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NEW UNIQUE EXISTENCE RESULT OF APPROXIMATE SOLUTION TO INITIAL VALUE PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATION OF VARIABLE ORDER

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Abstract. In this paper, we introduce the concept of continuous approximate solution which is new in term of variable order differential equations. By using the Banach Contraction Principle, we obtain the unique existence of approximate solution to an initial value problem for differential equation of variable order involving the derivative argument on half-axis. Finally, we give an example to illustrate our results.

1. Introduction

In this paper, we study the unique existence of approximate solution to the following initial value problem of variable order

$$\begin{cases} D_{0+}^{p(t)}x(t) = f(t, x, D_{0+}^{q(t)}x), 0 < t < +\infty, \\ x(0) = 0, \end{cases} \tag{1}$$

where $0 < q(t) < p(t) < 1$, $f(t, x, D_{0+}^{q(t)}x)$ are given real functions, and $D_{0+}^{p(t)}$, $D_{0+}^{q(t)}$ denote the Riemann-Liouville fractional derivatives of variable order $p(t)$ and $q(t)$ [1]-[4]

$$D_{0+}^{p(t)}x(t) = \frac{d}{dt}I_{0+}^{1-p(t)}x(t) = \frac{d}{dt}\int_0^t \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))}x(s)ds, t > 0. \tag{2}$$

$$D_{0+}^{q(t)}x(t) = \frac{d}{dt}I_{0+}^{1-q(t)}x(t) = \frac{d}{dt}\int_0^t \frac{(t-s)^{-q(t)}}{\Gamma(1-q(t))}x(s)ds,\ t > 0,$$

and $I_{0+}^{1-p(t)}x(t)$ is the Riemann-Liouville fractional integral of variable order $1-p(t)$ for function $x(t)$ [1]-[4], defined by

$$I_{0+}^{1-p(t)}x(t) = \frac{1}{\Gamma(1-p(t))}\int_0^t (t-s)^{-p(t)}x(s)ds,\ t > 0, \tag{3}$$

$I_{0+}^{1-q(t)}x(t)$ has the same meaning, for details, please refer to [1]-[4].

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The operators of variable order, which fall into a more complex category, are the derivatives and integrals whose orders are the functions of certain variables. There are several definitions of variable order fractional integrals and derivatives [1]-[4].

The problems denoted by the operator of variable order are apparently more complicated than the ones denoted by the operator of constant order. Recently, some authors have considered the applications of derivatives of variable order in various sciences such as anomalous diffusion modeling, mechanical applications, and multi-fractional Gaussian noises. Among these, there are many works that deal with the operator of variable order and numerical methods for some class of variable order fractional differential equations, for instance, [1]-[20]. But, to the best of our knowledge, there are few works dealing with the existence of solutions to differential equations of variable order, [21], [22], [26].

We notice that, in (2) and (3), if the order $p(t)$ is a constant function q, then the Riemann-Liouville variable order fractional derivatives and integrals are the Riemann-Liouville fractional derivative and integral, respectively [27]. It is well known that the Riemann-Liouville fractional integral has the the law of exponents. With this the law of exponents, one obtains the transformation between the Riemann-Liouville fractional derivative and integral [27]. Using these properties, one can transform differential equations of fractional order into equivalent integral equations, so that some nonlinear functional analysis (for instance, some fixed point theorems) have been applied in considering existence of solution of the differential equations of fractional order [28]-[30]. However from [1], [2], [8], [23], [24], [25] and [26], we notice that the law of exponents doesn't hold. Thus, we are not sure, for general function $p(t), f(t)$, what $D_{0+}^{p(t)} I_{0+}^{p(t)} f(t)$ and $I_{0+}^{p(t)} D_{0+}^{p(t)} f(t)$ equal. Consequently, we can't transform the initial value problem (1) into an integral equation, so that we can hardly consider the existence of solutions of (1), by means of nonlinear functional analysis (for instance, some fixed point theorems).

In [13], authors consider the variable order fractional functional boundary value problems of the form

$$\begin{cases} D_{0+}^{\alpha(x)} u(x) + \cos(x) u'(x) + 4u(x) + 5u(x^2) = f(x), 0 \le x \le 1, \\ u(0) = 0, u(1) = 1, \end{cases} \tag{4}$$

where $\alpha(x) = \frac{6+\cos(x)}{4}$, $D_{0+}^{\alpha(x)}$ is the Riemann-Liouville variable order derivative of order $1 < \alpha(x) < 2$ [1]-[4]

$$D_{0+}^{\alpha(t)} u(x)) = \frac{d^2}{dx^2} \int_0^x \frac{(x-s)^{1-\alpha(t)}}{\Gamma(2-\alpha(t))} u(s) ds, \ x > 0, \tag{5}$$

When $f(x) = \frac{2x^{2-\alpha(x)}}{\Gamma(3-\alpha(x))} + 5x^4 + 4x^2 + 2x\cos(x)$, the exact solution of boundary value problem (4) is

$$u(x) = x^2.$$

Also, in (4), according to (5), if we take $f(x) = 6x^6 + 5x^4 + 4x^2 + 2x\cos(x)$ or $f(x) = x^3$, then, we can't obtain its exact solution. And we don't even known whether the solution exists or not exist.

In [13], the variable order fractional functional boundary value problems of the form is also considered

$$\begin{cases} D^{\alpha(x)} u(x) + e^x u'(x) + 2u(x) + 8u(e^{x-1}) = f(x), 0 \le x \le 1, \\ u(0) = 4, u(1) = 9, \end{cases} \tag{6}$$

where $\alpha(x) = \frac{6+\cos(x)}{4}$, $D^{\alpha(x)}$ is the Riemann-Liouville variable order derivative defined by (5). When $f(x) = \frac{2x^{2-\alpha(x)}}{\Gamma(3-\alpha(x))} + 2(x^2+4x+4) + 8(4e^{x-1} + e^{2x-2} + 4) + e^2(2x+4)$, the exact solution of boundary value problem (6) is

$$u(x) = x^2 + 4x + 4.$$

Also, in (6), if we take $f(x) = x^2\alpha(x)+2(x^2+4x+4)+8(4e^{x-1}+e^{2x-2}+4)+e^2(2x+4)$ or $f(x) = 1$, then, we can't obtain its exact solution, we can't even known whether the solution exists or not exist.

Hence, an important question arises: how to solve the existence result of solutions to differential equations of variable order? In this paper, we will answer this question. Based on some facts on the solution of differential equations of integer order (fractional order) and some analysis of the initial value problem (1), we introduce the concept of continuous approximate solution to the initial value problem (1). And then, according to our discussion and analysis, we explore the unique existence of continuous approximate solution of the initial value problem (1). This paper is organized as follows. In section 2, we state some results which will play a very important role in obtaining our main results. In section 3, we set forth our main result. Finally, some examples are given.

2. Some preliminaries on approximate solution

we notice that the law of exponents doesn't hold for variable order fractional integrals (3). This leads to the fact that the Riemann-Liouville type variable order fractional derivative and integral of variable order are not inverse to each other, which is in contrast to the case of constant order fractional calculus. For examples,

Example 2.1. Let $p(t) = \frac{t}{4}+\frac{1}{4}$, $q(t) = \frac{3}{4}-\frac{t}{4}$, $f(t) = 1, 0 \le t \le 4$. Now, we calculate $I_{0+}^{p(t)}I_{0+}^{q(t)}f(t)|_{t=2}$, $I_{0+}^{q(t)}I_{0+}^{p(t)}f(t)|_{t=2}$ and $I_{0+}^{p(t)+q(t)}f(t)|_{t=2}$ which are defined in (3). For $0 \le t \le 4$, we have

$$I_{0+}^{p(t)}I_{0+}^{q(t)}f(t) = \int_0^t \frac{(t-s)^{\frac{t}{4}+\frac{1}{4}-1}}{\Gamma(\frac{t}{4}+\frac{1}{4})}\int_0^s \frac{(s-\tau)^{\frac{3}{4}-\frac{s}{4}-1}}{\Gamma(\frac{3}{4}-\frac{s}{4})}d\tau ds = \int_0^t \frac{(t-s)^{\frac{t}{4}-\frac{3}{4}}s^{\frac{3}{4}-\frac{s}{4}}}{\Gamma(\frac{t}{4}+\frac{1}{4})\Gamma(\frac{7}{4}-\frac{s}{4})}ds.$$

We get

$$I_{0+}^{p(t)}I_{0+}^{q(t)}f(t)|_{t=2} = \int_0^2 \frac{(2-s)^{-\frac{1}{4}}s^{\frac{3}{4}-\frac{s}{4}}}{\Gamma(\frac{3}{4})\Gamma(\frac{7}{4}-\frac{s}{4})}ds \approx 1.91596,$$

and

$$I_{0+}^{q(t)}I_{0+}^{p(t)}f(t) = \int_0^t \frac{(t-s)^{\frac{3}{4}-\frac{t}{4}-1}}{\Gamma(\frac{3}{4}-\frac{t}{4})}\int_0^s \frac{(s-\tau)^{\frac{s}{4}+\frac{1}{4}-1}}{\Gamma(\frac{s}{4}+\frac{1}{4})}d\tau ds = \int_0^t \frac{(t-s)^{-\frac{1}{4}-\frac{t}{4}}s^{\frac{s}{4}+\frac{1}{4}}}{\Gamma(\frac{3}{4}-\frac{t}{4})\Gamma(\frac{5}{4}+\frac{s}{4})}ds.$$

So

$$I_{0+}^{q(t)}I_{0+}^{p(t)}f(t)|_{t=2} = \int_0^2 \frac{(2-s)^{-\frac{3}{4}}s^{\frac{s}{4}+\frac{1}{4}}}{\Gamma(\frac{1}{4})\Gamma(\frac{5}{4}+\frac{s}{4})}ds \approx 2.02906,$$

and

$$I_{0+}^{p(t)+q(t)}f(t)|_{t=2} = \int_0^2 \frac{(2-s)^{p(2)+q(2)-1}}{\Gamma(p(2)+q(2))}ds = \int_0^2 ds = 2.$$

Therefore,

$$I_{0+}^{q(t)}I_{0+}^{p(t)}f(t)|_{t=2} \neq I_{0+}^{p(t)}I_{0+}^{q(t)}f(t)|_{t=2} \neq I_{0+}^{p(t)+q(t)}f(t)|_{t=2},$$

$$I_{0+}^{q(t)} I_{0+}^{p(t)} f(t)|_{t=2} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=2},$$

which illustrate that the law of exponentsof the Riemann-Liouville variable order fractional integral of continuous function doesn't holds for constant function.

Example 2.2. Let $p(t) = \begin{cases} \frac{1}{2}, & 0 \leq t \leq 1, \\ \frac{1}{3}, & 1 < t \leq 4, \end{cases}$ $q(t) = \begin{cases} \frac{1}{2}, & 0 \leq t \leq 1, \\ \frac{2}{3}, & 1 < t \leq 4, \end{cases}$ and $f(t) = 1, 0 \leq t \leq 4$. We'll calculate $I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3}$, $I_{0+}^{q(t)} I_{0+}^{p(t)} f(t)|_{t=3}$ and $I_{0+}^{p(t)+q(t)} f(t)|_{t=3}$ which are defined in (3).
For $1 \leq t \leq 4$, we have

$$\begin{aligned} I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) \quad &= \int_0^1 \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{\frac{1}{2}-1}}{\Gamma(\frac{1}{2})} d\tau ds \\ &+ \int_1^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} (\int_0^s \frac{(s-\tau)^{\frac{2}{3}-1}}{\Gamma(\frac{2}{3})} d\tau) ds \\ &= \int_0^1 \frac{(t-s)^{p(t)-1} s^{\frac{1}{2}}}{\Gamma(p(t))\Gamma(\frac{3}{2})} ds + \int_1^t \frac{(t-s)^{p(t)-1} s^{\frac{2}{3}}}{\Gamma(\frac{5}{3})\Gamma(p(t))} ds. \end{aligned}$$

Thus, we have

$$\begin{aligned} I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3} \quad &= \int_0^1 \frac{(3-s)^{-\frac{2}{3}} s^{\frac{1}{2}}}{\Gamma(\frac{1}{3})\Gamma(\frac{3}{2})} ds + \int_1^3 \frac{(3-s)^{-\frac{2}{3}} s^{\frac{2}{3}}}{\Gamma(\frac{5}{3})\Gamma(\frac{1}{3})} ds \\ &\approx 3.01744. \end{aligned}$$

By the same way, we get

$$\begin{aligned} I_{0+}^{q(t)} I_{0+}^{p(t)} f(t)|_{t=3} \quad &= \int_0^1 \frac{(3-s)^{-\frac{1}{3}} s^{\frac{1}{2}}}{\Gamma(\frac{2}{3}))\Gamma(\frac{3}{2})} ds + \int_1^3 \frac{(3-s)^{-\frac{1}{3}} s^{\frac{2}{3}}}{\Gamma(\frac{5}{3})\Gamma(\frac{2}{3})} ds \\ &\approx 3.68119 \end{aligned}$$

and

$$I_{0+}^{p(t)+q(t)} f(t)|_{t=3} = \int_0^3 \frac{(3-s)^{p(3)+q(3)-1}}{\Gamma(p(3)+q(3))} ds = \int_0^3 ds = 3.$$

Therefore, we obtain

$$\begin{aligned} &I_{0+}^{q(t)} I_{0+}^{p(t)} f(t)|_{t=3} \neq I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=3}, \\ &I_{0+}^{q(t)} I_{0+}^{p(t)} f(t)|_{t=3} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=3}, \end{aligned}$$

which illustrate that the law of exponentsof the Riemann-Liouville variable order fractional integral of piecewise constant function defined in the same partition, doesn't hold for constant function.

Without the the law of exponents, we can assure that the Riemann-Liouville variable order fractional integral of non-constant continuous functions $p(t)$ for $x(t)$ doesn't have the properties for the Riemann-Liouville fractional derivative and integral. In fact, from Examples 2.1, 2.2, we could verify this result.

