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A CLASS OF STARLIKE FUNCTIONS OF COMPLEX ORDER DEFINED BY $q-$DIFFERENCE OPERATOR

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Abstract. Utilizing the theory of quantum calculus, we define a $q-$difference operator and used it to define a class of univalent functions and obtained Fekete-Szego inequality for functions in this class.

1. Introduction

Let $\mathcal{S}$ be the family of functions:

$$\mathcal{F}(\varsigma)=\varsigma+\sum_{k=2}^{\infty}d_k\varsigma^k,\ \varsigma\in\mathcal{E}=\{\varsigma\in\mathbb{C}:|\varsigma|<1\},\tag{1}$$

which are univalent in $\mathcal{E}$.

It is known that the calculus without the notion of limits is called $q-$calculus which has influenced many scientific fields due to its important applications. The generalization of derivative in $q-$calculus that is $q-$derivative was defined and studied by Jackson [13] as:

for $\mathcal{F}\in S,\ 0<q<1$, the $q-$derivative operator ∇_q is given by:

$$\nabla_q\mathcal{F}(\varsigma)=\begin{cases}\dfrac{\mathcal{F}(\varsigma)-\mathcal{F}(q\varsigma)}{(1-q)\varsigma} & ,\varsigma\neq 0\\ \mathcal{F}'(0) & ,\varsigma=0\end{cases},$$

that is

$$\nabla_q\mathcal{F}(\varsigma)=1+\sum_{k=2}^{\infty}[k]_q d_k\varsigma^{k-1},\tag{2}$$

where

$$[j]_q=\frac{1-q^j}{1-q},\ [0]_q=0.\tag{3}$$

As $q\rightarrow 1^-$, $[j]_q=j$ and $\nabla_q\mathcal{F}(\varsigma)=\mathcal{F}'(\varsigma)$. For more studies of $q-$derivative operators one refer for example to [2, 3, 4, 5, 6, 8, 11, 21, 22, 23].

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Using the $q-$derivative operator, we define the following operator. For $\lambda \geq \mu \geq 0$, $0 < q < 1$, let

$$\mathcal{H}^0_{\lambda,\mu,q}\mathcal{F}(\varsigma) = \mathcal{F}(\varsigma),$$

$$\mathcal{H}^1_{\lambda,\mu,q}\mathcal{F}(\varsigma) = \mathcal{H}_{\lambda,\mu,q}\mathcal{F}(\varsigma) = (1-\lambda+\mu)\mathcal{F}(\varsigma) + (\lambda-\mu)\varsigma\nabla_q\mathcal{F}(\varsigma) + \lambda\mu\varsigma^2\nabla_q(\nabla_q\mathcal{F}(\varsigma)),$$

$$\mathcal{H}^2_{\lambda,\mu,q}\mathcal{F}(\varsigma) = \mathcal{H}_{\lambda,\mu,q}(\mathcal{H}_{\lambda,\mu,q}\mathcal{F}(\varsigma)),$$

and

$$\begin{aligned}\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(\varsigma) &= \mathcal{H}_{\lambda,\mu,q}(\mathcal{H}^{m-1}_{\lambda,\mu,q}\mathcal{F}(\varsigma)) \\ &= \varsigma + \sum_{k=2}^{\infty}[1-\lambda+\mu+[k]_q(\lambda-\mu+\lambda\mu[k-1]_q)]^m d_k\varsigma^k, m \in \mathbb{N} \\ &= \varsigma + \sum_{k=2}^{\infty}\Phi^m_{q,k}(\lambda,\mu)d_k\varsigma^k. \qquad (4)\end{aligned}$$

Note that

(i) $\lim_{q\to1-}\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(\varsigma) = \mathcal{H}^m_{\lambda,\mu}\mathcal{F}(\varsigma)$ see Orhan et al. [16] (see also [9], [15] and Răducanu and Orhan [19]);

(ii) $\mathcal{H}^m_{1,0,q}\mathcal{F}(\varsigma) = \mathcal{D}^m_q\mathcal{F}(\varsigma)$(see [12], [24] and [6]);

(iii) $\mathcal{H}^m_{\lambda,0,q}\mathcal{F}(\varsigma) = \mathcal{D}^m_{\lambda,q}\mathcal{F}(\varsigma)$ (see Aouf et al. [7]);

(iv) $\lim_{q\to1-}\mathcal{H}^m_{\lambda,0,q}\mathcal{F}(\varsigma) = \mathcal{D}^m_\lambda\mathcal{F}(\varsigma)$ (see Al-Oboudi [1]).

Now, by making use of the operator $\mathcal{H}^m_{\lambda,\mu,q}$, we have the following definition.

