = r_k = 0$ in Theorem 3, we have

$$I_{q,a}^{\alpha}\left\{\frac{(xbcdue^{i\theta}, xbcdue{-}i\theta, xb, xc, xd, xu, xf; q)_\infty}{(xe^{i\theta}, xe^{-i\theta}, xbcu, xbcd, xbdu, xcdu, xg; q)_\infty}\right\}$$
$$= \frac{(1-q^\alpha)(abcdue^{i\theta}, abcdue{-}i\theta, ab, ac, ad, au, af; q)_\infty}{ae^{i\theta}, ae^{-i\theta}, abcu, abcd, abdu, acdu, ag; q)_\infty}$$
$$\times \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}} {}_8\Phi_7\left[\begin{matrix} q^{-k}, ae^{i\theta}, ae^{-i\theta}, abcu, abcd, abdu, acdu, ag; \\ abcdue^{i\theta}, abcdue^{-i\theta}, ab, ac, ad, au, af; \end{matrix} q;q\right] \tag{40}$$

and

$$I_{q,a}^{\alpha}\left\{\frac{(xf;q)_\infty}{(xg;q)_\infty}\right\} = \frac{x^\alpha (a/x;q)_\alpha (af;q)_\infty}{\Gamma_q(\alpha+1)(ag;q)_\infty} \sum_{k=0}^{\infty} \frac{(aq^\alpha/x, f/g; q)_k (xg)^k}{(q^{1+\alpha}, af; q)_k}.$$

Combine these two equations into (39), we get the desired result. This completes the proof. □

3.5. **A generalization of a q-contour integral.**

Lemma 19. *([5], Theorem (2.1)) We have*

$$\frac{1}{2\pi i}\int_c \frac{(z^2, z^{-2};q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z;q)_\infty}\frac{dz}{z} = \frac{2(abcd;q)_\infty}{(q, ab, ac, ad, bc, bd, cd;q)_\infty} \quad (41)$$

Theorem 20. *We have*

$$\begin{aligned}&\frac{1}{2\pi i}\int_c \frac{(z^2, z^{-2};q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z;q)_\infty}\frac{dz}{z}\\&\qquad\times\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}{}_6\Phi_5\left[\begin{matrix}q^{-k}, az, a/z, abcd, au, av;\\ ab, ac, ad, auvy, avfu;\end{matrix}\; q;q\right]\frac{dz}{z}\\&= \frac{2(a/x;q)_\alpha(abcd;q)_\infty}{(q;q)_\alpha(q, ab, ac, ad, bc, bd, cd;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{k+\alpha}(aq^\alpha/x;q)_k}{a^k(q^{1+\alpha};q)_k}{}_3\Phi_2\left[\begin{matrix}q^{-k}, au, av;\\ auvy, avfu;\end{matrix}\; q;q\right].\end{aligned} \quad (42)$$

Remark 21. *For $u = v = 0$ in Theorem 20,equation (42) reduces to (41).*

Proof. We rewrite (41) as follows

$$\begin{aligned}&\frac{1}{2\pi i}\int_c \frac{(z^2, z^{-2};q)_\infty}{(bz, b/z, cz, c/z, dz, d/z;q)_\infty}\frac{(xb, xc, xd, xuvy, xvfu;q)_\infty}{(xz, x/z, xbcd, xu, xv;q)_\infty}\frac{dz}{z}\\&\qquad = \frac{2}{(q, bc, bd, cd;q)_\infty}\frac{(xuvy, xvfu;q)_\infty}{(xu, xv;q)_\infty}.\end{aligned} \quad (43)$$

Next, apply the operator $I^\alpha_{q,a}$ with respect to the variable x on the both sides of (43), we get

$$\begin{aligned}&\frac{1}{2\pi i}\int_c \frac{(z^2, z^{-2};q)_\infty}{(bz, b/z, cz, c/z, dz, d/z;q)_\infty}I^\alpha_{q,a}\left\{\frac{(xb, xc, xd, xuvy, xvfu;q)_\infty}{(xz, x/z, xbcd, xu, xv;q)_\infty}\right\}\frac{dz}{z}\\&\qquad = \frac{2}{(q, bc, bd, cd;q)_\infty}I^\alpha_{q,a}\left\{\frac{(xuvy, xvfu;q)_\infty}{(xu, xv;q)_\infty}\right\}.\end{aligned} \quad (44)$$