Example 2.3. Let $p(t) = \frac{t}{4} + \frac{1}{4}$, $f(t) = 1, 0 \leq t \leq 3$. Now, we consider $I_{0+}^{p(t)} D_{0+}^{p(t)} f(t)|_{t=2}$ and $D_{0+}^{p(t)} I_{0+}^{p(t)} f(t)|_{t=2}$.

By (2) and (3), we have

$$
\begin{aligned}
I_{0+}^{p(t)}D_{0+}^{p(t)}f(t) = & \int_0^t \frac{(t-s)^{\frac{t}{4}+\frac{1}{4}-1}}{\Gamma(\frac{t}{4}+\frac{1}{4})}\frac{d}{ds}\int_0^s \frac{(s-\tau)^{-\frac{s}{4}-\frac{1}{4}}}{\Gamma(\frac{3}{4}-\frac{s}{4})}d\tau ds \\
= & \int_0^t \frac{(t-s)^{\frac{t}{4}-\frac{3}{4}}}{\Gamma(\frac{t}{4}+\frac{1}{4})}\frac{d}{ds}\frac{s^{\frac{3}{4}-\frac{s}{4}}}{\Gamma(\frac{7}{4}-\frac{s}{4})}ds \\
= & -\int_0^t \frac{(t-s)^{\frac{t}{4}-\frac{3}{4}}}{\Gamma(\frac{t}{4}+\frac{1}{4})}\frac{s^{\frac{3}{4}-\frac{s}{4}}(s\ln s-3+s-\frac{s\Gamma'(\frac{7}{4}-\frac{s}{4})}{\Gamma(\frac{7}{4}-\frac{s}{4})})}{4\Gamma(\frac{7}{4}-\frac{s}{4})}ds,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
I_{0+}^{p(t)}D_{0+}^{p(t)}f(t)|_{t=2} = & -\int_0^2 \frac{(2-s)^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})}\frac{s^{\frac{3}{4}-\frac{s}{4}}(s\ln s-3+s-\frac{s\Gamma'(\frac{7}{4}-\frac{s}{4})}{\Gamma(\frac{7}{4}-\frac{s}{4})})}{4\Gamma(\frac{7}{4}-\frac{s}{4})}ds \\
= & \, 0.62725 \neq f(t)|_{t=2} = 1,
\end{aligned}
$$

which implies that we hardly say for sure that $I_{0+}^{p(t)}D_{0+}^{p(t)}$ has similar result for the Riemann-Liouville fractional derivative and integral[27], that is,

$$I_{0+}^{\alpha}D_{0+}^{\alpha}g(t) = g(t), 0 \leq t \leq b, \tag{7}$$

where $0 < \alpha < 1$, $g \in C[0,b], 0 < b < +\infty$.

On the other hand, from Example 2.1, we know

$$I_{0+}^{1-p(t)}I_{0+}^{p(t)}f(t) = \int_0^t \frac{(t-s)^{\frac{t}{4}-\frac{3}{4}}s^{\frac{3}{4}-\frac{s}{4}}}{\Gamma(\frac{t}{4}+\frac{1}{4})\Gamma(\frac{7}{4}-\frac{s}{4})}ds.$$

Thus, we get

$$D_{0+}^{p(t)}I_{0+}^{p(t)}f(t) = \frac{d}{dt}I_{0+}^{1-p(t)}I_{0+}^{p(t)}f(t) = \frac{d}{dt}\int_0^t \frac{(t-s)^{\frac{t}{4}-\frac{3}{4}}s^{\frac{3}{4}-\frac{s}{4}}}{\Gamma(\frac{t}{4}+\frac{1}{4})\Gamma(\frac{7}{4}-\frac{s}{4})}ds, t > 0,$$

which illustrates that $D_{0+}^{p(t)}I_{0+}^{p(t)}f(t)|_{t=2}$ is not clear.

Hence, we don't be sure whether $D_{0+}^{p(t)}I_{0+}^{p(t)}$ has similar result for the Riemann-Liouville fractional derivative and integral[27], that is,

$$D_{0+}^{\alpha}I_{0+}^{\alpha}h(t) = h(t), 0 < t \leq b, \tag{8}$$

where $0 < \alpha < 1$, $h \in L(0,b), 0 < b < +\infty$.

Example 2.4. Let $p(t) = \begin{cases} \frac{1}{2}, & 0 \leq t \leq 1, \\ \frac{1}{3}, & 1 < t \leq 4, \end{cases}$ $f(t) = 1, 0 \leq t \leq 4$. Now, we consider $I_{0+}^{p(t)}D_{0+}^{p(t)}f(t)|_{t=3}$ and $D_{0+}^{p(t)}I_{0+}^{p(t)}f(t)|_{t=3}$.

By (2) and (3), for $2 \leq t \leq 4$, we have

$$I_{0+}^{p(t)}D_{0+}^{p(t)}f(t) = \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))}\frac{d}{ds}\int_0^s \frac{(s-\tau)^{-p(s)}}{\Gamma(1-p(s))}d\tau ds$$

$$= \int_0^1 \frac{(t-s)^{p(t)-1}}{\Gamma(p(t)}\frac{d}{ds}\int_0^s \frac{(s-\tau)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}d\tau ds + \int_1^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))}\frac{d}{ds}\int_0^s \frac{(s-\tau)^{-\frac{1}{3}}}{\Gamma(\frac{2}{3})}d\tau ds$$

$$= \int_0^t \frac{(t-s)^{p(t)-1}s^{-\frac{1}{2}}}{\Gamma(p(t))\Gamma(\frac{1}{2})}ds + \int_1^t \frac{(t-s)^{p(t)-1}s^{-\frac{1}{3}}}{\Gamma(p(t))\Gamma(\frac{2}{3})}ds,$$

thus, we get

$$I_{0+}^{p(t)}D_{0+}^{p(t)}f(t)|_{t=3} = \int_0^1 \frac{(3-s)^{-\frac{2}{3}}s^{-\frac{1}{2}}}{\Gamma(\frac{1}{3})\Gamma(\frac{1}{2})}ds + \int_1^3 \frac{(3-s)^{-\frac{2}{3}}s^{-\frac{1}{3}}}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}ds$$

$$\approx 1.00029 \neq f(t)|_{t=3} = 1,$$

which implies that (7) is invalid for $I_{0+}^{p(t)}D_{0+}^{p(t)}$.

On the other hand, from Example 2.2 and similar arguments, it is difficult to say with certainly (8) is valid for $D_{0+}^{p(t)}I_{0+}^{p(t)}$.

According to the definition of solution of differential equations of integer order (fractional order), function $x : [0, +\infty) \to R$ is called a solution of the initial value problem (1.1), if $x(t)$ satisfies the equation of (1.1) with $x(0) = 0$. However, by the arguments above, we have extreme difficulties in considering the existence of solutions of variable order fractional differential equations. So, we will consider the existence of its continuous approximate solution. In this section, we give some preliminaries on approximate solution to the initial value problem (1). The following result is necessary in our next analysis of the main result.

Lemma 2.5. Let $p : [0, +\infty) \to (0, 1)$ and $q : [0, +\infty) \to (0, 1)$ be continuous functions, and that $p(t), q(t)$ satisfy

$$\lim_{t\to+\infty} p(t) = \eta_1, \lim_{t\to+\infty} q(t) = \eta_2, 0 \le \eta_1 < 1, 0 \le \eta_2 < 1. \tag{9}$$

Then there are positive constant T, natural number n^* and intervals $[0, T_1], (T_1, T_2], \cdots, (T_{n^*-1}, T]$ $(T, +\infty)(n^* \in N)$ and piecewise constant functions $\alpha : [0, +\infty) \to (0, 1)$ and $\beta : [0, +\infty) \to (0, 1)$ defined

$$\alpha(t) = \begin{cases} p_1, & t \in [0, T_1], \\ p_2, & t \in (T_1, T_2], \\ \vdots \\ p_{n^*}, & t \in (T_{n^*-1}, T], \\ \rho_1, & t \in (T, +\infty), \end{cases} \qquad \beta(t) = \begin{cases} q_1, & t \in [0, T_1], \\ q_2, & t \in (T_1, T_2], \\ \vdots \\ q_{n^*}, & t \in (T_{n^*-1}, T], \\ \rho_2, & t \in (T, +\infty), \end{cases} \tag{10}$$

where $0 < q_i < p_i < 1, 0 < \rho_2 < \rho_1 < 1$, $i = 1, 2, \cdots, n^*$, such that for arbitrary small $\varepsilon > 0$,

$$|p(t) - \alpha(t)| < \varepsilon, |q(t) - \beta(t)| < \varepsilon, \quad 0 \le t < +\infty. \tag{11}$$

Proof. By (9), for $\forall\ \varepsilon > 0$, there exists $\overline{T}_1, \overline{T}_2 > 0$, such that

$$|p(t) - \eta_1| < \frac{\varepsilon}{2}, t > \overline{T}_1; |q(t) - \eta_2| < \frac{\varepsilon}{2}, t > \overline{T}_2.$$

We take

$$T = \max\{\overline{T}_1, \overline{T}_2\}, \tag{12}$$

then, for $\forall\ \varepsilon > 0$, we have that

$$|p(t) - \eta_1| < \frac{\varepsilon}{2}, |q(t) - \eta_2| < \frac{\varepsilon}{2}, t > T. \tag{13}$$

For $\forall\ \varepsilon > 0$, we take

$$\rho_1 = p(T+1), \rho_2 = q(T+1). \tag{14}$$

For $\forall\ \varepsilon > 0$, by (13)-(14), we have

$$\begin{cases} |p(t)-\rho_1| = |p(t)-p(T+1)| \le |p(t)-\eta_1| + |\eta_1 - p(T+1)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, t > T, \\ |q(t)-\rho_2| = |q(t)-q(T+1)| \le |q(t)-\eta_2| + |\eta_2 - q(T+1)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, t > T. \end{cases}$$

We know that $p : [0,T] \to (0,1)$, $q : [0,T] \to (0,1)$ are continuous functions. Since $p(t), q(t)$ are right continuous at point 0, then, for arbitrary small $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$|p(t) - p(0)| < \varepsilon, \quad |q(t) - q(0)| < \varepsilon, \quad \text{for} \quad 0 \le t \le \delta_0.$$

We take point $\delta_0 \doteq T_1$ (if $T_1 < T$, we consider continuities of $p(t), q(t)$ at point T_1, otherwise, we end this procedure). Since $p(t), q(t)$ are right continuous at point T_1, so for arbitrary small $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|p(t) - p(T_1)| < \varepsilon, \quad |q(t) - q(T_1)| < \varepsilon, \quad \text{for} \quad T_1 \le t \le T_1 + \delta_1.$$

We take point $T_1 + \delta_1 \doteq T_2$ (if $T_2 < T$, we consider continuities of $p(t), q(t)$ at point T_2, otherwise, we end this procedure). Since $p(t), q(t)$ are right continuous at point T_2, so, for arbitrary small $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$$|p(t) - p(T_2)| < \varepsilon, \quad |q(t) - q(T_2)| < \varepsilon, \quad \text{for} \quad T_2 \le t \le T_2 + \delta_2.$$

We take point $T_2 + \delta_2 \doteq T_3$ (if $T_3 < T$, we consider continuities of $p(t), q(t)$ at point T_3, otherwise, we end this procedure). Since $p(t), q(t)$ are right continuous at point T_3, so, for arbitrary small $\varepsilon > 0$, there exists $\delta_3 > 0$ such that

$$|p(t) - p(T_3)| < \varepsilon, \quad |q(t) - q(T_3)| < \varepsilon, \quad \text{for} \quad T_3 \le t \le T_3 + \delta_3.$$

Since $[0,T]$ is a finite interval, then, continuing this analysis procedure, we can obtain there are $\delta_{n^*-2} > 0$, $\delta_{n^*-1} > 0$ ($n^* \in N$) such that $T_{n^*-2} + \delta_{n^*-2} \doteq T_{n^*-1} < T$, $T_{n^*-1} + \delta_{n^*-1} \ge T$, such that for arbitrary small $\varepsilon > 0$, it holds

$$|p(t) - p(T_{n^*-1})| < \varepsilon, |q(t) - q(T_{n^*-1})| < \varepsilon \quad \text{for} \quad T_{n^*-1} \le t \le T_{n^*} = T,$$

From previous arguments, we let

$$p(0) \doteq p_1, p(T_1) \doteq p_2, p(T_2) \doteq p_3, p(T_3) \doteq p_4, \cdots, p(T_{n^*-1}) \doteq p_{n^*}, \tag{15}$$

$$q(0) \doteq q_1, q(T_1) \doteq q_2, q(T_2) \doteq q_3, q(T_3) \doteq q_4, \cdots, q(T_{n^*-1}) \doteq q_{n^*}. \tag{16}$$

Thus, by (15)-(16) and arguments above, we get piecewise constant functions $\alpha, \beta : [0, +\infty) \to (0,1)$ as following

$$\alpha(t) = \begin{cases} p_1, & t \in [0, T_1], \\ p_2, & t \in (T_1, T_2], \\ \vdots & \\ p_{n^*}, & t \in (T_{n^*-1}, T], \\ \rho_1, & t \in (T, +\infty), \end{cases} \qquad \beta(t) = \begin{cases} q_1, & t \in [0, T_1], \\ q_2, & t \in (T_1, T_2], \\ \vdots & \\ q_{n^*}, & t \in (T_{n^*-1}, T], \\ \rho_2, & t \in (T, +\infty), \end{cases}$$

and that, for arbitrary small $\varepsilon > 0$, $\alpha(t), \beta(t)$ satisfy

$$\begin{cases} |p(t) - p_1| < \varepsilon, |q(t) - q_1| < \varepsilon, & \text{for } t \in [0, T_1], \\ |p(t) - p_2| < \varepsilon, |q(t) - q_2| < \varepsilon, & \text{for } t \in (T_1, T_2], \\ \quad \vdots \\ |p(t) - p_{n^*}| < \varepsilon, |q(t) - q_{n^*}| < \varepsilon, & \text{for } t \in (T_{n^*-1}, T], \\ |p(t) - \rho_1| < \varepsilon, |q(t) - \rho_2| < \varepsilon, & \text{for } t \in (T, +\infty), \end{cases}$$

which implies that (11) holds. Thus, we complete this proof. □

For $\alpha(t), \beta(t)$ obtained in Lemma 2.1, we get the following initial value problem

$$\begin{cases} D_{0+}^{\alpha(t)} x(t) = f(t, x, D_{0+}^{\beta(t)} x), 0 < t < +\infty, \\ x(0) = 0. \end{cases} \tag{17}$$

According to Example 2.3, we can't transform the initial value problem (1) into an integral equation, we have obstacles in consider existence of solutions of differential equations of variable order. Hence, here, we consider its approximate solutions of (1) in the following sense: If $p(t), q(t), \alpha(t), \beta(t)$ satisfy (11), then a solution $x : [0, +\infty) \to R$ of (17) is called a approximate solution of the initial value problem (1).