Definition 1. Let $\tau \in \mathbb{C}^* = \mathbb{C}/\{0\}$, $\lambda \geq \mu \geq 0$, $0 < q < 1$, $m \in \mathbb{N}_0$ and $\mathcal{F} \in \mathcal{A}$, such that $\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(\varsigma) \neq 0$ for $\varsigma \in \mathcal{E}/\{0\}$. We say that $\mathcal{F} \in \mathbb{K}^m_q(\tau,\lambda,\mu)$ if

$$Re\{1 + \frac{1}{\tau}(\frac{\varsigma\nabla_q(\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(\varsigma))}{\mathcal{H}^m_{\lambda,\mu,q}\mathcal{F}(\varsigma)} - 1)\} > 0. \qquad (5)$$

Note that:

(i) $\lim_{q\to1-}\mathbb{K}^0_q(\tau,1,0) = \mathbb{K}^*(\tau)$ (Nasr and Aouf ([14]));

(ii) $\lim_{q\to1-}\mathbb{K}^m_q(1-\alpha,1,0) = \mathbb{K}^m(\alpha)$ (Sălăgean ([20]));

(iii) $\lim_{q\to1-}\mathbb{K}^m_q(\tau,\lambda,\mu) = \mathbb{K}^m(\tau,\lambda,\mu) = \left\{\mathcal{F}(\varsigma) : \left|1 + \frac{1}{\tau}(\frac{\varsigma(\mathcal{H}^m_{\lambda,\mu}\mathcal{F}(\varsigma))'}{\mathcal{H}^m_{\lambda,\mu}\mathcal{F}(\varsigma)} - 1)\right| > 0\right\}$;

(iv) $\mathbb{K}^m_q(\tau,\lambda,0) = \mathbb{K}^m_q(\tau,\lambda) = \left\{\mathcal{F}(\varsigma) : \left|1 + \frac{1}{\tau}(\frac{\varsigma\nabla_q(\mathcal{H}^m_{\lambda,q}\mathcal{F}(\varsigma))}{\mathcal{H}^m_{\lambda,q}\mathcal{F}(\varsigma)} - 1)\right| > 0\right\}$;

(v) $\mathbb{K}^m_q(\tau,1,0) = \mathbb{K}^m_q(\tau) = \left\{\mathcal{F}(\varsigma) : \left|1 + \frac{1}{\tau}(\frac{\varsigma\nabla_q(\mathcal{H}^m_q\mathcal{F}(\varsigma))}{\mathcal{H}^m_q\mathcal{F}(\varsigma)} - 1)\right| > 0\right\}$.

It is well-known that for $\mathcal{F} \in S$, $\left|d_3 - d_2^2\right| \leq 1$. Fekete and Szego ([10]) proved that for $\mathcal{F} \in \mathcal{S}$,

$$\left|d_3 - \eta d_2^2\right| \leq \left\{\begin{array}{cc} 3-4\eta & , \eta \leq 0 \\ 1+2\exp(\frac{-2\eta}{1-\eta}) & ,0 \leq \eta \leq 1 \\ 4\eta-3 & ,\eta \geq 1 \end{array}\right. ,$$

and the inequality is sharp in the sense that for each real η there exists a function in $\mathcal{S}$ such that equality holds. Later, for η complex, Pfluger ([17]) proved that

$$\left|d_3 - \eta d_2^2\right| \leq 1 + 2\left|\exp(\frac{-2\eta}{1-\eta})\right|.$$

After this several authors extended the above inequality to more general classes of analytic functions.

2. Main Results

Unless indicated, let $\tau \in \mathbb{C}^*$, $\eta \in \mathbb{C}$, $\lambda \geq \mu \geq 0$, $0 < q < 1$, $m \in \mathbb{N}_0$, $\mathcal{F}(\varsigma)$ given by (1) and $\mathcal{P}$ be the class of analytic functions with positive real part in $\mathcal{E}$ with $p(0) = 1$.

To derive our results, we recall the following lemma due to [18].[18] Let $p \in \mathcal{P}$ with $p(\varsigma) = 1 + c_1\varsigma + c_2\varsigma^2 + ...$, then

$$|c_n| \leq 2, \text{ for } n \geq 1.$$

Lemma 1. If $|c_1| = 2$, then $p(\varsigma) \equiv p_1(\varsigma) = (1 + e_1\varsigma)/(1 - e_1\varsigma)$ with $e_1 = c_1/2$. Conversely, if $p(\varsigma) \equiv p_1(\varsigma)$ for some $|e_1| = 1$, then $c_1 = 2e_1$, $|c_1| = 2$ and

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq c_2 - \frac{|c_1|^2}{2}.$$

If $|c_1| < 2$ and $\left| c_2 - \frac{c_1^2}{2} \right| = c_2 - \frac{|c_1|^2}{2}$, then $p(\varsigma) \equiv p_2(\varsigma)$, where

$$p_2(\varsigma) = \frac{1 + \varsigma \frac{e_2\varsigma + e_1}{1 + \bar{e}_1 e_2 \varsigma}}{1 - \varsigma \frac{e_2\varsigma + e_1}{1 + \bar{e}_1 e_2 \varsigma}},$$

and $e_1 = c_1/2$, $e_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$. Conversely, if $p(\varsigma) \equiv p_2(\varsigma)$ for some $|e_1| < 1$ and $|e_2| = 1$ then $e_1 = c_1/2$, $e_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$ and $\left| c_2 - \frac{c_1^2}{2} \right| = c_2 - \frac{|c_1|^2}{2}$.