Taking $(s, z, u_3, u_4, u_5, b, t, r_3, r_4, r_5) = (1/z, z, bcd, u, v, b/z, c, 1/bc, vy, fu)$ and $u_6 = \cdots = u_k = r_6 = \cdots = r_k = 0$ in Theorem 3,we have

$$\begin{aligned}&I^\alpha_{q,a}\left\{\frac{(xb, xc, xd, xuvy, xvfu;q)_\infty}{(xz, x/z, xbcd, xu, xv;q)_\infty}\right\}\\&= \frac{(1-q^\alpha)(ab, ac, ad, auvy, avfu;q)_\infty}{az, a/z, abcd, au, av;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}{}_6\Phi_5\left[\begin{matrix}q^{-k}, az, a/z, abcd, au, av;\\ ab, ac, ad, auvy, avfu;\end{matrix}\; q;q\right].\end{aligned} \quad (45)$$

and

$$I^\alpha_{q,a}\left\{\frac{(xuvy, xvfu;q)_\infty}{(xu, xv)_\infty}\right\} = \frac{x^\alpha(a/x;q)_\alpha(auvy, afuv;q)_\infty}{\Gamma_q(\alpha+1)(au, av;q)_\infty}\sum_{k=0}^{\infty}\frac{x^k(aq^\alpha/x;q)_k}{a^k(q^{1+\alpha};q)_k}{}_3\Phi_2\left[\begin{matrix}q^{-k}, au, av;\\ auvy, avfu;\end{matrix}\; q;q\right] \quad (46)$$

Combine the equations (45) and (46) into (44), we get the desired results. This completes the proof of Theorem 20. ∎

3.6. **A generalization of Andrews-Askey integrals.** The following famous formula is the Andrews-Askey integral,which can be derived from Ramanujan's ${}_1\psi_1$ summation.

Proposition 22. *([4],Eq.(2.1)) For* $\max\{|ac|,|ad|,|bc|,|bd|\}<1$*,we have*

$$\int_c^d \frac{(qt/c,qt/d;q)_\infty}{(at,bt;q)_\infty}d_qt=\frac{d(1-q)(q,dq/c,c/d,abcd;q)_\infty}{(ac,ad,bc,bd;q)_\infty}. \tag{47}$$

The Andrews-Askey integral is an important formula in q-series.In this part,we give the following a generalizations of Andrews-Askey integral by the method of q-difference equation.

Theorem 23. *For* $\alpha\in R^+$*,if max* $\{|ac|,|ad|,|bc|,|bd|\}<1$*,then we have*

$$\int_c^d \frac{(qt/c,qt/d;q)_\infty}{(at,bt;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}{}_5\Phi_4\left[\begin{matrix}q^{-k},at,abcd,au,av;\\ ac,ad,auvy,avfu;\end{matrix}q;q\right]d_qt$$
$$=\frac{d(1-q)(a/x;q)_\alpha(q,dq/c,c/d,abcd;q)_\infty}{(q;q)_\alpha(ac,ad,bc,bd;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{k+\alpha}(aq^\alpha/x;q)_k}{a^k(q^{1+\alpha};q)_k}{}_3\Phi_2\left[\begin{matrix}q^{-k},au,av;\\ auvy,avfu;\end{matrix}q;q\right] \tag{48}$$

Remark 24. *For* $u=v=0$ *in Theorem 23,equation* (48) *reduces to* (47)*.*

Proof of Theorem 23. We rewrite (47) as follows

$$\int_c^d \frac{(qt/c,qt/d;q)_\infty}{(bt;q)_\infty}\frac{(xc,xd,xuvy,xvfu;q)_\infty}{(xt,xbcd,xu,av;q)_\infty}d_qt=\frac{d(1-q)(q,dq/c,c/d;q)_\infty}{(bc,bd;q)_\infty}\frac{(xuvy,xvfu;q)_\infty}{(xu,xv;q)_\infty}. \tag{49}$$

Next, apply the operator $I_{q,a}^{\alpha}$ with respect to the variable x on the both sides of (49), we get

$$\int_c^d \frac{(qt/c,qt/d;q)_\infty}{(bt;q)_\infty}I_{q,a}^{\alpha}\left\{\frac{(xc,xd,xuvy,xvfu;q)_\infty}{(xt,xbcd,xu,av;q)_\infty}\right\}d_qt$$
$$=\frac{d(1-q)(q,dq/c,c/d;q)_\infty}{(bc,bd;q)_\infty}I_{q,a}^{\alpha}\left\{\frac{(xuvy,xvfu;q)_\infty}{(xu,xv;q)_\infty}.\right\} \tag{50}$$

