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APPLICATIONS OF ANALYTIC FUNCTIONS RELATED TO MITTAG-LEFFLER TYPE BOREL DISTRIBUTION SERIES

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Abstract. For analytic function f in the open unit disc E, a linear operator defined by Mittag-Leffler -type Borel distribution series is introduced. The object of the present paper is to study some properties for $B_\lambda(\alpha,\beta)f(z)$ belonging to some classes by applying the concept of Jack's lemma. Subordination relations are introduced.

1. Introduction

Let $\mathscr{A}$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

in the open the unit disk $E = \{z : |z| < 1\}$. Let S be the subclass of $\mathscr{A}$ consisting of univalent functions and satisfy the following usual normalization condition $f(0) = 0$ and $f'(0) = 1$. We denote by S the subclass of $\mathscr{A}$ consisting of $f(z)$ which are all univalent in E. A function $f \in \mathscr{A}$ is a starlike function of the order υ, $\upsilon(0 \le \upsilon < 1)$ if it satisfy

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \upsilon, \ (z \in E), \tag{2}$$

we denote by this class $S^*(\upsilon)$.

A function $f \in \mathscr{A}$ is a convex function of the order υ, $\upsilon(0 \le \upsilon < 1)$ if it satisfy

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \upsilon, \ (z \in E), \tag{3}$$

we denote this class with $K(\upsilon)$.

For $f \in \mathscr{A}$ given by (1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{4}$$

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their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z), \ (z \in E). \tag{5}$$

Note that $f * g \in \mathscr{A}$.

Mittag-Leffler function and Borel distribution

The study of operators is fundamental in geometric function theory, complex analysis, and related areas. Several derivative and integral operators can be expressed by convolution of certain analytic functions. It should be noted that this formalism helps future mathematical research as well as a better grasp of the geometric properties of such operators. Let $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ be functions defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (z \in \mathbb{C}, \Re(\alpha) > 0)$$

and

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (\alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0).$$

It can be written in other form

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + \sum_{n=2}^{\infty} \frac{z^{n-1}}{\Gamma(\alpha(n-1) + \beta)'} \quad (\alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0).$$

The function $E_\alpha(z)$ was introduced by Mittag-Leffler [11] and is, therefore, known as the Mittag-Leffler function. A more general function $E_{\alpha,\beta}$ generalizing $E_\alpha(z)$ was introduced by Wiman [15] and defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (z, \alpha, \beta \in \mathrm{C}, \Re(\alpha) > 0, \Re(\beta) > 0).$$

Observe that the function $E_{\alpha,\beta}$ contains many well-known functions as its special case, for example,

$$\begin{gathered} E_{1,1}(z) = \mathrm{e}^z, \ E_{1,2}(z) = \frac{\mathrm{e}^z - 1}{z}, \\ E_{2,1}\left(z^2\right) = \cosh z, \ E_{2,1}\left(-z^2\right) = \cos z, \ E_{2,2}\left(z^2\right) = \frac{\sinh z}{z}, \\ E_{2,2}\left(-z^2\right) = \frac{\sin z}{z}c, \ E_3(z) = \frac{1}{2}\left[e^{z^{1/3}} + 2e^{-\frac{1}{2}z^{1/3}} \cos\left(\frac{\sqrt{3}}{2} z^{1/3}\right)\right] \\ \text{and } E_4(z) = \frac{1}{2}\left[\cos z^{1/4} + \cosh z^{1/4}\right]. \end{gathered}$$

The Mittag-Leffler function appears naturally in the solution of fractional order differential and integral equations. In the study of complex systems and super diffusive transport, in particular, fractional generalisation of the kinetic equation, random walks, and Levy flights. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found, e.g., in [5, 6, 7, 8, 9, 10, 13]. Observe that MittagLeffler function $E_{\alpha,\beta}(z)$ does not belong to the family $\mathscr{A}$. Thus, it is natural to consider the following normalization of Mittag-Leffler functions as below:

$$\mathbb{E}_{\alpha,\beta}(z) = z\Gamma(\beta)E_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} z^n, \tag{6}$$

it holds for complex parameters α, β and $z \in \mathbb{C}$. In this paper, we shall restrict our attention to the case of real-valued α, β and $z \in E$.