Theorem 2. Let $\eta \in \mathbb{C}$ and $\mathcal{F} \in \mathbb{K}_q^m(\tau, \lambda, \mu)$, then

$$|d_2| \leq \frac{2|\tau|}{q\mathbb{A}^m}, \tag{6}$$

$$|d_3| \leq \frac{2|\tau|}{\mathbb{B}^m([3]_q - 1)} \max\{1; 1 + \frac{1}{q}(|1 + 2\tau| - 1)\}, \tag{7}$$

and

$$\left| d_3 - \eta d_2^2 \right| \leq \frac{2|\tau|}{\mathbb{B}^m([3]_q - 1)} \max\left\{ 1; 1 + \frac{1}{q}\left(\left| 1 + 2\tau - \frac{2\mathbb{B}^m([3]_q - 1)\tau}{q\mathbb{A}^{2m}}\eta \right| - 1 \right) \right\}, \tag{8}$$

where $\mathbb{A} = [1 - \lambda + \mu + [2]_q(\lambda - \mu + \lambda\mu)$ and $\mathbb{B} = [1 - \lambda + \mu + [2]_q(\lambda - \mu + \lambda\mu[2]_q)]$. Consider the functions

$$\frac{\varsigma\nabla_q(\mathcal{H}_{\lambda,\mu,q}^m \mathcal{F}(\varsigma))}{\mathcal{H}_{\lambda,\mu,q}^m \mathcal{F}(\varsigma)} = 1 + \tau(p_1(\varsigma) - 1), \tag{9}$$

and

$$\frac{\varsigma\nabla_q(\mathcal{H}_{\lambda,\mu,q}^m \mathcal{F}(\varsigma))}{\mathcal{H}_{\lambda,\mu,q}^m \mathcal{F}(\varsigma)} = 1 + \tau(p_2(\varsigma) - 1), \tag{10}$$

where $p_1(\varsigma)$, $p_2(\varsigma)$ are given in Lemma 1. Equality in (6) holds if (9); in (7) if (9) and (10); for each η in (8) if (9) and (10).

Proof. Let $\mathcal{H}_{\lambda,\mu,q}^m \mathcal{F}(\varsigma) = \varsigma + \beta_2\varsigma^2 + \beta_3\varsigma^3 + ...$, then

$$\beta_2 = \mathbb{A}^m d_2, \; \beta_3 = \mathbb{B}^m d_3. \tag{11}$$

By (5), there exists $p \in \mathcal{P}$ such that

$$\frac{\varsigma\nabla_q(\mathcal{H}_{\lambda,\mu,q}^m \mathcal{F}(\varsigma))}{\mathcal{H}_{\lambda,\mu,q}^m \mathcal{F}(\varsigma)} = 1 + \tau(p(\varsigma) - 1), \tag{12}$$

so that

$$\frac{1+\beta_2[2]_q\varsigma+\beta_3[3]_q\varsigma^2+...}{1+\beta_2\varsigma+\beta_3\varsigma^2+...}=1+\tau c_1\varsigma+\tau c_2\varsigma^2+..., \tag{13}$$

which implies

$$1+\beta_2[2]_q\varsigma+\beta_3[3]_q\varsigma^2+...=1+(\tau c_1+\beta_2)\varsigma+(\tau c_2+\beta_2\tau c_1+\beta_3)\varsigma^2+.... \tag{14}$$

Equating the coefficients of both sides we have

$$\beta_2=\frac{\tau c_1}{q},\ \beta_3=\frac{\tau c_2}{([3]_q-1)}+\frac{\tau^2 c_1^2}{([3]_q-1)q}, \tag{15}$$

so that, according to (11) and (15),

$$d_2=\frac{\tau c_1}{q\mathbb{A}^m},\ d_3=\frac{\tau}{\mathbb{B}^m([3]_q-1)}(c_2+\frac{\tau c_1^2}{q}). \tag{16}$$

Taking into account (16) and Lemma 1, we obtain

$$|d_2|=\left|\frac{\tau c_1}{q\mathbb{A}^m}\right|\le\frac{2\left|\tau\right|}{q\mathbb{A}^m}, \tag{17}$$

and

$$\begin{aligned}
|d_3| &= \left|\frac{\tau}{\mathbb{B}^m([3]_q-1)}(c_2-\frac{c_1^2}{2q}+\frac{(1+2\tau)}{2q}c_1^2)\right| \\
&\le \frac{|\tau|}{\mathbb{B}^m([3]_q-1)}\left[2-\frac{|c_1|^2}{2q}+|1+2\tau|\frac{|c_1|^2}{2q}\right] \\
&\le \frac{2\left|\tau\right|}{\mathbb{B}^m([3]_q-1)}\left[1+\frac{|1+2\tau|-1}{4q}\left|c_1\right|^2\right] \\
&\le \frac{2\left|\tau\right|}{\mathbb{B}^m([3]_q-1)}\max\left\{1;1+\frac{1}{q}\left(|1+2\tau|-1\right)\right\}.
\end{aligned} \tag{18}$$

Using Lemma 1, we obtain

$$\begin{aligned}
\left|d_3-\eta d_2^2\right| &= \left|\frac{\tau}{\mathbb{B}^m([3]_q-1)}(c_2-\frac{c_1^2}{2q}+\frac{(1+2\tau)}{2q}c_1^2)-\frac{\tau^2c_1^2}{\mathbb{A}^{2m}q^2}\eta\right| \\
&\le \frac{|\tau|}{\mathbb{B}^m([3]_q-1)}\left[\left|c_2-\frac{c_1^2}{2q}\right|+\frac{|c_1|^2}{2q}\left|1+2\tau-\frac{2\mathbb{B}^m([3]_q-1)\tau}{q\mathbb{A}^{2m}}\eta\right|\right] \\
&\le \frac{|\tau|}{\mathbb{B}^m([3]_q-1)}\left[2-\frac{|c_1|^2}{2q}+\frac{|c_1|^2}{2q}\left|1+2\tau-\frac{2\mathbb{B}^m([3]_q-1)\tau}{q\mathbb{A}^{2m}}\eta\right|\right] \\
&= \frac{2\left|\tau\right|}{\mathbb{B}^m([3]_q-1)}\left[1+\frac{|c_1|^2}{4q}\left(\left|1+2\tau-\frac{2\mathbb{B}^m([3]_q-1)\tau}{q\mathbb{A}^{2m}}\eta\right|-1\right)\right] \\
&\le \frac{2\left|\tau\right|}{\mathbb{B}^m([3]_q-1)}\max\left\{1;1+\frac{1}{q}\left(\left|1+2\tau-\frac{2\mathbb{B}^m([3]_q-1)\tau}{q\mathbb{A}^{2m}}\eta\right|-1\right)\right\}.
\end{aligned} \tag{19}$$