Letting $(s,z,u_3,u_4,b,t,r_3,r_4)=(t,bcd,u,v,1/bd,d,vy,fu)$ and $u_5=\cdots=u_k=r_5=\cdots=r_k=0$ in Theorem 3, we have

$$I_{q,a}^{\alpha}\left\{\frac{(xc,xd,xuvy,xvfu;q)_\infty}{(xt,xbcd,xu,xv;q)_\infty}\right\}$$
$$=\frac{(1-q^\alpha)(ac,ad,auvy,avfu;q)_\infty}{at,abcd,au,av;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}{}_5\Phi_4\left[\begin{matrix}q^{-k},at,abcd,au,av;\\ ac,ad,auvy,avfu;\end{matrix}q;q\right] \tag{51}$$

and

$$I_{q,a}^{\alpha}\left\{\frac{(xuvy, xvfu;q)_\infty}{(xu,xv)_\infty}\right\}$$
$$=\frac{x^\alpha(a/x;q)_\alpha(auvy,afuv;q)_\infty}{\Gamma_q(\alpha+1)(au,av;q)_\infty}\sum_{k=0}^{\infty}\frac{x^k(aq^\alpha/x;q)_k}{a^k(q^{1+\alpha};q)_k}{}_3\Phi_2\left[\begin{matrix} q^{-k}, au, av; \\ auvy, avfu; \end{matrix} q;q\right]. \quad (52)$$

Combine the above two equations into (50), we get the desired result. This completes the proof. □

Acknowledgments

This work was supported by the Zhejiang Provincial Natural Science Foundation of China (No. LY21A010019).

References

[1] W.A. Al-Salam, *Some fractional q-integral and q-derivatives*, Proc. Edinburgh Math. Soc. **15** (1966), 135–140.

[2] W.A. Al-Salam, *q-Analogues of Cauchy's formulas*, Proc. Amer. Math. Soc. **17** (1966), 616–621.

[3] G.E. Andrews, *Applications of basic hypergeometric series*, SIAM Rev. **16** (1974), 441–484.

[4] G.E. Andrews,R.Askey *Another q-extension of the beta function*, Proc.Amer.Math.Soc. **81** (1981), 97–100.

[5] R. Askey, J.A. Willson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. **319** (1985) 55.

[6] R. Askey, *Bate integrals and q-extensions*, in: Proceedings of the Ramanujan Centennial International Conference, Anna-malainagar, 15-18 December 1987, pp. 85-102.

[7] N.M. Atakishiyev, *On the Askey–Wilson q-beta integral*, Theoretical and Mathematical Physics, **99** (1994), 155–159.

[8] D. Baleanu, A. Fernandez, *On some new properties of fractional derivatives with Mittag–Leffler kernel*, Commun Nonlinear Sci Numer Simulat. **59** (2018), 444–462.

[9] G. Bangerezako, *Variational calculus on q-nonuniform lattices*, J. Math. Anal. Appl. **306** (2005), 161–179.

[10] J. Cao and S. Arjika, A note on fractional Askey–Wilson integrals, J. Frac. Cal. Appl. **12** (2) (2021), 133–140.

[11] J. Cao, *A note on generalized q-difference equations for q-beta and Andrews-Askey integral*, J. Math. Anal. Appl. **412** (2014), 841–851.

[12] J. Cao and D.-W. Niu, *q-Difference equations for Askey–Wilson type integrals via q-polynomials*, J. Math. Anal. Appl. **452** (2017), 830–845.

[13] J. Cao, H. M. Srivastava, Z. G. Liu, *Some iterated fractional q-integrals and their applications*, Fractional Calculus and Applied Analysis. **21** (2018), 672–695.

[14] J. Cao, *A note on q-difference equations for Ramanujan's integrals*, The Ramanujan J. **(**2018), 1–11.

[15] J. Cao, *A note on fractional q-integrals and applications to generating functions and q-Mittag–Leffler functions*, J. Frac. Calc. Appl. **10** (2019), 136–146.

[16] V. Y. B. Chen and S. S. Gu, *The Cauchy operator for basic hypergeometric series*, Adv. Appl. Math.**41** (2008), 177–196.

[17] W. Y. C. Chen and Z.-G. Liu, *Parameter augmenting for basic hypergeometric series. II*, J. Combin. Theory Ser. A **80** (1997), 175–195.

[18] M. El-Shahed, M. Gaber and M. Al-Yami, *The fractional q-differential transformation and its application*, Commun Nonlinear Sci Numer Simulat. **18** (2013), 42–55.