For the initial value problem (17), according to the definition of solution of differential equations of integer order (fractional order), function $x : [0, +\infty) \to R$ is called a solution of the initial value problem (17), if $x(t)$ satisfies the equation of (17) with $x(0) = 0$. However, it follows from Example 2.4 that we also can't transform the initial value problem (17) into an integral equation, hence in order to obtain our main results, we need to carry on essential analysis to the equation of (17).

For the initial value problem (17), by (10), in the interval $[0, T_1]$, we have the initial value problem

$$\begin{cases} D_{0+}^{p_1} x(t) = f(t, x, D_{0+}^{q_1} x), 0 < t \le T_1, \\ x(0) = 0. \end{cases} \tag{18}$$

By (10), the equation (17) in the interval $(T_1, T_2]$ can be written by

$$\begin{aligned} D_{0+}^{\alpha(t)} x(t) \quad &= \frac{d}{dt} \int_0^t \frac{(t-s)^{-p_2}}{\Gamma(1-p_2)} x(s) ds \\ &= \frac{d}{dt} \int_0^{T_1} \frac{(t-s)^{-p_2} x(s)}{\Gamma(1-p_2)} ds + \frac{d}{dt} \int_{T_1}^t \frac{(t-s)^{-p_2} x(s)}{\Gamma(1-p_2)} ds \\ &= f\Big(t, x, \frac{d}{dt} \int_0^{T_1} \frac{(t-s)^{-q_2} x(s)}{\Gamma(1-q_2)} ds + \frac{d}{dt} \int_{T_1}^t \frac{(t-s)^{-q_2} x(s)}{\Gamma(1-q_2)} ds \Big). \end{aligned}$$

In order to consider the existence of solution to (17) in the interval $[T_1, T_2]$, we let

$$x(t) = \begin{cases} 0, & \text{if } x_1(t) \equiv 0 \text{ for } t \in [0, T_1], \\ \frac{x_1(T_1) \int_0^t |x_1(s)| ds}{\int_0^{T_1} |x_1(s)| ds}, & \text{if } x_1(t) \text{ is not identically vanishing for } t \in [0, T_1], \end{cases} \tag{19}$$

here, $x_1 : [0, T_1] \to \mathbb{R}$ is continuous solution of the initial value problem (18). Thus, if $x_1(s) \equiv 0, 0 \le s \le T_1$, then by (19), we let

$$\begin{cases} \frac{d}{dt}\int_0^{T_1} \frac{(t-s)^{-q_2}x(s)}{\Gamma(1-q_2)}ds = 0 \doteq \varphi_{x_1}(t), \\ \frac{d}{dt}\int_0^{T_1} \frac{(t-s)^{-p_2}x(s)}{\Gamma(1-p_2)}ds = 0 \doteq \psi_{x_1}(t), \end{cases} \tag{20}$$

if $x_1(s)$ is not identically vanishing for $0 \le s \le T_1$, by (19) and integration by parts, we let

$$\begin{cases} \frac{d}{dt}\int_0^{T_1} \frac{(t-s)^{-q_2}x(s)}{\Gamma(1-q_2)}ds = \frac{x_1(T_1)\int_0^{T_1}(t-s)^{-q_2}|x_1(s)|ds}{\Gamma(1-q_2)\int_0^{T_1}|x_1(s)|ds} - \frac{x_1(T_1)(t-T_1)^{-q_2}}{\Gamma(1-q_2)} \doteq \varphi_{x_1}(t), \\ \\ \frac{d}{dt}\int_0^{T_1} \frac{(t-s)^{-p_2}x(s)}{\Gamma(1-p_2)}ds = \frac{x_1(T_1)\int_0^{T_1}(t-s)^{-p_2}|x_1(s)|ds}{\Gamma(1-p_2)\int_0^{T_1}|x_1(s)|ds} - \frac{x_1(T_1)(t-T_1)^{-p_2}}{\Gamma(1-p_2)} \doteq \psi_{x_1}(t). \end{cases} \tag{21}$$

Hence, we may consider the initial value problem defined in the interval $[T_1, T_2]$ as following

$$\begin{cases} D^{p_2}_{T_1+}x(t) = f(t, x, D^{q_2}_{T_1+}x + \varphi_{x_1}(t)) - \psi_{x_1}(t), T_1 < t \le T_2, \\ x(T_1) = x_1(T_1). \end{cases} \tag{22}$$

By (10), the equation (17) in the interval $(T_2, T_3]$ can be written by

$$\begin{aligned} D^{\alpha(t)}_{0+}x(t) \quad &= \frac{d}{dt}\int_0^t \frac{(t-s)^{-p_3}}{\Gamma(1-p_3)}x(s)ds \\ &= \frac{d}{dt}\int_0^{T_2} \frac{(t-s)^{-p_3}x(s)}{\Gamma(1-p_3)}ds + \frac{d}{dt}\int_{T_2}^t \frac{(t-s)^{-p_3}x(s)}{\Gamma(1-p_3)}ds \\ &= f(t, x, \frac{d}{dt}\int_0^{T_2} \frac{(t-s)^{-q_3}x(s)}{\Gamma(1-q_3)}ds + \frac{d}{dt}\int_{T_2}^t \frac{(t-s)^{-q_3}x(s)}{\Gamma(1-q_3)}ds). \end{aligned}$$

In order to consider the existence of solution to (17) in the interval $[T_2, T_3]$, we let

$$x(t) = \begin{cases} 0, & \text{if } x_1(t) \equiv 0 \text{ for } t \in [0, T_1], \\ \frac{x_1(T_1)\int_0^t |x_1(s)|ds}{\int_0^{T_1}|x_1(s)|ds}, & \text{if } x_1(t) \text{ is not identically vanishing for } t \in [0, T_1], \\ 0, & \text{if } x_2(t) \equiv 0 \text{ for } t \in [T_1, T_2], \\ \frac{x_2(T_2)\int_{T_1}^t |x_2(s)|ds}{\int_{T_1}^{T_2}|x_2(s)|ds}, & \text{if } x_2(t) \text{ is not identically vanishing for } t \in [T_1, T_2], \end{cases} \tag{23}$$

here, $x_1 : [0, T_1] \to \mathbb{R}$ is continuous solution of the initial value problem (18) and $x_2 : [T_1, T_2] \to \mathbb{R}$ is continuous solution of the initial value problem (22). Thus, if $x_1(s) \equiv 0, 0 \le s \le T_1$, $x_2(s) \equiv 0, T_1 \le s \le T_2$, then by (23), we let

$$\begin{cases} \frac{d}{dt}\int_{T_{i-1}}^{T_i} \frac{(t-s)^{-q_3}x(s)}{\Gamma(1-q_3)}ds = 0 \doteq \varphi_{x_i}(t), i = 1, 2, T_0 = 0, \\ \frac{d}{dt}\int_{T_{i-1}}^{T_i} \frac{(t-s)^{-p_3}x(s)}{\Gamma(1-p_3)}ds = 0 \doteq \psi_{x_i}(t), i = 1, 2, T_0 = 0, \end{cases} \tag{24}$$

if $x_1(s)$ is not identically vanishing for $0 \le s \le T_1$, $x_2(s)$ is not identically vanishing for $T_1 \le s \le T_2$, then by (23) and integration by parts, we let

$$\begin{cases} \frac{d}{dt}\int_{T_{i-1}}^{T_i} \frac{(t-s)^{-q_3}x(s)}{\Gamma(1-q_3)}ds = \frac{x_i(T_i)\int_{T_{i-1}}^{T_i}(t-s)^{-q_3}|x_i(s)|ds}{\Gamma(1-q_3)\int_{T_{i-1}}^{T_i}|x_i(s)|ds} - \frac{x_i(T_i)(t-T_i)^{-q_3}}{\Gamma(1-q_3)} \doteq \varphi_{x_i}(t), \\ \frac{d}{dt}\int_{T_{i-1}}^{T_i} \frac{(t-s)^{-p_3}x(s)}{\Gamma(1-p_3)}ds = \frac{x_i(T_i)\int_{T_{i-1}}^{T_i}(t-s)^{-p_3}|x_i(s)|ds}{\Gamma(1-p_3)\int_{T_{i-1}}^{T_i}|x_i(s)|ds} - \frac{x_i(T_i)(t-T_i)^{-p_3}}{\Gamma(1-p_3)} \doteq \psi_{x_i}(t), \end{cases} \tag{25}$$

$i = 1, 2, T_0 = 0$.

Hence, we may consider the initial value problem defined in the interval $[T_2, T_3]$ as following

$$\begin{cases} D_{T_2+}^{p_3}x(t) = f(t, x, D_{T_2+}^{q_3}x + \varphi_{x_1}(t) + \varphi_{x_2}(t)) - \psi_{x_1}(t) - \psi_{x_2}(t), T_2 < t \le T_3, \\ x(T_2) = x_2(T_2). \end{cases} \tag{26}$$

Similarly, the equation of (17) in the interval $(T_{i-1}, T_i]$ $(i = 4, 5, \cdots, n^*, T_{n^*} = T)$ can be written by

$$\begin{aligned} D_{0+}^{\alpha(t)}x(t) &= \frac{d}{dt}\int_0^t \frac{(t-s)^{-p_i}}{\Gamma(1-p_i)}x(s)ds \\ &= \frac{d}{dt}\int_0^{T_{i-1}} \frac{(t-s)^{-p_i}x(s)}{\Gamma(1-p_i)}ds + \frac{d}{dt}\int_{T_{i-1}}^t \frac{(t-s)^{-p_i}x(s)}{\Gamma(1-p_i)}ds \\ &= f\left(t, x, \frac{d}{dt}\int_0^{T_{i-1}} \frac{(t-s)^{-q_i}x(s)}{\Gamma(1-q_i)}ds + \frac{d}{dt}\int_{T_{i-1}}^t \frac{(t-s)^{-q_i}x(s)}{\Gamma(1-q_i)}ds\right). \end{aligned}$$

By the same reasons and ways, we let

$$x(t) = \begin{cases} 0, & \text{if } x_1(t) \equiv 0 \text{ for } t \in [0, T_1], \\ \frac{x_1(T_1)\int_0^t |x_1(s)|ds}{\int_0^{T_1}|x_1(s)|ds}, & \text{if } x_1(t) \text{ is not identically vanishing for } t \in [0, T_1] \\ 0, & \text{if } x_2(t) \equiv 0 \text{ for } t \in [T_1, T_2], \\ \frac{x_2(T_2)\int_{T_1}^t |x_2(s)|ds}{\int_{T_1}^{T_2}|x_2(s)|ds}, & \text{if } x_2(t) \text{ is not identically vanishing for } , t \in [T_1, T_2], \\ \cdots, & \\ 0, & \text{if } x_{i-1}1(t) \equiv 0 \text{ for } t \in [T_{i-2}, T_{i-1}], \\ \frac{x_{i-1}(T_{i-1})\int_{T_{i-2}}^t |x_{i-}(s)|ds}{\int_{T_{i-2}}^{T_{i-1}}|x_{i-1}(s)|ds}, & \text{if } x_{i-1}(t) \text{ is not identically vanishing for } t \in [T_{i-2}, T_{i-1}], \end{cases}$$

here, $x_1 : [0, T_1] \to \mathbb{R}$ is continuous solution of the initial value problem (18), $x_2 : [T_1, T_2] \to \mathbb{R}$ is continuous solution of the initial value problem (22), $\cdots$, $x_j : [T_{j-1}, T_j] \to \mathbb{R}$ is continuous solution of the initial value problem defined in $[T_{j-1}, T_j]$ $(j = 3, \cdots, i-1)$. Thus, if $x_j(s) \equiv 0, T_{j-1} \le s \le T_j$, then by expression of $x(t)$ above, we let

$$\begin{cases} \frac{d}{dt}\int_{T_{j-1}}^{T_j} \frac{(t-s)^{-q_i}x(s)}{\Gamma(1-q_i)}ds = 0 \doteq \varphi_{x_j}(t), \\ \frac{d}{dt}\int_{T_{j-1}}^{T_j} \frac{(t-s)^{-p_i}x(s)}{\Gamma(1-p_i)}ds = 0 \doteq \psi_{x_j}(t), \end{cases}$$

if $x_j(s)$ is not identically vanishing for $T_{j-1} \le s \le T_j$, by expression of $x(t)$ above and integration by parts, we let

$$\begin{cases} \frac{d}{dt}\int_{T_{j-1}}^{T_j} \frac{(t-s)^{-q_i}x(s)}{\Gamma(1-q_i)}ds = \frac{x_j(T_j)\int_{T_{j-1}}^{T_j}(t-s)^{-q_i}|x_j(s)|ds}{\Gamma(1-q_i)\int_{T_{j-1}}^{T_j}|x_j(s)|ds} - \frac{x_j(T_j)(t-T_j)^{-q_i}}{\Gamma(1-q_i)} \doteq \varphi_{x_j}(t), \\ \frac{d}{dt}\int_{T_{j-1}}^{T_j} \frac{(t-s)^{-p_i}x(s)}{\Gamma(1-p_i)}ds = \frac{x_j(T_j)\int_{T_{j-1}}^{T_j}(t-s)^{-p_i}|x_j(s)|ds}{\Gamma(1-p_i)\int_{T_{j-1}}^{T_j}|x_j(s)|ds} - \frac{x_j(T_j)(t-T_j)^{-p_i}}{\Gamma(1-p_i)} \doteq \psi_{x_j}(t), \end{cases}$$

$j = 1, 2, \cdots, i-1$, $i = 4, 5, \cdots, n^*$ ($T_0 = 0$).