Now we obtain sharpness of (6), (7) and (8).

Firstly, in (6) the equality holds if $c_1 = 2$. Equivalently, we have $p(\varsigma) \equiv p_1(\varsigma) = \frac{(1+\varsigma)}{(1-\varsigma)}$. Therefore, the extremal function in $\mathbb{K}_q^m(\tau, \lambda, \mu)$ is given by

$$\frac{\varsigma \nabla_q(\mathcal{H}_{\lambda,\mu,q}^m \mathcal{F}(\varsigma))}{\mathcal{H}_{\lambda,\mu,q}^m \mathcal{F}(\varsigma)} = \frac{1 + (2\tau - 1)\varsigma}{1 - \varsigma}. \tag{20}$$

Next, in (7), for first case, the equality holds if $c_1 = c_2 = 2$. Therefore, the extremal functions in $\mathbb{K}_q^m(\tau, \lambda, \mu)$ is given by (20) and for second case, the equality holds if $c_1 = 0, c_2 = 2$. Equivalently, we have $p(\varsigma) \equiv p_2(\varsigma) = \frac{(1+\varsigma^2)}{(1-\varsigma^2)}$. Therefore, the extremal function in $\mathbb{K}_q^m(\tau, \lambda, \mu)$ is given by

$$\frac{\varsigma \nabla_q(\mathcal{H}_{\lambda,\mu,q}^m \mathcal{F}(\varsigma))}{\mathcal{H}_{\lambda,\mu,q}^m \mathcal{F}(\varsigma)} = \frac{1 + (2\tau - 1)\varsigma^2}{1 - \varsigma^2}. \tag{21}$$

Finally, in (8), the equality holds. Obtained extremal function for (7) is also valid for (8).

If η and τ are real, then we have:

Theorem 3. Let $\tau > 0$ and let $\mathcal{F} \in \mathbb{K}_q^m(\tau, \lambda, \mu)$. Then for $\eta \in \mathbb{R}$, $\mathbb{A}$ and $\mathbb{B}$ as in Theorem 2, we have

$$\left|d_3 - \eta d_2^2\right| \leq \begin{cases} \frac{2\tau}{\mathbb{B}^m([3]_q-1)}\left[1 + \frac{2\tau}{q}\left(1 - \frac{\mathbb{B}^m([3]_q-1)}{q\mathbb{A}^{2m}}\eta\right)\right] & ,\eta \leq \frac{q\mathbb{A}^{2m}}{\mathbb{B}^m([3]_q-1)} \\ \frac{2\tau}{\mathbb{B}^m([3]_q-1)} & ,\frac{q\mathbb{A}^{2m}}{\mathbb{B}^m([3]_q-1)} \leq \eta \leq \frac{(1+2\tau)q\mathbb{A}^{2m}}{2\mathbb{B}^m([3]_q-1)\tau} \\ \frac{2\tau}{\mathbb{B}^m([3]_q-1)}\left[1 + \frac{2}{q}\left(\frac{\mathbb{B}^m([3]_q-1)\tau}{q\mathbb{A}^{2m}}\eta - \tau - 1\right)\right] & ,\eta \geq \frac{(1+2\tau)q\mathbb{A}^{2m}}{2\mathbb{B}^m([3]_q-1)\tau} \end{cases}, \tag{22}$$

For each η, the equality holds for functions in (9) and (10).

Proof. First, let $\eta \leq \frac{q\mathbb{A}^{2m}}{\mathbb{B}^m([3]_q-1)} \leq \frac{(1+2\tau)q\mathbb{A}^{2m}}{2\mathbb{B}^m([3]_q-1)\tau}$. In this case (16) and Lemma 1 gives

$$\begin{aligned} \left|d_3 - \eta d_2^2\right| &\leq \frac{\tau}{\mathbb{B}^m([3]_q - 1)}\left[2 - \frac{|c_1|^2}{2q} + \frac{|c_1|^2}{2q}\left(1 + 2\tau - \frac{2\mathbb{B}^m([3]_q - 1)\tau}{q\mathbb{A}^{2m}}\eta\right)\right] \\ &\leq \frac{2\tau}{\mathbb{B}^m([3]_q - 1)}\left[1 - \frac{1}{q} + \frac{1}{q}\left(1 + 2\tau - \frac{2\mathbb{B}^m([3]_q - 1)\tau}{q\mathbb{A}^{2m}}\eta\right)\right] \\ &= \frac{2\tau}{\mathbb{B}^m([3]_q - 1)}\left[1 + \frac{2\tau}{q}\left(1 - \frac{\mathbb{B}^m([3]_q - 1)}{q\mathbb{A}^{2m}}\eta\right)\right]. \end{aligned} \tag{23}$$