A discrete random variable x is said to have a Borel distribution if it takes the values $1, 2, 3, \cdots$ with the probabilities $\frac{e^{-\lambda}}{1!}, \frac{2\lambda e^{-2\lambda}}{2!}, \frac{9\lambda^2 e^{-3}}{3!}, \cdots$, respectively, where λ is called the parameter.

Very recently, Wanas and Khuttar [14] introduced the Borel distribution (BD) whose probability mass function is

$$P(x = \rho) = \frac{(\rho\lambda)^{\rho-1} e^{-\lambda\rho}}{\rho!}, \quad \rho = 1, 2, 3, \cdots.$$

Wanas and Khuttar introduced a series $\mathscr{M}_\lambda(z)$ whose coefficients are probabilities of the Borel distribution (BD)

$$\mathscr{M}_\lambda(z) = z + \sum_{n=2}^{\infty} \frac{[\lambda(n-1)]^{n-2} e^{-\lambda(n-1)}}{(n-1)!} z^n, (0 < \lambda \leq 1). \tag{7}$$

In [12], Murugusundaramoorthy and El-Deeb defined the Mittag-Leffler-type Borel distribution as follows:

$$\mathscr{P}_\lambda(\alpha, \beta; \rho) = \frac{(\lambda\rho)^{\rho-1}}{E_{\alpha,\beta}(\lambda\rho)\Gamma(\alpha\rho+\beta)}, \quad \rho = 0, 1, 2, \cdots$$

where

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad (\alpha,\ \beta \in \mathbb{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0).$$

Thus by using (6) and (7) and by convolution operator, the Mittag-Leffler-type Borel distribution series defined as below

$$B_\lambda(\alpha, \beta)(z) = z + \sum_{n=2}^{\infty} \frac{[\lambda(n-1)]![\lambda(n-1)]^{n-2} e^{-\lambda(n-1)}}{(n-1)! E_{\alpha,\beta}(\lambda(n-1))\Gamma(\alpha(n-1)+\beta)} z^n, (0 < \lambda \leq 1).$$

Further, by the convolution operator, we define

$$\begin{aligned} B_\lambda(\alpha, \beta)f(z) &= B_\lambda(\alpha, \beta)(z) * f(z) \\ &= z + \sum_{n-2}^{\infty} \frac{[\lambda(n-1)]![\lambda(n-1)]^{n-2} e^{-\lambda(n-1)}}{(n-1)! E_{\alpha,\beta}(\lambda(n-1))\Gamma(\alpha(n-1)+\beta)} a_n z^n \\ &= z + \sum_{n=2}^{\infty} \phi_n a_n z^n, \quad (\alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, 0 < \lambda \leq 1) \quad (8) \end{aligned}$$

$$\text{where } \phi_n = \frac{[\lambda(n-1)]![\lambda(n-1)]^{n-2} \mathrm{e}^{-\lambda(n-1)}}{(n-1)! E_{\alpha,\beta}(\lambda(n-1))\Gamma(\alpha(n-1)+\beta)}. \tag{9}$$

Now, by making use of the Mittag-Leffler-type Borel distribution series $B_\lambda(\alpha, \beta)f$, we define a new subclass of functions belonging to the class $\mathscr{A}$.

Definition 1.1. Let a function $f \in A$. Then $f \in B_\lambda(\alpha,\beta)f(z)$ if and only if

$$\Re\left\{\frac{z\left(B_\lambda(\alpha,\beta)f(z)\right)'}{B_\lambda(\alpha,\beta)f(z)}\right\} > \varrho,\ \ z \in E,\ 0 \le \varrho \le 1. \tag{10}$$

Let f and g be analytic in E. Then f is said to be subordinate to g if there exists an analytic function ω satisfying $\omega(0) = 0$ and $\omega(z) < 1$, such that $f(z) = g(\omega z), z \in E$. We denote this subordination as $f(z) \prec g(z)$ or $(f \prec g), z \in E$.

The basic idea in proving our result is the following lemma due to Jack [1] (also, due to Miller and Mocannu [2]).

Lemma 1.2. *Let $\omega(z)$ be analytic in E with $\omega(0) = 0$. Then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0 in E then we have $z_0\ \omega\ '(z) = k\ \omega(z_0)$, where $k \ge 1$ is a real number.*

2. Main Results

In the present paper, we follow similar works done by Shireishi and Owa [4] and Ochiai et al. [3], we derive the following result.