Hence, we may consider the initial value problem in the interval $[T_{i-1}, T_i]$ as following

$$\begin{cases} D^{p_i}_{T_{i-1}+}x(t) = f(t, x, D^{q_i}_{T_{i-1}+}x + \sum_{j=1}^{i-1}\varphi_{x_j}) - \sum_{j=1}^{i-1}\psi_{x_j}(t), \\ x(T_{i-1}) = x_{i-1}(T_{i-1}). \end{cases} \tag{27}$$

By (10), the equation of (17) in the interval $(T, +\infty)$ can be written by

$$\begin{aligned} D^{\alpha(t)}_{0+}x(t) &= \frac{d}{dt}\int_0^t \frac{(t-s)^{-\rho_1}}{\Gamma(1-\rho_1)}x(s)ds \\ &= \frac{d}{dt}\int_0^T \frac{(t-s)^{-\rho_1}x(s)}{\Gamma(1-\rho_1)}ds + \frac{d}{dt}\int_T^t \frac{(t-s)^{-\rho_1}x(s)}{\Gamma(1-\rho_1)}ds \\ &= f\Big(t, x, \frac{d}{dt}\int_0^T \frac{(t-s)^{-\rho_2}x(s)}{\Gamma(1-\rho_2)}ds + \frac{d}{dt}\int_T^t \frac{(t-s)^{-\rho_2}x(s)}{\Gamma(1-\rho_2)}ds\Big). \end{aligned}$$

By the same reasons and ways, we let

$$x(t) = \begin{cases} 0, & \text{if } x_1(t) \equiv 0 \text{ for } t \in [0, T_1], \\ \frac{x_1(T_1)\int_0^t |x_1(s)|ds}{\int_0^{T_1}|x_1(s)|ds}, & \text{if } x_1(t) \text{ is not identically vanishing for } , t \in [0, T_1] \\ 0, & \text{if } x_2(t) \equiv 0 \text{ for } t \in [T_1, T_2], \\ \frac{x_2(T_2)\int_{T_1}^t |x_2(s)|ds}{\int_{T_1}^{T_2}|x_2(s)|ds}, & \text{if } x_2(t) \text{ is not identically vanishing for } t \in [T_1, T_2], \\ \cdots, & \\ 0, & \text{if } x_{n^*}(t) \equiv 0 \text{ for } , t \in [T_{n^*-1}, T], \\ \frac{x_{n^*}(T)\int_{T_{n^*-1}}^t |x_{n^*}(s)|ds}{\int_{T_{n^*-1}}^{T}|x_{n^*}(s)|ds}, & \text{if } x_{n^*}(t) \text{ is not identically vanishing for } t \in [T_{n^*-1}, T], \end{cases}$$

here, $x_1 : [0, T_1] \to \mathbb{R}$ is continuous solution of the initial value problem (18), $x_i : [T_{i-1}, T_i] \to \mathbb{R}(i = 2, 3, \cdots, n^*(T_{n^*} = T))$ are continuous solutions of the initial value problems (22), (26), (27). Thus, if $x_i(s) \equiv 0, T_{i-1} \le s \le T_i$, then by expression of $x(t)$ above, we let

$$\begin{cases} \frac{d}{dt}\int_{T_{i-1}}^{T_i} \frac{(t-s)^{-\rho_2}x(s)}{\Gamma(1-\rho_2)}ds = 0 \doteq \varphi_{x_i}(t), \\ \frac{d}{dt}\int_{T_{i-1}}^{T_i} \frac{(t-s)^{-\rho_1}x(s)}{\Gamma(1-\rho_1)}ds = 0 \doteq \psi_{x_i}(t), \end{cases} \tag{28}$$

if $x_i(s)$ is not identically vanishing for $T_{i-1} \le s \le T_i$, by expression of $x(t)$ above and integration by parts, we let

$$\begin{cases} \frac{d}{dt}\int_{T_{i-1}}^{T_i} \frac{(t-s)^{-\rho_2}x(s)}{\Gamma(1-\rho_2)}ds = \frac{x_i(T_i)\int_{T_{i-1}}^{T_i}(t-s)^{-\rho_2}|x_i(s)|ds}{\Gamma(1-\rho_1)\int_{T_{i-1}}^{T_i}|x_i(s)|ds} - \frac{x_i(T_i)(t-T_i)^{-\rho_2}}{\Gamma(1-\rho_2)} \doteq \varphi_{x_i}(t), \\ \frac{d}{dt}\int_{T_{i-1}}^{T_i} \frac{(t-s)^{-\rho_1}x(s)}{\Gamma(1-\rho_1)}ds = \frac{x_i(T_i)\int_{T_{i-1}}^{T_i}(t-s)^{-\rho_1}|x_i(s)|ds}{\Gamma(1-\rho_1)\int_{T_{i-1}}^{T_i}|x_i(s)|ds} - \frac{x_i(T_i)(t-T_i)^{-\rho_1}}{\Gamma(1-\rho_1)} \doteq \psi_{x_i}(t), \end{cases} \tag{29}$$

$i = 1, 2, \cdots, n^*(T_0 = 0, T_{n^*} = T)$.

Hence, we may consider the initial value problem defined in the interval $[T, +\infty)$ as following

$$\begin{cases} D_{T+}^{\rho_1}x(t) = f(t, x, D_{T+}^{\rho_2}x + \sum_{i=1}^{n^*}\varphi_{x_i}) - \sum_{i=1}^{n^*}\psi_{x_i}(t), \\ x(T) = x_{n^*}(T). \end{cases} \tag{30}$$

Now, based on the arguments above, we present the definition of solution to the initial value problem (17), which is crucial in our work.

Definition 2.6. If the initial value problems (18), (22), (26), (27) and (30) exist solutions $x_1 : [0, T_1] \to \mathbb{R}$, $x_2 : [T_1, T_2] \to \mathbb{R}$, $x_3 : [T_2, T_3] \to \mathbb{R}$, $x_i : [T_{i-1}, T_i] \to \mathbb{R}(i = 4, \cdots, n^*, T_{n^*} = T)$and $x_{n^*+1} : [T, +\infty) \to \mathbb{R}$, respectively, then we call function $x : [0, +\infty) \to (-\infty, +\infty)$ defined by

$$x(t) = \begin{cases} x_1(t), 0 \le t \le T_1, \\ x_2(t), T_1 \le t \le T_2, \\ x_3(t), T_2 \le t \le T_3, \\ \vdots \\ x_{n^*}(t), T_{n^*-1} \le t \le T, \\ x_{n^*+1}(t), T \le t < +\infty \end{cases} \tag{31}$$

is a solution of the initial value problem (17).

Definition 2.7. If $x_1(t), x_2(t), \cdots, x_{n^*+1}(t)$ are unique, then we say $x(t)$ defined in (31) is one unique solution of the initial value problem (17).

The following is the definition of the (unique)approximate solution of the initial value problem (1).

Definition 2.8. If there are $T > 0$, natural number $n^* \in N$ and intervals $[0, T_1]$, $(T_1, T_2]$, $\cdots$, $(T_{n^*-1}, T]$, $(T, +\infty)$ and piecewise functions defined in (10) satisfying (11), and that the initial value problem (17) exists one (unique) solution, then, we say this solution of the initial value problem (17) is one (unique) approximate solution of the initial value problem (1).

Remark 2.9. The differential equations of fractional order or integer order, in general, its solution in given interval is affected by the state of the solution in the preceding intervals. For instance, the differential equation

$$D_{0+}^{\frac{1}{2}}x(t) = t^{-\frac{1}{3}} + \frac{t^4x^3}{1 + x^2}, 0 < t \le 2. \tag{32}$$

We know that

$$x(t)=\frac{1}{\Gamma(\frac{1}{2})}\int_0^t(t-s)^{-\frac{1}{2}}(s^{-\frac{1}{3}}+\frac{s^4x^3(s)}{1+x^2(s)})ds, 0\le t\le 2 \tag{33}$$

is a solution of (32). Then by (33), in the interval $[1,\frac{3}{2}]$, $x(t)$ should be

$$\begin{aligned}x(t)&=\int_0^t\frac{(t-s)^{-\frac{1}{2}}(s^{-\frac{1}{3}}+\frac{s^4x^3(s)}{1+x^2(s)})}{\Gamma(\frac{1}{2})}ds\\&=\int_0^1\frac{(t-s)^{-\frac{1}{2}}(s^{-\frac{1}{3}}+\frac{s^4x^3}{1+x^2(s)})}{\Gamma(\frac{1}{2})}ds+\int_1^t\frac{(t-s)^{-\frac{1}{3}}(s^{-\frac{1}{2}}+\frac{s^4x^3(s)}{1+x^2(s)})}{\Gamma(\frac{1}{2})}ds,\end{aligned}$$

which illstrates that the state of $x(t)$ in $[1,\frac{3}{2}]$, must be affected by the state of $x(t)$ in $[0,1]$.

Remark 2.10. From the Definition 2.6, we see that, in interval $[T_2,T_3]$, solution $x(t)$ of the initial value problem (17), is $x_3(t)$, which is solution of the equation of (26), obvious, the state of $x_3(t)$ is affected by the state of $x_1(t),x_2(t)$, that is, the state of $x_3(t)$ is affected by he state of $x(t)$ in interval $[0,T_2]$. Thus, according the Definition 2.8, in the interval $[T_2,T_3]$, the state of approximate solution $x(t)$ of the equation of (1) must be affected by the state of $x(t)$ in interval $[0,T_2]$. Hence, according to Remark 2.9, Definitions 2.6 and 2.8 are suitable and reasonable.

Remark 2.11. In our previous analysis, we chose functions (20)((21)), (24) ((25)) etc, so that we obtain the initial value problems (22), (26) etc. Such choosing must meet the following two reasons at the same time. The first reason is operability, for instance, choosing function (20)((21)), we can calculate out functions $\varphi_{x_1}(t),\varphi_{x_2}(t)$ and $\psi_{x_1}(t),\psi_{x_2}(t)$, and then we obtain the initial value problem (22) defined in $[T_1,T_2]$. The second reason is for fitting Remark 2.9. If we take $x(t)=x_1(T_1)$ for $0\le t\le T_1$ (here $x_1(t)$ is the solution of the initial value problem (18)), then, although we may easily calculate out functions $\varphi_{x_1}(t)=\frac{x_1(T_1)(t^{-q_1}-(t-T_1)^{-q_1})}{\Gamma(1-q_1)}$ and $\psi_{x_1}(t)=\frac{x_1(T_1)(t^{-p_1}-(t-T_1)^{-p_1})}{\Gamma(1-p_1)}$, and then we have the initial value problem (22) with such $\varphi_{x_1}(t)$ and $\psi_{x_1}(t)$, But, as a result, we see that the state of solution of the equation of (22) is only affected by $x_1(T_1)$, and doesn't by affected by the state of $x_1(t),0\le t\le T_1$. Hence this choosing of functions $x(t)$ in known intervals, is suitable.

3. Existence of approximate solution

In this section, according to arguments and analysis in the section 2, we consider the unique existence of continuous approximate solution of the initial value problem (1). Throughout this paper, we assume that

(A_1) Let $p:[0,+\infty)\to(0,1)$ and $q:[0,+\infty)\to(0,1)$ be continuous functions, $2q(t)<p(t)$ for all $t\in[0,+\infty)$, and that $p(t),q(t)$ satisfy

$$\lim_{t\to+\infty}p(t)=\eta_1,\lim_{t\to+\infty}q(t)=\eta_2, 0\le 2\eta_2<\eta_1<1.$$

(A_2) let $t^rf:[0,+\infty)\times R^2\to R$ be a continuous function, $0\le r<p(t)-2q(t)$, $0\le t<+\infty$. Assume that there are positive constants $c_1>0,c_2>0,\lambda>1$ such that

$t^r|f(t,(1+t^\lambda)x_1,(1+t^\lambda)y_1)-f(t,(1+t^\lambda)x_2,(1+t^\lambda)y_2)| \le c_1|x_1-x_2|+c_2|y_1-y_2|.$

(A_3) there exists $0 \le \mu < \lambda - 1$ such that $\lim_{t\to+\infty} \frac{t^r|f(t,0,0)|}{1+t^\mu} < +\infty$, where r, λ are the constants appearing in (A_2).

Our main result is as following.

Theorem 3.1. Let conditions $(A_1),(A_2),(A_3)$ hold, then the initial value problem (1.1) exists one unique approximate solution.

Proof. From the Definition 2.8, we only need to consider the unique existence of the solution of the initial value problem (17). And by the Definition 2.6, we only need to consider the unique existence of the solution of the initial value problems (18), (22), (26), (27) and (30).

For convenience, we let

$$\begin{cases} M_q = \max\{\max_{0\le t\le T}|\Gamma(1-q(t))|, \Gamma(1-\rho_2)\}, \\ M_{p,q} = \max\{\max_{0\le t\le T}|\frac{1}{\Gamma(p(t)-q(t))}|, \Gamma(\rho_1-\rho_2)\} \\ M_{p,q,r} = \max\{\max_{0\le t\le T}|\frac{1}{p(t)-q(t)-r}|, \frac{1}{\rho_1-\rho_2-r}\}, \\ m_{p,q} = \min_{0\le t\le T+1}(p(t)-2q(t)), \end{cases} \tag{34}$$

where T, ρ_1, ρ_2 are the constants in (12), (14).