Now, let $\frac{q\mathbb{A}^{2m}}{\mathbb{B}^m([3]_q-1)} \leq \eta \leq \frac{(1+2\tau)q\mathbb{A}^{2m}}{2\mathbb{B}^m([3]_q-1)\tau}$. Then, using the above calculations, we obtain

$$\left|d_3 - \eta d_2^2\right| \leq \frac{2\tau}{\mathbb{B}^m([3]_q - 1)}. \tag{24}$$

Finally, if $\eta \geq \frac{(1+2\tau)q\mathbb{A}^{2m}}{2\mathbb{B}^m([3]_q-1)\tau}$, then

$$\begin{aligned} \left|d_3 - \eta d_2^2\right| &\leq \frac{\tau}{\mathbb{B}^m([3]_q - 1)}\left[2 - \frac{|c_1|^2}{2q} + \frac{|c_1|^2}{2q}\left(\frac{2\mathbb{B}^m([3]_q - 1)\tau}{q\mathbb{A}^{2m}}\eta - 1 - 2\tau\right)\right] \\ &= \frac{\tau}{\mathbb{B}^m([3]_q - 1)}\left[2 + \frac{|c_1|^2}{2q}\left(\frac{2\mathbb{B}^m([3]_q - 1)\tau}{q\mathbb{A}^{2m}}\eta - 2\tau - 2\right)\right] \\ &\leq \frac{2\tau}{\mathbb{B}^m([3]_q - 1)}\left[1 + \frac{2}{q}\left(\frac{\mathbb{B}^m([3]_q - 1)\tau}{q\mathbb{A}^{2m}}\eta - \tau - 1\right)\right]. \end{aligned} \tag{25}$$

Finally, considering the case, when $\tau \in \mathbb{C}^*$ and $\eta \in \mathbb{R}$. Then we get:

Theorem 4. Let $\tau \in \mathbb{C}^*$ and $\mathcal{F} \in \mathbb{K}_q^m(\tau, \lambda, \mu)$. Then for $\eta \in \mathbb{R}$, $\mathbb{A}$ and $\mathbb{B}$ are as in Theorem 2, we have

$$\left|d_3 - \eta d_2^2\right| \leq \begin{cases} \frac{4|\tau|^2}{\mathbb{A}^{2m}q^2}\left[\mathcal{R}(\mathcal{K}_1) - \eta\right] + \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)}\left[1 - \frac{1}{q}(1-|\sin\theta|)\right] & ,\eta \leq \mathcal{N}_1 \\ \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)} & ,\mathcal{N}_1 \leq \eta \leq R_1 \\ \frac{4|\tau|^2}{\mathbb{A}^{2m}q^2}\left[\eta - \mathcal{R}(\mathcal{K}_1)\right] + \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)}\left[1 - \frac{1}{q}(1-|\sin\theta|)\right] & ,\eta \geq R_1 \end{cases}, \tag{26}$$

where, $|\tau| = \tau e^{i\theta}$, $\mathcal{K}_1 = \frac{q\mathbb{A}^{2m}}{\mathbb{B}^m([3]_q-1)} + \frac{q\mathbb{A}^{2m}e^{i\theta}}{2\mathbb{B}^m([3]_q-1)|\tau|}$, $\mathcal{L}_1 = \frac{q\mathbb{A}^{2m}}{2\mathbb{B}^m([3]_q-1)|\tau|}$, $\mathcal{N}_1 = \mathcal{R}(\mathcal{K}_1) - \mathcal{L}_1(1-|\sin\theta|)$ and $R_1 = \mathcal{R}(\mathcal{K}_1) + \mathcal{L}_1(1-|\sin\theta|)$. For each η there is a function in $\mathbb{K}_q^m(\tau, \lambda, \mu)$ such that the equality holds.

Proof. From (19), we have

$$\begin{aligned} \left|d_3 - \eta d_2^2\right| &\leq \frac{|\tau|}{\mathbb{B}^m([3]_q-1)}\left[\left|c_2 - \frac{c_1^2}{2q}\right| + \frac{|c_1|^2}{2q}\left|1 + 2\tau - \frac{2\mathbb{B}^m([3]_q-1)\tau}{q\mathbb{A}^{2m}}\eta\right|\right] \\ &\leq \frac{|\tau|}{\mathbb{B}^m([3]_q-1)}\left[2 - \frac{|c_1|^2}{2q} + \frac{|c_1|^2}{2q}\left|1 + 2\tau - \frac{2\mathbb{B}^m([3]_q-1)\tau}{q\mathbb{A}^{2m}}\eta\right|\right] \\ &= \frac{|\tau|}{\mathbb{B}^m([3]_q-1)}\left[\frac{|c_1|^2}{2q}\left(\left|1 + 2\tau - \frac{2\mathbb{B}^m([3]_q-1)\tau}{q\mathbb{A}^{2m}}\eta\right| - 1\right) + 2\right] \\ &= \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)} + \frac{|\tau|}{2q\mathbb{B}^m([3]_q-1)}\left[\left|\frac{2\mathbb{B}^m([3]_q-1)\tau}{q\mathbb{A}^{2m}}\eta - 2\tau - 1\right| - 1\right]|c_1|^2 \\ &= \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)} + \frac{|\tau|^2}{\mathbb{A}^{2m}q^2}\left[\begin{array}{c}\left|\eta - \frac{q\mathbb{A}^{2m}}{\mathbb{B}^m([3]_q-1)} - \frac{q\mathbb{A}^{2m}}{2\mathbb{B}^m([3]_q-1)\tau}\right| \\ -\frac{q\mathbb{A}^{2m}}{2\mathbb{B}^m([3]_q-1)|\tau|}\end{array}\right]|c_1|^2. \end{aligned} \tag{27}$$