[19] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Camberdge: Camberdge Univ Press, 2004.

[20] F.H. Jackson, *On q-definite integrals*, Q. J. Pure Appl. Math. **41** (1910), 193–203.

[21] A.A Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential E quations.* North-Holland Math. Studies, Vol. 204, Elevier(North-Holland)Sci.Publ, Amsterdam,London and New York, 2006.

[22] Z.-G. Liu, *An identity of Andrews and the Askey–Wilson integral*, J. Ramanujan Jouranl. **19** (2009), 115–119.

[23] Z.-G. Liu, *Two expansion formulas involving the Rogers–Szegö polynomials with applications*, J. International Journal of Number Theory **11** (2015), 507–525.

[24] Z.-G. Liu, *On a reduction formula for a kind of double q-integrals identities*, Symmetry **8** (2016), Article ID 44, 1–16.

[25] D.-Q. Lu, *q-difference equation and cauchy operator identities*, J. Math.Anal.Appl. **359** (2009), 265–274.

[26] I. Podlubny, *Fractional Differential Equations, An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications*, Academic Press, San Diego-Boston-New York-London-Tokyo-Toronto, 1999.

[27] Predrag M. Rajković, Sladjana D. Marinković and Miomir S. Stanković, *On q-analogues of Caputo derivative and Mittag–Leffler function*, Fract. Calc. Appl. Anal. **10** (2007), 359–373.

[28] Predrag M. Rajković, Sladjana D. Marinković and Miomir S. Stanković, *Fractional integrals and derivatives in q-calculus*, Appl. Anal. Discrete Math. **1** (2007), 311–323.

[29] Predrag M. Rajković, Sladjana D. Marinković and Miomir S. Stanković, *A generalization of the concept of q-fractional integrals*, Acta Math. Sin., Engl. Ser., **25** (2009), 1635–1646.

[30] S. Roman, *The theory of the umbral calculus. I*, J. Math. Anal. Appl. **87** (1982), 58–115.

[31] H. L. Saad and A. A. Sukhi, *Another homogeneous q-difference operator*, Appl. Math. Comput. **215** (2010), 4332–4339.

[32] K. Sayevand, J.T. Machado and D. Baleanu, *A new glance on the Leibniz rule for fractional derivatives*, Commun Nonlinear Sci Numer Simulat. **62** (2018), 244–249.

[33] J. Thomae, *Beitraäge zur Theorie der durch die Heinesche Reihe: Darstellbaren Function*, J. Reine Angew. Math. **70** (1869), 258–281.

[34] M. Wang, *A recurring q-integral formula*, Appl. Math.Lett. **23** (2009), 256–260.

[35] M. Wang, *A remark on Andrews–Askey integral*, J. Math.Anal.Appl. **341** (2008), 1487–1494.

[36] M. Wang, *Generalizations of Milne's $U(n+1)$ q-binomial theorem*, Comput. Math. Appl. **58** (2009), 80–87.

[37] M. Wang, *q-integral representation of the Al-Salam–Carlitz polynomials*, Appl. Math.Lett. **22** (2009), 943–945.

[38] M. Wang, *An extension of q-beta integral with application*, J. Math. Anal. Appl. **365** (2010), 653–658.

[39] R.P. Agarwal, *Certain fractional q-integrals and q-derivatives*, Proc.Cambridge Philos. Soc. 66 (1969), 365–370.

[40] M.H. Annaby and Z.S. Mansour, *q-Fractional Calculus and Equations*,Lecture Notes in Math, Vol. 2056, Springer-Verlag, Berlin, 2012.

[41] H.M. Srivastava, *Some families of Mittag-Leffler type functions and associated operators of fractional calculus (Survey)* TWMS J. Pure Appl. Math. 7 (2016), 123–145.

[42] H.M. Srivastava, R. Agarwal and S. Jain, *Integral transform and fractional derivative formulas involving the extended generalized hypergeometric functions and probability distributions*, Math. Methods Appl. Sci. 40 (2017), 255–273.

Hong-Li Zhou

Department of Mathematics, Hangzhou Normal University, Hangzhou City, Zhejiang Province, 311121, PR China

E-mail address: 739839551@qq.com

Jian Cao*

Department of Mathematics, Hangzhou Normal University, Hangzhou City, Zhejiang Province, 311121, PR China

E-mail address: 21caojian@163.com, 21caojian@hznu.edu.cn

Sama Arjika

Department of Mathematics and Informatics, University of Agadez, Niger

E-mail address: rjksama2008@gmail.com