It follows from the continuities of functions $p(t), q(t)$ and Gamma function that M_p ,$M_{p,q}$, $M_{p,q,r}$, $m_{p,q}$ exist. By (A_1) and (A_2), we know that $0 \le r < m_{p,q}$.

Take $R \in N$ such that

$$R > \left\{1, \left[4(T+1)^2\big(c_2+c_1M_q\big)M_{p,q}M_{p,q,r}\right]^{\frac{1}{m_{p,q}-r}}\right\}, \tag{35}$$

where T, ρ_1, ρ_2 are the constants in (12), (14), c_1, c_2, r are the constants in (A_2), $M_p, M_{p,q}, M_{p,q,r}$, $m_{p,q}$ are the constants in (34).

Let $C[T_{i-1}, T_i]$ denote the Banach spaces of continuous functions on $[T_{i-1}, T_i]$ with the norm

$$\|x\| = \max_{T_{i-1}\le t\le T_i}|x(t)|, x \in C[T_{i-1}T_i],$$

$C_{q_i}[T_{i-1}, T_i] = \{x|x \in C(T_{i-1}, T_i], (t-T_{i-1})^{q_i}x \in C[T_{i-1}, T_i]\}$ denote the Banach spaces with the norm

$$\|x\|_{C_{q_i}} = \max_{t\in[T_{i-1},T_i]}(t-T_{i-1})^{q_i}e^{-R^2(t-T_{i-1})^{p_i-q_i-r}}|x(t)|, x \in C_{q_i}[T_{i-1}, T_i],$$

T_i $(T_0 = 0, T_{n^*} = T)$ are the constants obtained in the Lemma 2.5, p_i, q_i are the constants in (15), (16), $i = 1, 2\cdots, n^*$, $n^* \in N$. r is the constant in (A_2), R is the positive integer satisfying (35).

Let

$$E = \Big\{x|x \in C(T,+\infty), \\ (t-t)^{\rho_2}x \in C[T,+\infty), \sup_{t\ge T}\frac{(t-T)^{\rho_2}e^{-R^2(t-T)^{\rho_1-\rho_2-r}}|x(t)|}{1+t^\lambda} < \infty\Big\}$$

with the norm

$$\|x\|_E = \sup_{t\geq T} \frac{(t-T)^{\rho_2} e^{-R^2(t-T)^{\rho_1-\rho_2-r}} |x(t)|}{1+t^\lambda},$$

where T, ρ_1, ρ_2 are the constants in (12), (14), λ is the constant in (A_2), R is the positive integer satisfying (35). By the same arguments as in Lemma 2.2 [30], we know that $(E, \|\cdot\|_E)$ is a Banach space.

We first investigative the initial value problem (18)

$$\begin{cases} D_{0+}^{p_1} x(t) = f(t, x, D_{0+}^{q_1} x), 0 < t \leq T_1, \\ x(0) = 0. \end{cases}$$

We have the following claim.

Claim 1. If $y \in C_{q_1}[0, T_1]$ is a solution of the following integral equation

$$y(t) = I_{0+}^{p_1-q_1} f(t, I_{0+}^{q_1} y(t), y(t)), 0 < t \leq T_1, \tag{36}$$

then, $x(t) = I_{0+}^{q_1} y(t) \in C[0, T_1]$ must be a solution of the initial value problem (18).

In fact, if $y \in C_{q_1}[0, T_1]$ is a solution of the integral (36), then, applying operator $I_{0+}^{q_1}$ on both sides of (36), from property of the Rieamnn-Liouville calculus, it holds

$$I_{0+}^{q_1} y(t) = I_{0+}^{q_1} I_{0+}^{p_1-q_1} f(t, I_{0+}^{q_1} y(t), y(t)) = I_{0+}^{p_1} f(t, I_{0+}^{q_1} y(t), y(t)).$$

Let

$$I_{0+}^{q_1} y(t) = x(t), 0 \leq t \leq T,$$

thus, $x \in C[0, T_1]$, and $y(t) = D_{0+}^{q_1} x(t) \in C_{q_1}[0, T_1]$. As a result, we have that

$$x(t) = I_{0+}^{p_1} f(t, x(t), D_{0+}^{q_1} x(t)),\ 0 \leq t \leq T_1,$$

according to assumptions of function f, we get $x(0) = 0$ and

$$D_{0+}^{p_1} x(t) = f(t, x(t), D_{0+}^{q_1} x(t)), 0 < t \leq T_1,$$

that is, $x \in C[0, T_1]$ is a solution of the initial value problem (18).

Define operator $F : C_{q_1}[0, T_1] \to C_{q_1}[0, T_1]$ by

$$Fy(t) = I_{T0+}^{p_1-q_1} f(t, I_{0+}^{q_1} y(t), y(t)), 0 \leq t \leq T_1.$$

By the assumptions of function f, we know that $F : C_{q_1}[0, T_1] \to C_{q_1}[0, T_1]$ is well defined. Next, we will verify that F is a contraction operator.

For $y_1, y_2 \in C_{q_1}[0, T_1]$, by (A_2) and (34), we get

$$\begin{aligned}
& |Fy_1(t) - Fy_2(t)| \\
\leq \quad & \frac{1}{\Gamma(p_1-q_1)} \int_0^t (t-s)^{p_1-q_1-1} \frac{s^{-r}}{1+s^\lambda} \big(c_1 |I_{0+}^{q_2} y_1(s) - I_{0+}^{q_2} y_2(s)| \\
& + c_2 |y_1(s) - y_2(s)|\big) ds \\
\leq \quad & \frac{1}{\Gamma(p_1-q_1)} \int_0^t (t-s)^{p_1-q_1-1} s^{-r} \\
& \big(c_1 |I_{0+}^{q_2} y_1(s) - I_{0+}^{q_2} y_2(s)| + c_2 |y_1(s) - y_2(s)|\big) ds \\
\leq \quad & \frac{c_1 \|y_1 - y_2\|_{C_{q_1}}}{\Gamma(q_1)\Gamma(p_1-q_1)} \int_0^t (t-s)^{p_1-q_1-1} s^{-r} \int_0^s (s-\tau)^{q_1-1} \tau^{-q_1} e^{R^2 \tau^{p_1-q_1-r}} d\tau ds
\end{aligned}$$

$$
\begin{aligned}
& +\frac{c_2\|y_1-y_2\|_{C_{q_1}}}{\Gamma(p_1-q_1)}\int_0^t (t-s)^{p_1-q_1-1}s^{-q_1-r}e^{R^2s^{p_1-q_1-r}}ds \\
\le\quad & \frac{c_1\|y_1-y_2\|_{C_{q_1}}\Gamma(1-q_1)}{\Gamma(p_1-q_1)}\int_0^t (t-s)^{p_1-q_1-1}s^{q_1}s^{-q_1-r}e^{R^2s^{p_1-q_1-r}}ds \\
& +\frac{c_2\|y_1-y_2\|_{C_{q_1}}}{\Gamma(p_1-q_1)}\int_0^t (t-s)^{p_1-q_1-1}s^{-q_1-r}e^{R^2s^{p_1-q_1-r}}ds \\
\le\quad & (c_1M_qT^{q_1}+c_2)M_{p,q}\|y_1-y_2\|_{C_{q_1}}\int_0^t (t-s)^{p_1-q_1-1}s^{-q_1-r}e^{R^2s^{p_1-q_1-r}}ds \\
\le\quad & (c_1M_q+c_2)(T+1)M_{p,q}\|y_1-y_2\|_{C_{q_1}}\int_0^t (t-s)^{p_1-q_1-1}s^{-q_1-r}e^{R^2s^{p_1-q_1-r}}ds.
\end{aligned}
$$

Next, using a similar method as in [25], by (A_1) and (34), we estimate the integral above.

$$
\begin{aligned}
& \int_0^t (t-s)^{p_1-q_1-1}s^{-q_1-r}e^{R^2s^{p_1-q_1-r}}ds \\
=\quad & \sum_{i=1}^{R-1}\int_{\frac{(i-1)t}{R}}^{\frac{it}{R}}(t-s)^{p_1-q_1-1}s^{-q_1-r}e^{R^2s^{p_1-q_1-r}}ds \\
& +\int_{\frac{(R-1)t}{R}}^{t}(t-s)^{p_1-q_1-1}s^{-q_1-r}e^{R^2s^{p_1-q_1-r}}ds \\
\le\quad & \sum_{i=1}^{R-1}\int_{\frac{(i-1)t}{R}}^{\frac{it}{R}}R^{1-p_1+q_1}(R-i)^{p_1-q_1-1}t^{p_1-q_1-1}s^{-q_1-r}e^{R^2s^{p_1-q_1-r}}ds \\
& +\left(\frac{R-1}{R}\right)^{-q_1-r}t^{-q_1-r}e^{R^2t^{p_1-q_1-r}}\int_{\frac{(R-1)t}{R}}^{t}(t-s)^{p_1-q_1-1}ds \\
\le\quad & \sum_{i=1}^{R-1}\int_{\frac{(i-1)t}{R}}^{\frac{it}{R}}R^{1-p_1+q_1}t^{p_1-q_1-1}s^{-q_1-r}e^{R^2s^{p_1-q_1-r}}ds \\
& +\frac{R^{q_1+r}e^{R^2t^{p_1-q_1-r}}}{t^{q_1+r}}\int_{\frac{(R-1)t}{R}}^{t}(t-s)^{p_1-q_1-1}ds \\
=\quad & R^{1-p_1+q_1}\int_0^{\frac{(R-1)t}{R}}t^{p_1-q_1-1}s^{-q_1-r}e^{R^2s^{p_1-q_1-r}}ds \\
& +\frac{R^{r-p_1+2q_1}t^{p_1-2q_1-r}}{p_1-q_1}e^{R^2t^{p_1-q_1-r}} \\
\le\quad & R^{1-p_1+q_1}t^{-q_1}\int_0^{\frac{(R-1)t}{R}}s^{p_1-q_1-r-1}e^{R^2s^{p_1-q_1-r}}ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{R^{r-p_1+2q_1}t^{-q_1}t^{p_1-q_1-r}}{p_1-q_1-r}e^{R^2t^{p_1-q_1-r}}\\
\leq\ & \frac{R^{1-p_1+2q_1}t^{-q_1}}{R^2(p_1-q_1-r)}e^{R^2(\frac{(R-1)t}{R})^{p_1-q_1-r}}+\frac{R^{r-p_1+2q_1}t^{-q_1}(T+1)}{p_1-q_1-r}e^{R^2t^{p_1-q_1-r}}\\
\leq\ & M_{p,q,r}R^{r-p_1+2q_1}t^{-q_1}e^{R^2t^{p_1-q_1-r}}+M_{p,q,r}R^{r-p_1+2q_1}t^{-q_1}(T+1)e^{R^2t^{p_1-q_1-r}}\\
\leq\ & 2(T+1)M_{p,q,r}t^{-q_1}R^{r-m_{p,q}}e^{R^2t^{p_1-q_1-r}}.
\end{aligned}
$$

By (35), we have

$$
\begin{aligned}
& t^{q_1}e^{-R^2t^{p_1-q_1-r}}|Fy_1(t)-Fy_2(t)|\\
\leq\ & 2(c_1M_q+c_2)(T+1)^2M_{p,q}M_{p,q,r}R^{r-m_{p,q}}\|y_1-y_2\|_{C_{q_1}}\\
\leq\ & \frac{1}{2}\|y_1-y_2\|_{C_{q_1}},
\end{aligned}
$$

which implies that

$$\|Fy_1-Fy_2\|_{C_{q_1}}\leq\frac{1}{2}\|y_1-y_2\|_{C_{q_1}}.$$

Hence, F has one unique fixed point $y_1\in C_{q_1}[0,T_1]$. Thus, by the arguments above, we obtain $x_1(t)=I_{0+}^{q_1}y_1(t)\in C[0,T_1]$ is one unique solution of the initial value problem (18).

In the next analysis, without loss of generality, we assume $x_1(t)$ is not identically vanishing in $t\in[0,T_1]$. Thus, by previous arguments, Definitions 2.6 and 2.8, we consider the initial value problem equation (22)

$$
\begin{cases}
D_{T_1+}^{p_2}x(t)=f(t,x,D_{T_1+}^{q_2}x+\varphi_{x_1}(t))-\psi_{x_1}(t), T_1\leq t\leq T_2,\\
x(T_1)=x_1(T_1),
\end{cases}
$$

where $x_1=I_{0+}^{q_1}y_1\in C[0,T_1]$ is the unique solution of the initial value problem (18), $y_1\in C_{q_1}[0,T_1]$ is the unique solution of the integral equation (36),

$$\varphi_{x_1}(t)=\frac{x_1(T_1)\int_0^{T_1}(t-s)^{-q_2}|x_1(s)|ds}{\Gamma(1-q_2)\int_0^{T_1}|x_1(s)|ds}-\frac{x_1(T_1)(t-T_1)^{-q_2}}{\Gamma(1-q_2)},$$

$$\psi_{x_1}(t)=\frac{x_1(T_1)\int_0^{T_1}(t-s)^{-p_2}|x_1(s)|ds}{\Gamma(1-p_2)\int_0^{T_1}|x_1(s)|ds}-\frac{x_1(T_1)(t-T_1)^{-p_2}}{\Gamma(1-p_2)}.$$

Let

$$h_{x_1}(t)=\frac{x_1(T_1)\int_0^{T_1}(t-s)^{-p_2}|x_1(s)|ds}{\Gamma(1-p_2)\int_0^{T_1}|x_1(s)|ds},$$

by calculating, we get

$$I_{T_1+}^{p_2}\psi_{x_1}(t)=I_{T_1+}^{p_2}h_{x_1}(t)-x_1(T_1).\tag{37}$$

Similar to the previous arguments, we can obtain the following result.