Taking $|\tau| = \tau e^{i\theta}$ (or $\tau = |\tau|e^{-i\theta}$), $\mathcal{K}_1 = \frac{q\mathbb{A}^{2m}}{\mathbb{B}^m([3]_q-1)} + \frac{q\mathbb{A}^{2m}e^{i\theta}}{2\mathbb{B}^m([3]_q-1)|\tau|}$ and $\mathcal{L}_1 = \frac{q\mathbb{A}^{2m}}{2\mathbb{B}^m([3]_q-1)|\tau|}$ in (27), we get

$$\begin{aligned} \left|d_3 - \eta d_2^2\right| &\leq \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)} + \frac{|\tau|^2}{\mathbb{A}^{2m}q^2}\left[|\eta - \mathcal{K}_1| - \mathcal{L}_1\right]|c_1|^2 \\ &\leq \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)} + \frac{|\tau|^2}{\mathbb{A}^{2m}q^2}\left[|\eta - \mathcal{R}(\mathcal{K}_1)| + \mathcal{L}_1|\sin\theta| - \mathcal{L}_1\right]|c_1|^2 \\ &\leq \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)} + \frac{|\tau|^2}{\mathbb{A}^{2m}q^2}\left[|\eta - \mathcal{R}(\mathcal{K}_1)| - \mathcal{L}_1(1-|\sin\theta|)\right]|c_1|^2. \end{aligned} \tag{28}$$

We consider the following cases for (28). Suppose $\eta \leq \mathcal{R}(\mathcal{K}_1)$. Then

$$\begin{aligned} \left|d_3 - \eta d_2^2\right| &\leq \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)} + \frac{|\tau|^2}{\mathbb{A}^{2m}q^2}\left[\mathcal{R}(\mathcal{K}_1) - \mathcal{L}_1(1-|\sin\theta|) - \eta\right]|c_1|^2 \\ &= \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)} + \frac{|\tau|^2}{\mathbb{A}^{2m}q^2}\left[\mathcal{N}_1 - \eta\right]|c_1|^2. \end{aligned} \tag{29}$$

Let $\eta \leq \mathcal{N}_1 = \mathcal{R}(\mathcal{K}_1) - \mathcal{L}_1(1 - |\sin\theta|)$. By using Lemma 1 and $\mathcal{L}_1 = \frac{q\mathbb{A}^{2m}}{2\mathbb{B}^m([3]_q-1)|\tau|}$ in (29), we get

$$\begin{aligned}\left|d_3 - \eta d_2^2\right| &\leq \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)} + \frac{4|\tau|^2}{\mathbb{A}^{2m}q^2}\left[\mathcal{R}(\mathcal{K}_1) - \eta\right] - \frac{4|\tau|^2}{\mathbb{A}^{2m}q^2}\frac{q\mathbb{A}^{2m}}{2\mathbb{B}^m([3]_q-1)|\tau|}(1 - |\sin\theta|) \\ &= \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)} + \frac{4|\tau|^2}{\mathbb{A}^{2m}q^2}\left[\mathcal{R}(\mathcal{K}_1) - \eta\right] - \frac{2|\tau|}{q\mathbb{B}^m([3]_q-1)}(1 - |\sin\theta|) \\ &= \frac{4|\tau|^2}{\mathbb{A}^{2m}q^2}\left[\mathcal{R}(\mathcal{K}_1) - \eta\right] + \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)}\left[1 - \frac{1}{q}(1 - |\sin\theta|)\right].\end{aligned}$$

If we take $\mathcal{N}_1 = \mathcal{R}(\mathcal{K}_1) - \mathcal{L}_1(1 - |\sin\theta|) \leq \eta \leq \mathcal{R}(\mathcal{K}_1)$, then (29) gives

$$\left|d_3 - \eta d_2^2\right| \leq \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)}.$$

Let $\eta \geq \mathcal{R}(\mathcal{K}_1)$. From (28) we get

$$\begin{aligned}\left|d_3 - \eta d_2^2\right| &\leq \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)} + \frac{|\tau|^2}{\mathbb{A}^{2m}q^2}\left[\eta - (\mathcal{R}(\mathcal{K}_1) + \mathcal{L}_1(1 - |\sin\theta|))\right]|c_1|^2 \\ &= \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)} + \frac{|\tau|^2}{\mathbb{A}^{2m}q^2}\left[\eta - R_1\right]|c_1|^2. \qquad (30)\end{aligned}$$

Let $\eta \leq R_1 = \mathcal{R}(\mathcal{K}_1) + \mathcal{L}_1(1 - |\sin\theta|)$. Applying (30) we obtain

$$\left|d_3 - \eta d_2^2\right| \leq \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)}.$$