Claim 2. If $y\in C_{q_2}[T_1,T_2]$ is a solution of the following integral equation

$$y(t)=I_{T_1+}^{p_2-q_2}f(t,I_{T_1+}^{q_2}y(t),y(t)+\varphi_{x_1}(t))-I_{T_1+}^{p_2-q_2}h_{x_1}(t)+\frac{x_1(T_1)(t-T_1)^{-q_2}}{\Gamma(1-q_2)},\tag{38}$$

then, $x = I^{q_2}_{T_1+}y \in C[T_1, T_2]$ must be a solution of the initial value problem (22).

In fact, if $y \in C_{q_2}[T_1, T_2]$ is a solution of the integral (38), then, applying operator $I^{q_2}_{T_1+}$ on both sides of (38), from property of the Riemann-Liouville fractional calculus, it holds

$$I^{q_2}_{T_1+}y(t) = I^{p_2}_{T_1+}f(t, I^{q_2}_{T_1+}y(t), y(t) + \varphi_{x_1}(t)) - I^{p_2}_{T_1+}h_{x_1}(t) + x_1(T_1),$$

let

$$I^{q_2}_{T_1+}y(t) = x(t), T_1 \le t \le T_2,$$

thus, $x \in C[T-1, T_2]$ and $D^{q_2}_{T_1+}x = y \in C_{q_2}[T_1, T_2]$, as a result, we have

$$x(t) = I^{p_2}_{T_1+}f(t, x(t), D^{q_2}_{0+}x(t) + \varphi_{x_1}) - I^{p_2}_{T_1+}h_{x_1}(t) + x_1(T_1),\ T_1 \le t \le T_2,$$

according to assumptions of function f and continuity of function $x_1(t)$, we get $x(T_1) = x_1(T_1)$, and by (37), it holds

$$D^{p_2}_{T_1+}x(t) = f(t, , x(t), D^{q_2}_{T_1+}x(t) + \varphi_{x_1}(t)) - \psi_{x_1}(t), T_1 < t \le T_2,$$

that is, $x \in C[T_1, T_2]$ is a solution of the initial value problem (22).

Define operator $F : C_{q_2}[T_1, T_2] \to C_{q_2}[T_1, T_2]$ by

$$Fy(t) = I^{p_2-q_2}_{T_1+}f(t, I^{q_2}_{T_1+}y(t), y(t) + \varphi_{x_1}(t)) - I^{p_2-q_2}_{T_1+}h_{x_1}(t) + \frac{x_1(T_1)(t-T_1)^{-q_2}}{\Gamma(1-q_2)}.$$

By the assumptions of function f and $x_1 \in C[0, T_1]$, we know that $F : C_{q_2}[T_1, T_2] \to C_{q_2}[T_1, T_2]$ is well defined. Next, we will verify that F is a contraction operator.

For $y_1, y_2 \in C_{q_2}[T_1, T_2]$, by $(A_1), (A_2)$, (34), (35), using ways similar to the ways used previously, we get

$$\begin{aligned}
&|Fy_1(t) - Fy_2(t)| \\
\le\ & \frac{c_1\|y_1 - y_2\|_{C_{q_2}}}{\Gamma(q_2)\Gamma(p_2-q_2)}\int_{T_1}^{t}(t-s)^{p_2-q_2-1}s^{-r}\int_{T_1}^{s}\frac{(s-\tau)^{q_2-1}e^{R^2(\tau-T_1)^{p_2-q_2-r}}d\tau}{(\tau-T_1)^{q_2}}ds \\
&+\frac{c_2\|y_1 - y_2\|_{C_{q_2}}}{\Gamma(p_2-q_2)}\int_{T_1}^{t}(t-s)^{p_2-q_2-1}s^{-r}(s-T_1)^{-q_2}e^{R^2(s-T_1)^{p_2-q_2-r}}ds \\
\le\ & \frac{c_1\|y_1 - y_2\|_{C_{q_2}}\Gamma(1-q_2)}{\Gamma(p_2-q_2)}\int_{T_1}^{t}(t-s)^{p_2-q_2-1}s^{q_2}s^{-q_2-r}e^{R^2(s-T_1)^{p_2-q_2-r}}ds \\
&+\frac{c_2\|y_1 - y_2\|_{C_{q_2}}}{\Gamma(p_2-q_2)}\int_{T_1}^{t}(t-s)^{p_2-q_2-1}(s-T_1)^{-q_2-r}e^{R^2(s-T_1)^{p_2-q_2-r}}ds \\
\le\ & (c_1M_qT^{q_2} + c_2)M_{p,q}\|y_1 - y_2\|_{C_{q_2}}\int_{T_1}^{t}\frac{(t-s)^{p_2-q_2-1}e^{R^2(s-T_1)^{p_2-q_2-r}}}{(s-T_1)^{q_2+r}}ds \\
\le\ & (c_1M_q + c_2)(T+1)M_{p,q}\|y_1 - y_2\|_{C_{q_2}}\int_{T_1}^{t}\frac{(t-s)^{p_2-q_2-1}e^{R^2(s-T_1)^{p_2-q_2-r}}}{(s-T_1)^{q_2+r}}ds,
\end{aligned}$$

and

$$\int_{T_1}^{t}(t-s)^{p_2-q_2-1}(s-T_1)^{-q_2-r}e^{R^2(s-T_1)^{p_2-q_2-r}}ds$$

$$
\begin{aligned}
&= \sum_{i=1}^{R-1}\int_{\frac{(i-1)(t-T_1)}{R}+T_1}^{\frac{i(t-T_1)}{R}+T_1}(t-s)^{p_2-q_2-1}(s-T_1)^{-q_2-r}e^{R^2(s-T_1)^{p_2-q_2-r}}ds\\
&\quad+\int_{\frac{(R-1)(t-T_1)}{R}+T_1}^{t}(t-s)^{p_2-q_2-1}(s-T_1)^{-q_2-r}e^{R^2(s-T_1)^{p_2-q_2-r}}ds\\
&\leq 2(T+1)M_{p,q,r}(t-T_1)^{-q_2}R^{r-m_{p,q}}e^{R^2(t-T_1)^{p_2-q_2-r}},
\end{aligned}
$$

and

$$
\begin{aligned}
&(t-T_1)^{q_2}e^{-R^2(t-T_1)^{p_2-q_2-r}}|Fy_1(t)-Fy_2(t)|\\
\leq\ & 2(c_1M_q+c_2)(T+1)^2M_{p,q}M_{p,q,r}R^{r-m_{p,q}}\|y_1-y_2\|_{C_{q_2}}\leq\frac{1}{2}\|y_1-y_2\|_{C_{q_2}},
\end{aligned}
$$

which implies that

$$\|Fy_1-Fy_2\|_{C_{q_2}}\leq\frac{1}{2}\|y_1-y_2\|_{C_{q_2}}.$$

Hence, F has one unique fixed point $y_2\in C_{q_2}[T_1,T_2]$. Thus, by previous arguments, we obtain $x_2=I^{q_2}_{T_1+}y_2\in C[T_1,T_2]$ is one unique solution of the initial value problem (22).

By the similar way, we obtain that the initial value problem (26) has one solution $x_3\in C[T_2,T_3]$, and the initial value problem (27) has one unique solution $x_i=I^{q_i}_{T_{i-1}+}y_i\in C[T_{i-1},T_i]$, where $y_i\in C_{q_i}[T_{i-1},T_i]$ is one unique solution of the integral equation defined in the interval $[T_{i-1},T_i]$, $i=4,\cdots,n^*$, $T_{n^*}=T$.

In the next analysis, without loss of generality, we assume $x_i(t)$ is not identically vanishing in $t\in[T_{i-1},T_i]$, $i=1,2,\cdots,n^*$, $T_0=0,T_{n^*}=T$. By the previous arguments and Definitions 2.6 and 2.8, now we consider the initial value problem (30)

$$
\begin{cases}
D^{\rho_1}_{T+}x(t)=f(t,x,D^{\rho_2}_{T+}x+\varphi_{x_1}(t)+\cdots+\varphi_{x_{n^*}}(t))-\psi_{x_1}(t)-\cdots-\psi_{x_{n^*}}(t),\\
x(T)=x_{n^*}(T),
\end{cases}
$$

where

$$\varphi_{x_j}(t)=\frac{x_j(T_j)\int_{T_{j-1}}^{T_j}(t-s)^{-\rho_2}|x_j(s)|ds}{\Gamma(1-\rho_2)\int_{T_{j-1}}^{T_j}|x_i(s)|ds}-\frac{x_j(T_j)(t-T_j)^{-\rho_2}}{\Gamma(1-\rho_2)}$$

and

$$\psi_{x_j}(t)=\frac{x_j(T_j)\int_{T_{j-1}}^{T_j}(t-s)^{-\rho_1}|x_j(s)|ds}{\Gamma(1-\rho_1)\int_{T_{j-1}}^{T_j}|x_i(s)|ds}-\frac{x_j(T_j)(t-T_j)^{-\rho_1}}{\Gamma(1-\rho_1)},$$

where $j=1,2,\cdots,n^*$ ($T_0=0,T_{n^*}=T$).
Let

$$h_{x_{n^*}}(t)=\frac{x_{n^*}(T)\int_{T_{n^*-1}}^{T}(t-s)^{-\rho_1}|x_{n^*}(s)|ds}{\Gamma(1-\rho_1)\int_{T_{n^*-1}}^{T}|x_{n^*}(s)|ds},$$

by calculating, we get

$$I^{\rho_1}_{T+}\psi_{x_{n^*}}(t)=I^{\rho_1}_{T+}h_{x_{n^*}}(t)-x_{n^*}(T).$$

By a similar way, we can obtain the following result.

Claim 3. If $y \in E$ is a fixed point of the operator $F : E \to E$ defined as following

$$\begin{aligned} Fy(t) =& I_{T+}^{\rho_1-\rho_2} f(t, I_{T+}^{\rho_2} y(t), y(t) + \varphi_{x_1}(t) + \cdots + \varphi_{x_{n^*}}(t)) \\ & - I_{T+}^{\rho_1-\rho_2}(\psi_{x_1} + \cdots + \psi_{x_{n^*-1}})(t) \\ & - I_{T+}^{\rho_1-\rho_2} h_{x_{n^*}}(t) + \frac{x_{n^*}(T)(t-T)^{-\rho_2}}{\Gamma(1-\rho_2)}, T \le t < +\infty. \end{aligned}$$

Then, $x = I_{T+}^{\rho_2} y \in C[T, +\infty)$ must be a solution of the initial value problem (30).

Now, we verify that $F : E \to E$ is well defined. First, by the standard arguments, we know that $Fx \in C(T, +\infty), (t-T)^{\rho_2} Fx \in C[T, +\infty)$ for $x \in E$. Second, we will verify that $\sup_{t \ge T} \frac{(t-T)^{\rho_2} e^{-R^2(t-T)^{\rho_1-\rho_2-r}} |Fx(t)|}{1+t^\lambda} < +\infty$ for $x \in E$.

In fact, for $y \in E$, it holds

$$\begin{aligned} |\varphi_{x_j}(t)| \le & \quad \frac{\|x_j\|^2_{C[T_{j-1},T_j]} \int_{T_{j-1}}^{T_j} (t-s)^{-\rho_2} ds}{\Gamma(1-\rho_2) \int_{T_{j-1}}^{T_j} |x_j(s)| ds} + \frac{\|x_j\|_{C[T_{j-1},T_j]} (t-T_j)^{-\rho_2}}{\Gamma(1-\rho_2)} \\ \le & \quad \frac{\|x_j\|^2_{C[T_{j-1},T_j]} \int_{T_{j-1}}^{T_j} (t-T)^{-\rho_2} ds}{\Gamma(1-\rho_2) \int_{T_{j-1}}^{T_j} |x_j(s)| ds} + \frac{\|x_j\|_{C[T_{j-1},T_j]} (t-T)^{-\rho_2}}{\Gamma(1-\rho_2)} \\ \le & \quad \left[\frac{T\|x_j\|^2_{C[T_{j-1},T_j]}}{\Gamma(1-\rho_2) \int_{T_{j-1}}^{T_j} |x_j(s)| ds} + \frac{\|x_j\|_{C[T_{j-1},T_j]}}{\Gamma(1-\rho_2)} \right] (t-T)^{-\rho_2} \\ \doteq & \quad L_j (t-T)^{-\rho_2}. \end{aligned}$$

Using the same analysis, we get

$$|\psi_{x_j}(t)| \le [\frac{T\|x_j\|^2_{C[T_{j-1},T_j]}}{\Gamma(1-\rho_1) \int_{T_{j-1}}^{T_j} |x_j(s)| ds} + \frac{\|x_j\|_{C[T_{j-1},T_j]}}{\Gamma(1-\rho_1)}](t-T)^{-\rho_1} \doteq K_j (t-T)^{-\rho_1}.$$

From (A_3), there exists positive M such that

$$\frac{t^r |f(t,0,0)|}{1+t^\mu} \le M, t \ge 0. \tag{39}$$

Thus, by (A_2), it holds

$$\begin{aligned} & |f(t, I_{T+}^{\rho_2} y(t), y(t) + \varphi_{x_1}(t) + \cdots + \varphi_{x_{n^*}}(t))| \\ \le & \quad \frac{t^{-r}}{1+t^\lambda} [c_1 |I_{T+}^{\rho_2} y(t)| + c_2(|y(t)| + |\varphi_{x_1}(t)| + \cdots + |\varphi_{x_{n^*}}(t)|)] + |f(t,0,0)| \\ \le & \quad \frac{c_1 t^{-r}}{\Gamma(\rho_2)} \int_T^t \frac{(t-s)^{\rho_2-1}}{1+s^\lambda} |y(s)| ds + \frac{c_2 t^{-r}(|y(t)| + |\varphi_{x_1}(t)| + \cdots + |\varphi_{x_{n^*}}(t)|}{1+t^\lambda} \\ & + |f(t,0,0)| \\ \le & \quad \frac{c_1 t^{-r} \|y\|_E}{\Gamma(\rho_2)} \int_T^t (t-s)^{\rho_2-1} (s-T)^{-\rho_2} e^{R^2(s-T)^{\rho_1-\rho_2-r}} ds \\ & + c_2 \|y\|_E t^{-r} (t-T)^{-\rho_2} e^{R^2(t-T)^{\rho_1-\rho_2-r}} \end{aligned}$$

$$
\begin{aligned}
& +\frac{c_2(t-T)^{-r}}{1+t^{\lambda}}(|\varphi_{x_1}(t)|+\cdots+|\varphi_{x_{n^*}}(t)|)+|f(t,0,0)| \\
\leq \quad & c_1(t-T)^{-r}\|y\|_E e^{R^2(t-T)^{\rho_1-\rho_2-r}}\Gamma(1-\rho_2) \\
& +c_2\|y\|_E(t-T)^{-\rho_2-r}e^{R^2(t-T)^{\rho_1-\rho_2-r}} \\
& +\frac{L_1c_2(t-T)^{-r-\rho_2}}{1+t^{\lambda}}+\cdots+\frac{L_{n^*}c_2(t-T)^{-r-\rho_2}}{1+t^{\lambda}}+|f(t,0,0)|.
\end{aligned}
$$

Next, we estimate these terms above,

$$
\begin{aligned}
& \frac{(t-T)^{\rho_2}e^{-R^2(t-T)^{\rho_1-\rho_2-r}}I_{T+}^{\rho_1-\rho_2}|(t-T)^{-r}e^{R^2(t-T)^{\rho_1-\rho_2-r}}|}{1+t^{\lambda}} \\
= \quad & \frac{(t-T)^{\rho_2}e^{-R^2(t-T)^{\rho_1-\rho_2-r}}}{1+t^{\lambda}}\int_T^t\frac{(t-s)^{\rho_1-\rho_2-1}(s-T)^{-r}e^{R^2(s-T)^{\rho_1-\rho_2-r}}}{\Gamma(\rho_1-\rho_2)}ds \\
\leq \quad & \frac{\Gamma(1-r)}{(1+(t-T)^{\lambda})\Gamma(1-\rho_2-r+\rho_1)}(t-T)^{\rho_1-r} \\
\leq \quad & \frac{\Gamma(1-r)}{\Gamma(1-\rho_2-r+\rho_1)}<\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{(t-T)^{\rho_2}e^{-R^2(t-T)^{\rho_1-\rho_2-r}}I_{T+}^{\rho_1-\rho_2}|(t-T)^{-\rho_2-r}e^{R^2(t-T)^{\rho_1-\rho_2-r}}|}{1+t^{\lambda}} \\
\leq \quad & \frac{\Gamma(1-\rho_2-r)}{\Gamma(1-2\rho_2-r+\rho_1)}<\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{(t-T)^{\rho_2}e^{-R^2(t-T)^{\rho_1-\rho_2-r}}I_{T+}^{\rho_1-\rho_2}|\frac{(t-T)^{-r-\rho_2}}{1+t^{\lambda}}|}{1+t^{\lambda}} \\
= \quad & \frac{(t-T)^{\rho_2}e^{-R^2(t-T)^{\rho_1-\rho_2-r}}}{(1+t^{\lambda})\Gamma(\rho_1-\rho_2)}\int_T^t(t-s)^{\rho_1-\rho_2-1}\frac{(s-T)^{-r-\rho_2}}{1+s^{\lambda}}ds \\
\leq \quad & \frac{(t-T)^{\rho_2}}{(1+(t-T)^{\lambda})\Gamma(\rho_1-\rho_2)}\int_T^t(t-s)^{\rho_1-\rho_2-1}(s-T)^{-r-\rho_2}ds \\
= \quad & \frac{(t-T)^{\rho_1-\rho_2-r}\Gamma(1-r-\rho_2)}{(1+(t-T)^{\lambda})\Gamma(1-r-2\rho_2+\rho_1)} \\
\leq \quad & \frac{\Gamma(1-r-\rho_2)}{\Gamma(1-r-2\rho_2+\rho_1)}<+\infty,
\end{aligned}
$$

and

$$
\frac{(t-T)^{\rho_2}e^{-R^2(t-T)^{\rho_1-\rho_2-r}}I_{T+}^{\rho_1-\rho_2}|f(t,0,0)|}{1+t^{\lambda}}
$$

$$\begin{aligned}
&\leq \frac{M(t-T)^{\rho_2}}{(1+t^\lambda)\Gamma(\rho_1-\rho_2)}\int_T^t (t-s)^{\rho_1-\rho_2-1}s^{-r}(1+s^\mu)ds \\
&\leq \frac{M(t-T)^{\rho_2}}{(1+t^\lambda)\Gamma(\rho_1-\rho_2)}\int_T^t (t-s)^{\rho_1-\rho_2-1}(s-T)^{-r}(1+s^\mu)ds \\
&\leq \frac{M(1+t^\mu)\Gamma(1-r)(t-T)^{\rho_1-r}}{(1+t^\lambda)\Gamma(1+\rho_1-\rho_2-r)} \\
&\leq \frac{M(1+t^\mu)\Gamma(1-r)t^{\rho_1-r}}{(1+t^\lambda)\Gamma(1+\rho_1-\rho_2-r)} \\
&= \frac{M\Gamma(1-r)t^{\rho_1-r}}{(1+t^\lambda)\Gamma(1+\rho_1-\rho_2-r)} + \frac{M\Gamma(1-r)t^{\rho_1-r+\mu}}{(1+t^\lambda)\Gamma(1+\rho_1-\rho_2-r)} < +\infty.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{(t-T)^{\rho_2}e^{-R^2(t-T)^{\rho_1-\rho_2-r}}I_{T+}^{\rho_1-\rho_2}|\psi_{x_j}(t)|}{1+t^\lambda} \\
\leq\ & \frac{K_j(t-T)^{\rho_2}e^{-R^2(t-T)^{\rho_1-\rho_2-r}}}{(1+(t-T)^\lambda)\Gamma(\rho_1-\rho_2)}\int_T^t (t-s)^{\rho_1-\rho_2-1}(s-T)^{-\rho_1}ds \\
\leq\ & \frac{K_j\Gamma(1-\rho_1)}{(1+(t-T)^\lambda)\Gamma(1-\rho_2)} \leq \frac{K_j\Gamma(1-\rho_1)}{\Gamma(1-\rho_2)} < +\infty.
\end{aligned}$$

All these estimations imply that

$$\sup_{t\geq T}\frac{(t-T)^{\rho_2}e^{-R^2(t-T)^{\rho_1-\rho_2-r}}|Fx(t)|}{1+t^\lambda} < +\infty.$$

Hence, $F : E \to E$ is well defined.

Now, for $y_1, y_2 \in E$, by a similar way, we get

$$\begin{aligned}
&|Fy_1(t) - Fy_2(t)| \\
\leq & \frac{c_1\Gamma(1-\rho_2)\|y_1-y_2\|_E}{\Gamma(\rho_1-\rho_2)}\int_T^t (t-s)^{\rho_1-\rho_2-1}s^{-r}e^{R^2(s-T)^{\rho_1-\rho_2-r}}ds \\
&+ \frac{c_2\|y_1-y_2\|_E}{\Gamma(\rho_1-\rho_2)}\int_T^t (t-s)^{\rho_1-\rho_2-1}s^{-r}(s-T)^{-\rho_2}e^{R^2(s-T)^{\rho_1-\rho_2-r}}ds \\
\leq & \frac{c_1\Gamma(1-\rho_2)\|y_1-y_2\|_E}{\Gamma(\rho_1-\rho_2)}\int_T^t (t-s)^{\rho_1-\rho_2-1}(s-T)^{-r}e^{R^2(s-T)^{\rho_1-\rho_2-r}}ds \\
&+ \frac{c_2\|y_1-y_2\|_E}{\Gamma(\rho_1-\rho_2)}\int_T^t (t-s)^{\rho_1-\rho_2-1}(s-T)^{-r}(s-T)^{-\rho_2}e^{R^2(s-T)^{\rho_1-\rho_2-r}}ds \\
\leq & (c_1M_q(t-T)^{\rho_2}+c_2)M_{p,q}\|y_1-y_2\|_E \\
&\quad\cdot\int_T^t (t-s)^{\rho_1-\rho_2-1}(s-T)^{-\rho_2-r}e^{R^2(s-T)^{\rho_1-\rho_2-r}}ds.
\end{aligned}$$

By the similar arguments, we get the estimation

$$\int_T^t (t-s)^{\rho_1-\rho_2-1}(s-T)^{-\rho_2-r}e^{R^2(s-T)^{\rho_1-\rho_2-r}}ds$$

$$\begin{aligned}
&\leq \sum_{i=1}^{R-1}\int_{\frac{(i-1)(t-T)}{R}+T}^{\frac{i(t-T)}{R}+T} R^{1-\rho_1+\rho_2}(t-T)^{\rho_1-\rho_2-1}(s-T)^{-\rho_2-r}e^{R^2(s-T)^{\rho_1-\rho_2-r}}ds \\
&\quad +\frac{R^{\rho_2+r}e^{R^2(t-T)^{\rho_1-\rho_2-r}}}{(t-T)^{\rho_2+r}}\int_{\frac{(R-1)(t-T)}{R}+T}^{t}(t-s)^{\rho_1-\rho_2-1}ds \\
&= R^{1-\rho_1+\rho_2}(t-T)^{-\rho_2}\int_0^{\frac{(R-1)(t-T)}{R}+T}(s-T)^{\rho_1-\rho_2-r-1}e^{R^2(s-T)^{\rho_1-\rho_2-r}}ds \\
&\quad +\frac{R^{r-\rho_1+2\rho_2}(t-T)^{-\rho_2}(t-T)^{\rho_1-\rho_2-r}}{\rho_1-\rho_2}e^{R^2(t-T)^{\rho_1-\rho_2-r}} \\
&\leq \frac{R^{1-\rho_1+\rho_2}(t-T)^{-\rho_2}}{R^2(\rho_1-\rho_2-r)}e^{R^2(t-T)^{\rho_1-\rho_2-r}} \\
&\quad +\frac{R^{r-\rho_1+2\rho_2}(t-T)^{-\rho_2}(t-T)^{\rho_1-\rho_2-r}}{\rho_1-\rho_2-r}e^{R^2(t-T)^{\rho_1-\rho_2-r}} \\
&\leq ((t-T)^{\rho_1-\rho_2-r}+1)(t-T)^{-\rho_2}M_{p,q,r}R^{r-m_{p,q}}e^{R^2(t-T)^{\rho_1-\rho_2-r}}.
\end{aligned}$$

As a result, we have

$$\begin{aligned}
&\frac{(t-T)^{\rho_2}e^{-R^2(t-T)^{\rho_1-\rho_2-r}}}{1+t^\lambda}|Fy_1(t)-Fy_2(t)| \\
&\leq \frac{(t-T)^{\rho_2}e^{-R^2(t-T)^{\rho_1-\rho_2-r}}}{1+(t-T)^\lambda}|Fy_1(t)-Fy_2(t)| \\
&\leq \frac{M_{p,q}M_{p,q,r}(c_1M_q(t-T)^{\rho_2}+c_2)((t-T)^{\rho_1-\rho_2-r}+1)}{1+(t-T)^\lambda}R^{r-m_{p,q}}\|y_1-y_2\|_E \\
&= \frac{c_1M_{p,q}M_{p,q,r}M_q((t-T)^{\rho_1-r}+(t-T)^{\rho_2})}{1+(t-T)^\lambda}R^{r-m_{p,q}}\|y_1-y_2\|_E \\
&\quad +\frac{c_2M_{p,q}M_{p,q,r}((t-T)^{\rho_1-\rho_2-r}+1)}{1+(t-T)^\lambda}R^{r-m_{p,q}}\|y_1-y_2\|_E \\
&\leq 2M_{p,q}M_{p,q,r}(c_1M_q+c_2)R^{r-m_{p,q}}\|y_1-y_2\|_E \\
&\leq 2(T+1)^2M_{p,q}M_{p,q,r}(c_1M_q+c_2)R^{r-m_{p,q}}\|y_1-y_2\|_E \leq \frac{1}{2}\|y_1-y_2\|_E.
\end{aligned}$$

Hence, F has one unique fixed point $y_{n^*+1}\in E$. Thus, by the previous arguments, we obtain $x_{n^*+1}=I_{T_+}^{\rho_2}y_{n^*+1}$ is one unique solution of the initial value problem (30).

Thus, according to the Definition 2.8, we obtain that the initial value problem (1) has one unique approximate solution $x \in C[0, +\infty)$ as following

$$x(t) = \begin{cases} x_1(t), 0 \le t \le T_1, \\ x_2(t), T_1 \le t \le T_2, \\ x_3(t), T_2 \le t \le T_3, \\ \vdots \\ x_{n^*}(t), T_{n^*-1} \le t \le T, \\ x_{n^*+1}(t), T \le t < +\infty. \end{cases}$$

Thus we complete this proof. □

Example 3.2. Now, we consider the following initial value problem for linear equation

$$D_{0+}^{\frac{1}{2}+\frac{t}{700000(1+t^2)}} x(t) = t^{\frac{1}{4}}, x(0) = 0, 0 < t < +\infty. \tag{40}$$

By the definition of the Riemann-Liouville variable order fractional derivative, we don't have a way to obtain its exact solution, we don't even have method to study the existence result of solution. Next, according to the Definition 2.8, we seek its continuous approximate solution.