Let $\eta \geq R_1 = \mathcal{R}(\mathcal{K}_1) + \mathcal{L}_1(1 - |\sin\theta|)$ By using Lemma 1 and $\mathcal{L}_1 = \frac{q\mathbb{A}^{2m}}{2\mathbb{B}^m([3]_q-1)|\tau|}$ in (30), we get

$$\begin{aligned}\left|d_3 - \eta d_2^2\right| &\leq \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)} + \frac{4|\tau|^2}{\mathbb{A}^{2m}q^2}\left[\eta - \mathcal{R}(K_1)\right] - \frac{4|\tau|^2}{\mathbb{A}^{2m}q^2}\frac{q\mathbb{A}^{2m}}{2\mathbb{B}^m([3]_q-1)|\tau|}(1 - |\sin\theta|) \\ &= \frac{2|\tau|}{\mathbb{B}^m([3]_q-1)}\left[1 - \frac{1}{q}(1 - |\sin\theta|)\right] + \frac{4|\tau|^2}{\mathbb{A}^{2m}q^2}\left[\eta - \mathcal{R}(K_1)\right].\end{aligned}$$

Remark 1. Letting $q \to 1-$ in Theorems 2-4, we have the results obtained by Orhan et al. [16].

Taking $\lambda = 1$ and $\mu = 0$ in Theorems 2-4, we obtain the following corollaries with equalities for $\lambda = 1$, $\mu = 0$ of (9) and (10), respectively.

Corollary 5. Let $\tau \in \mathbb{C}^*$ and $\mathcal{F} \in \mathbb{K}_q^m(\tau, \lambda, \mu)$. Then for $\eta \in \mathbb{C}$:

$$|d_2| \leq \frac{|\tau|}{q2^{m-1}}, \ \ |d_3| \leq \frac{2|\tau|}{3^m([3]_q-1)}\max\{1; 1 + \frac{1}{q}(|1 + 2\tau| - 1)\},$$

and

$$\left|d_3 - \eta d_2^2\right| \leq \frac{2|\tau|}{3^m([3]_q-1)}\max\left\{1; 1 + \frac{1}{q}\left(\left|1 + 2\tau - \frac{2([3]_q-1)\tau}{q}(\frac{3}{4})^m\eta\right| - 1\right)\right\}.$$

Corollary 6. Let $\tau > 0$ and $\mathcal{F} \in \mathbb{K}_q^m(\tau, \lambda, \mu)$. Then for $\eta \in \mathbb{R}$:

$$\left|d_3 - \eta d_2^2\right| \leq \begin{cases} \frac{2\tau}{3^m([3]_q-1)}\left[1 + \frac{2\tau}{q}\left(1 - \frac{([3]_q-1)}{q}(\frac{3}{4})^m\eta\right)\right] & , \eta \leq \frac{q}{([3]_q-1)}(\frac{4}{3})^m \\ \frac{2\tau}{3^m([3]_q-1)} & , \frac{q}{([3]_q-1)}(\frac{4}{3})^m \leq \eta \leq \frac{(1+2\tau)q}{2([3]_q-1)\tau}(\frac{4}{3})^m \\ \frac{2\tau}{3^m([3]_q-1)}\left[1 + \frac{2}{q}\left(\frac{2([3]_q-1)\tau}{q}(\frac{3}{4})^m\eta - \tau - 1\right)\right] & , \eta \geq \frac{(1+2\tau)q}{2([3]_q-1)\tau}(\frac{4}{3})^m \end{cases}.$$

Corollary 7. Let $\tau \in \mathbb{C}^*$ and $\mathcal{F} \in \mathbb{K}_q^m(\tau,\lambda,\mu)$. Then for $\eta \in \mathbb{R}$:

$$\left|d_3 - \eta d_2^2\right| \leq \begin{cases} \frac{|\tau|^2}{4^{m-1}q^2}\left[\mathcal{R}(\mathcal{K}_1)-\eta\right] + \frac{2|\tau|}{3^m([3]_q-1)}\left[1-\frac{1}{q}(1-|\sin\theta|)\right] & ,\eta \leq \mathcal{N}_1 \\ \frac{2|\tau|}{3^m([3]_q-1)} & ,\mathcal{N}_1 \leq \eta \leq R_1 \\ \frac{|\tau|^2}{4^{m-1}q^2}\left[\eta - \mathcal{R}(\mathcal{K}_1)\right] + \frac{2|\tau|}{3^m([3]_q-1)}\left[1-\frac{1}{q}(1-|\sin\theta|)\right] & ,\eta \geq R_1 \end{cases},$$

where $|\tau| = \tau e^{i\theta}$, $\mathcal{K}_1 = \frac{q}{([3]_q-1)}(\frac{4}{3})^m + \frac{qe^{i\theta}}{2([3]_q-1)|\tau|}(\frac{4}{3})^m$, $\mathcal{L}_1 = \frac{q}{2([3]_q-1)|\tau|}(\frac{4}{3})^m$, $\mathcal{N}_1 = \mathcal{R}(\mathcal{K}_1) - \mathcal{L}_1(1-|\sin\theta|)$ and $R_1 = \mathcal{R}(\mathcal{K}_1) + \mathcal{L}_1(1-|\sin\theta|)$.