For given arbitrary small $\varepsilon = \frac{1}{1000}$, there exists $T = \frac{1.8}{\varepsilon} = 1800$, so that

$$\left| p(t) - \frac{1}{2} \right| = \frac{t}{700000(1+t^2)} < \frac{1}{t} \le \frac{1}{T} = \frac{\varepsilon}{1.8} < \varepsilon, t \ge T,$$

Now, we consider the function $p(t)$ restricted on the interval $[0, T] = [0, 1800]$. By the right continuity of function $p(t)$ at point 0, for $\varepsilon = \frac{1}{1000}$, taking $\delta_0 = 600$, when $0 \le t \le \delta_0 = 600$, we have

$$|p(t) - p(0)| = \left| \frac{t}{700000(1+t^2)} \right| \le \frac{t}{700000} \le \frac{\delta_0}{700000} < \frac{1}{1000} = \varepsilon.$$

We get $t_1 = \delta_0 = 600$. By the right continuity of function $p(t)$ at point t_1, for $\varepsilon = \frac{1}{1000}$, taking $\delta_1 = 600$, when $0 \le t - t_1 \le \delta_1$, by the differential mean value theorem, we have

$$\begin{aligned} |p(t) - p(t_1)| = & \left| \frac{t}{700000(1+t^2)} - \frac{t_1}{700000(1+t_1^2)} \right| \\ \le & \left| \frac{1-\xi^2}{700000(1+\xi^2)^2} \right| |t - t_1| \\ \le & \frac{1+\xi^2}{700000(1+\xi^2)^2} |t - t_1| \\ \le & \frac{1}{700000} |t - t_1| \\ \le & \frac{\delta_1}{700000} < \frac{1}{1000} = \varepsilon, \end{aligned}$$

where $t_1 < \xi < t$. We let $t_2 = t_1 + \delta_1 = 1200$. By the right continuity of function $p(t)$ at point t_2, for $\varepsilon = \frac{1}{1000}$, taking $\delta_2 = 600$, when $0 \le t - t_2 \le \delta_2$, by the same reasons above, we have

$$|p(t) - p(t_2)| = \left|\frac{t}{700000(1+t^2)} - \frac{t_2}{700000(1+t_2^2)}\right| \le \frac{\delta_2}{700000} < \frac{1}{1000} = \varepsilon,$$

we see that $t_3 = t_2 + \delta_2 = 1800 = T$, hence, we get intervals $[0, 600]$, $(600, 1200]$, $(1200, 1800]$, $(1800, +\infty)$ and piecewise constant function $\alpha(t)$ defined by

$$\alpha(t) = \begin{cases} p_1 = p(0) = \frac{1}{2}, & t \in [0, 600], \\ p_2 = p(600) = \frac{1}{2} + \frac{3}{3500 \times 360001}, & t \in (600, 1200], \\ p_3 = p(1200) = \frac{1}{2} + \frac{3}{1750 \times 1440001}, & t \in (1200, 1800], \\ \rho = \frac{1}{2}, & t \in (1800, +\infty). \end{cases}$$

Thus, according analysis above, first, we consider the initial value problem

$$\begin{cases} D_{0+}^{p_1} x(t) = t^{\frac{1}{4}}, 0 < t \le 600, \\ x(0) = 0, \end{cases} \tag{41}$$

by the fact of the Riemann-Liouville fractional calculus, we get solution of the initial value problem (41) is $x_1(t) = \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{4})} t^{\frac{3}{4}}$, $0 \le t \le 600$, obvious, $x_1 \in C[0, 600]$.

Second, we seek the solution of the initial value problem

$$\begin{cases} D_{600+}^{p_2} x(t) = t^{\frac{1}{4}} - \psi_{x_1}(t), 600 < t \le 1200, \\ x(600) = \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{4})} 600^{\frac{3}{4}}, \end{cases} \tag{42}$$

where $x_1(t) = \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{4})} t^{\frac{3}{4}}$ is the unique solution of the initial value problem (41),

$$\begin{aligned} \psi_{x_1}(t) &= \frac{x_1(600) \int_0^{600} (t-s)^{-p_2} |x_1(s)| ds}{\Gamma(1-p_2) \int_0^{600} |x_1(s)| ds} - \frac{x_1(600)(t-600)^{-p_2}}{\Gamma(1-p_2)} \\ &\doteq h_{x_1}(t) - \frac{x_1(600)(t-600)^{-p_2}}{\Gamma(1-p_2)}. \end{aligned}$$

Obvious, $h_{x_1} \in C[600, 1200]$, thus, the solution of the initial value problem (42) is

$$x_2(t) = \int_{600}^{t} \frac{(t-s)^{p_2-1}}{\Gamma(p_2)} (s^{\frac{1}{4}} - h_{x_1}(s)) ds + x_1(600).$$

Obvious, $x_2 \in C[600, 1200]$.

Third, we seek the solution of the initial value problem

$$\begin{cases} D_{1200+}^{p_3} x(t) = t^{\frac{1}{4}} - \psi_{x_1}(t) - \psi_{x_2}(t), 1200 < t \le 1800, \\ x(1200) = x_2(1200), \end{cases} \tag{43}$$

where $x_1 \in C[0,600]$ is the unique solution of the initial value problem (41), $x_2 \in C[600,1200]$ is the unique solution of the initial value problem (42),

$$\begin{cases} \psi_{x_1}(t) = \frac{x_1(600)\int_0^{600}(t-s)^{-p_3}|x_1(s)|ds}{\Gamma(1-p_3)\int_0^{600}|x_1(s)|ds} - \frac{x_1(600)(t-600)^{-p_3}}{\Gamma(1-p_3)}, \\ \psi_{x_2}(t) = \frac{x_2(1200)\int_{600}^{1200}(t-s)^{-p_3}|x_2(s)|ds}{\Gamma(1-p_3)\int_{600}^{1200}|x_2(s)|ds} - \frac{x_2(1200)(t-1200)^{-p_3}}{\Gamma(1-p_3)} \\ \qquad \doteq h_{x_2}(t) - \frac{x_2(1200)(t-1200)^{-p_3}}{\Gamma(1-p_3)}, \end{cases}$$

thus, the solution of the initial value problem (43) is

$$x_3(t) = \int_{1200}^{t} \frac{(t-s)^{p_3-1}}{\Gamma(p_3)}(s^{\frac{1}{4}} - \psi_{x_1}(s) - h_{x_2}(s))ds + x_2(1200).$$

Obvious, $x_3 \in C[1200,1800]$.

Finally, we seek the solution of the initial value problem

$$\begin{cases} D^{\rho}_{1800+}x(t) = t^{\frac{1}{4}} - \psi_{x_1}(t) - \psi_{x_2}(t) - \psi_{x_3}(t), 1800 < t < +\infty, \\ x(1800) = x_3(1800), \end{cases} \tag{44}$$

where $x_1 \in C[0,600]$ $x_2 \in C[600,1200]$ and $x_3 \in C[1200,1800]$ are unique solutions of the initial value problems (41)-(43),

$$\begin{cases} \psi_{x_1}(t) = \frac{x_1(600)\int_0^{600}(t-s)^{-\rho}|x_1(s)|ds}{\Gamma(1-\rho)\int_0^{600}|x_1(s)|ds} - \frac{x_1(600)(t-600)^{-\rho}}{\Gamma(1-\rho)}, \\ \psi_{x_2}(t) = \frac{x_2(1200)\int_{600}^{1200}(t-s)^{-\rho}|x_2(s)|ds}{\Gamma(1-\rho)\int_{600}^{1200}|x_2(s)|ds} - \frac{x_2(1200)(t-1200)^{-\rho}}{\Gamma(1-\rho)}, \\ \psi_{x_3}(t) = \frac{x_3(1800)\int_{1200}^{1800}(t-s)^{-\rho}|x_3(s)|ds}{\Gamma(1-\rho)\int_{1200}^{1800}|x_3(s)|ds} - \frac{x_3(1800)(t-1800)^{-\rho}}{\Gamma(1-\rho)} \\ \qquad \doteq h_{x_3}(t) - \frac{x_3(1800)(t-1800)^{-\rho}}{\Gamma(1-\rho)}, \end{cases}$$

thus, the solution of the initial value problem (44) is

$$x_4(t) = \int_{1800}^{t} \frac{(t-s)^{\rho-1}}{\Gamma(\rho)}(s^{\frac{1}{4}} - \psi_{x_1}(s) - \psi_{x_1}(s) - h_{x_3}(s))ds + x_3(1800),$$

obvious, $x_4 \in C[1800,+\infty)$.

Hence, by the Definition 2.8, we obtain that the initial value problem (40) exists the unique continuous approximate solution $x : [0, +\infty) \to \mathbb{R}$ defined by

$$x(t) = \begin{cases} x_1(t) = \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{4})} t^{\frac{3}{4}}, 0 \le t \le 600, \\ x_2(t) = \int_{600}^{t} \frac{(t-s)^{p_2-1}}{\Gamma(p_2)} (s^{\frac{1}{4}} - h_{x_1}(s))ds + x_1(600)], 600 \le t \le 1200, \\ x_3(t) = \int_{1200}^{t} \frac{(t-s)^{p_3-1}}{\Gamma(p_3)} (s^{\frac{1}{4}} - \psi_{x_1}(s) - h_{x_2}(s))ds \\ \qquad +x_2(1200), 1200 \le t \le 1800, \\ x_4(t) = \int_{1800}^{t} \frac{(t-s)^{\rho-1}}{\Gamma(\rho)} (s^{\frac{1}{4}} - \psi_{x_1}(s) - \psi_{x_1}(s) - h_{x_3}(s))ds \\ \qquad +x_3(1800), 1800 \le t < +\infty. \end{cases}$$

Example 3.3. Now, we consider the initial value problem as following

$$\begin{cases} D_{0+}^{\frac{1}{2}+\frac{t}{700000(1+t^2)}} x(t) = \frac{\Gamma(\frac{3}{2})x^4}{12(1+t^2)^4(1+x^4)} + \frac{\Gamma(\frac{7}{6})(D_{0+}^{\frac{1}{5}+\frac{t}{1400000(1+t^2+t^3)}} x)^2}{12(1+t^2)^2(1+(D_{0+}^{\frac{1}{5}+\frac{t}{1400000(1+t^2+t^3)}} x)^2)}, \\ x(0) = 0, \end{cases} \tag{45}$$

where $0 < t < +\infty$. We let

$$p(t) = \frac{1}{2} + \frac{t}{700000(1+t^2)}, q(t) = \frac{1}{5} + \frac{t}{1400000(1+t^2+t^3)},$$

$$f(t, x(t), y(t)) = \frac{\Gamma(\frac{3}{2})x^4(t)}{12(1+t^2)^4(1+x^4(t))} + \frac{\Gamma(\frac{7}{6})y^2(t)}{12(1+t^2)^2(1+y^2(t))},$$

where $0 < t < +\infty, x(t), y(t) \in \mathbb{R}$. Obviously, we get $p(t) > 2q(t), 0 \le t < +\infty$, $\lim_{t\to+\infty} p(t) = \frac{1}{2} > 2\lim_{t\to+\infty} q(t) = \frac{2}{5}$, thus, p satisfies (A_1) with $\eta_1 = \frac{1}{2}$, $\eta_2 = \frac{1}{5}$. And that, for all $0 \le t < +\infty, x(t), y(t) \in \mathbb{R}$, from the differentiation mean theorem, we get

$$\begin{aligned} &|f(t, (1+t^2)x_1, (1+t^2)y_1) - f(t, (1+t^2)x_2, (1+t^2)y_2)| \\ &\le \frac{\Gamma(\frac{3}{2})}{12} \left| \frac{x_1^4}{1+(1+t^2)^4x_1^4} - \frac{x_2^4}{1+(1+t^2)^4x_2^4} \right| \\ &+ \frac{\Gamma(\frac{7}{6})}{12} \left| \frac{y_1^2(t)}{1+(1+t^2)^2y_1^2} - \frac{y_2^2}{1+(1+t^2)^2y_2^2}) \right| \\ &\le \frac{\Gamma(\frac{3}{2})}{3}|x_1 - x_2| + \frac{\Gamma(\frac{7}{6})}{3}|y_1 - y_1|, \end{aligned}$$

which implies that f satisfies (A_2) with $r = 0, \lambda = 2, c_1 = \frac{\Gamma(\frac{3}{2})}{3}$, $c_2 = \frac{\Gamma(\frac{7}{6})}{3}$. In addition, $f(t, 0, 0) = 0$ satisfies $\lim_{t\to+\infty} \frac{t^r|f(t,0,0)|}{1+t^\mu} = 0$, which implies that f satisfies (A_3) with $r = 0, \mu = 0$. By the same arguments done in section 2, we get intervals $[0, 600], (600, 1200]$, $(1200, 1800]$, $(1800, +\infty)$ and piecewise constant functions $\alpha(t), \beta(t)$ defined by

$$\alpha(t) = \begin{cases} p_1 = p(0), & t \in [0, 600], \\ p_2 = p(600), & t \in (600, 1200], \\ p_3 = p(1200), & t \in (1200, 1800], \\ \rho_1 = \frac{1}{2}, & t \in (1800, +\infty), \end{cases}$$

$$\beta(t) = \begin{cases} q_1 = q(0), & t \in [0, 600], \\ q_2 = q(600), & t \in (600, 1200], \\ q_3 = q(1200), & t \in (1200, 1800], \\ \rho_2 = \frac{1}{5}, & t \in (1800, +\infty). \end{cases}$$

By Theorem 3.1, the initial value problem (45) has one continuous unique approximate solution $x(t), 0 \leq t < +\infty$.

4. CONCLUSION

Based on some known results, the Riemann-Liouville variable order fractional integral doesn't have semigroup property. Hence the transform between the variable order fractional integral and derivative is not clear, which brings us extreme difficulties in considering the solutions of variable order differential equations. It is interesting and meaningful for we to overcome the difficulties and obtain the solutions of variable order differential equation. To the best of the authors' knowledge, this is the first paper dealing with variable order fractional differential equations on half-axis. This paper enriches and extends the existing literatures. Finally, we give an example to illustrate our results.

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