Taking $q \to 1-$ in Corollary 6, we have **Corollary 8.** Let $\tau > 0$ and $\mathcal{F} \in \mathbb{K}^m(\tau,\lambda,\mu)$. Then for $\eta \in \mathbb{R}$:

$$\left|d_3 - \eta d_2^2\right| \leq \begin{cases} \frac{\tau}{3^m}\left[1+2\tau\left(1-2\eta(\frac{3}{4})^m\right)\right] & ,\eta \leq \frac{1}{2}(\frac{4}{3})^m \\ \frac{\tau}{3^m} & ,\frac{1}{2}(\frac{4}{3})^m \leq \eta \leq \frac{1+2\tau}{4\tau}(\frac{4}{3})^m \\ \frac{\tau}{3^m}\left[4\tau\eta(\frac{3}{4})^m - 2\tau - 1\right] & ,\eta \geq \frac{1+2\tau}{4\tau}(\frac{4}{3})^m \end{cases}.$$

Remark 2. Note that Corollary 8, modifies the result of [15] Corollary 5 case 2.

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References

[1] F. M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator, Int. J. Math. Math. Sci., 27 (2004), 1429–1436.

[2] M. H. Annby and Z. S. Mansour, $q-$Fractional Calculus Equations, Lecture Notes in Mathematics, Vol. 2056. Springer, Berlin 2012.

[3] M. K. Aouf, H. E. Darwish and G. S. Sălăgean, On a generalization of starlike functions with negative coefficients, Math., Tome 43 66 (2001), no. 1, 3–10.

[4] M. K. Aouf and A. O. Mostafa, Subordination results for analytic functions associated with fractional $q-$calculus operators with complex order, Afrika Math., 31 (2020), 1387–1396.

[5] M. K. Aouf and A. O. Mostafa, Some subordinating results for classes of functions defined by Sălăgean type $q-$derivative operator, Filomat, 34 (2020), no. 7, 2283–2292.

[6] M. K. Aouf, A. O. Mostafa and F. Y. AL-Quhali, Properties for class of $\beta-$ uniformly univalent functions defined by Sălăgean type $q-$difference operator, Int. J. Open Probl. Complex Anal., 11 (2019), no. 2, 1–16.

[7] M. K. Aouf, A. O. Mostafa and R. E. Elmorsy, Certain subclasses of analytic functions with varying arguments associated with $q-$difference operator, Afrika Math., 13(2022), no. 1, 58-64.

[8] A. Aral, V. Gupta and R. P. Agarwal, Applications of $q-$Calculus in Operator Theory, Springer, New York, 2013.

[9] E. Deniz and H. Orhan, The Fekete-Szego problem for a generalized subclass of analytic functions, Kyungpook Math. J., 50 (2010), 37–47.

[10] M. Fekete and G. Szego, Eine Bemerkung über ungerade schlichte Funktionen, J. Lond. Math. Soc., 8 (1933), 85–89.

[11] B. A. Frasin and G. Murugusundaramoorthy, A subordination results for a class of analytic functions defined by $q-$differential operator, Ann. Univ. Paedagog. Crac. Stud. Math., 19 (2020), 53-64.

[12] M. Govindaraj and S. Sivasubramanian, On a class of analytic function related to conic domains involving $q-$calculus, Anal. Math., 43 (2017), no. 3, 475–487.

[13] F. H. Jackson, On $q-$functions and a certain difference operator, Trans. R. Soc. Edinb., 46 (1908), 253–281.

[14] M. A. Nasr and M. K. Aouf, Starlike function of complex order, J. Natur. Sci. Math., 25 (1985), 1–12.

[15] H. Orhan, E. Deniz and M. Cağlar, Fekete-Szego problem for certain subclasses of analytic functions, Demonstratio. Math.,14(2012), no. 4, 835-846.

[16] H. Orhan, E. Deniz and D. Răducanu, The Fekete–Szego problem for subclasses of analytic functions defined by a differential operator related to conic domains, Comput. Math. Appl,. 59 (2010), 283–295.

[17] A. Pfluger, The Fekete-Szego inequality by a variational method, Ann. Acad. Sci. Fenn. Ser. A 1 Math., 10 (1984).

[18] C. Pommerenke, Univalent Functions, in: Studia Mathematica Mathematische Lehrbucher, Vandenhoeck and Ruprecht, 1975.

[19] D. Răducanu and H. Orhan, Subclasses of analytic functions defined by a generalized differential operator, Int. J. Math. Anal., 4 (2010), no. 1, 1–15.

[20] G. S. Sălăgean, Subclasses of univalent functions, Complex analysis, Proc. 5th Rom.- Finn. Semin., Bucharest 1981, Part 1, Lect. Notes Math., 1013 (1983), 362–372.

[21] T. M. Seoudy and M. K. Aouf, Coefficient estimates of new classes of $q-$starlike and $q-$convex functions of complex order, J. Math. Inequal., 10 (2016), no. 1, 135–145.

[22] H. M. Srivastava, Operators of basic (or $q-$) calculus and fractional $q-$calculus and their applications in geometric function theory of complex analysis, Iran. J. Sci. Technol. Trans. Sci., 44(2020), 327–344.

[23] H. M. Srivastava, A. O. Mostafa, M. K. Aouf and H. M. Zayed, Basic and fractional $q-$calculus and associated Fekete–Szego problem for $p-$valently $q-$starlike functions and $p-$valently $q-$convex functions of complex order, Miskolc Math. Notes, 20 (2019), no. 1, 489–509.

[24] K. Vijaya, M. Kasthuri and G. Murugusundaramoorthy, Coefficient bounds for subclasses of bi-univalent functions defined by the Sălăgean derivative operator, Boletin de la Asociaciton, Matematica Venezolana, 21(2014), no. 2, 1-9.

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