Theorem 2.1. *If $f \in A$ satisfies*

$$\Re\left\{\frac{z\left(B_\lambda(\alpha,\beta)f(z)\right)'}{B_\lambda(\alpha,\beta)f(z)}\right\} < \frac{\varrho-3}{2(\varrho-1)},\ z \in E$$

for some $\varrho(-1 < \varrho \le 0)$ then

$$\frac{B_\lambda(\alpha,\beta)f(z)}{z} \prec \frac{1+\varrho z}{1-z},\ z \in E.$$

This implies that

$$\Re\left\{\frac{B_\lambda(\alpha,\beta)f(z)}{z}\right\} > \frac{1-\varrho}{2}.$$

Proof. Let us define the function $\omega(z)$ by

$$\frac{B_\lambda(\alpha,\beta)f(z)}{z} = \frac{1-\varrho\omega(z)}{1-\omega(z)},\ (\omega(z) \ne 1).$$

Clearly, $\omega(z)$ is analytic in E and $\omega(0) = 0$. We want to prove that $|\omega(z)| < 1$ in E. Since

$$\frac{z\left(B_\lambda(\alpha,\beta)f(z)\right)'}{B_\lambda(\alpha,\beta)f(z)} = \frac{-\varrho z\omega'(z)}{1-\varrho\omega(z)} + \frac{z\omega'(z)}{1-\omega(z)} + 1,$$

we see that

$$\Re\left\{\frac{z\left(B_\lambda(\alpha,\beta)f(z)\right)'}{B_\lambda(\alpha,\beta)f(z)}\right\} = \Re\left\{\frac{-\varrho z\omega'(z)}{1-\varrho\omega(z)} + \frac{z\omega'(z)}{1-\omega(z)} + 1\right\}$$
$$< \frac{\varrho-3}{2(\varrho-1)},\ (z \in E)$$

for $-1 < \varrho \le 0$. If there exists a point $z_0 \in E$ such that

$$\max_{|z|\le|z_0|} |\omega(z)| = |\omega(z_0)| = 1,$$

then Lemma 1.2, gives us that $\omega(z_0) = e^{i\theta}$ and $z_0\omega'(z_0) = k\omega(z_0), k \ge 1$.

Thus we have

$$\begin{aligned}\frac{z_0\left(B_\lambda(\alpha,\beta)f(z_0)\right)'}{B_\lambda(\alpha,\beta)f(z_0)} &= \frac{-\varrho z_0\omega'(z_0)}{1-\varrho\omega(z_0)} + \frac{z_0\omega'(z_0)}{1-\omega(z_0)} + 1\\ &= 1 + \frac{k}{1-e^{i\theta}} - \frac{k}{1-\varrho e^{i\theta}}.\end{aligned}$$

It follows that

$$\Re\left\{\frac{1}{1-\omega(z_0)}\right\} = \Re\left\{\frac{1}{1-e^{i\theta}}\right\} = \frac{1}{2}$$
$$\text{and } \Re\left\{\frac{1}{1-\varrho\omega(z_0)}\right\} = \Re\left\{\frac{1}{1-\varrho e^{i\theta}}\right\} = \frac{1}{2} - \frac{1-\varrho^2}{2(1+\varrho^2-2\varrho cos\theta)}.$$

Therefore, we have

$$\Re\left\{\frac{z_0\left(B_\lambda(\alpha,\beta)f(z_0)\right)'}{B_\lambda(\alpha,\beta)f(z_0)}\right\} = 1 - \frac{k(\varrho^2-1)}{2(1+\varrho^2-2\varrho cos\theta)}.$$

This implies that $-1<\varrho\le 0$,

$$\Re\left\{\frac{z_0\left(B_\lambda(\alpha,\beta)f(z_0)\right)'}{B_\lambda(\alpha,\beta)f(z_0)}\right\} \ge 1 + \frac{(1-\varrho^2)}{2(\varrho-1)^2} = \frac{\varrho-3}{2(\varrho-1)}.$$

This contradicts the condition in the theorem. Then there is no $z_0\in E$ such that $|\omega(z_0)|=1$ for all $z\in E$, that is

$$\Re\left(\frac{B_\lambda(\alpha,\beta)f(z)}{z}\right) < \frac{1+\varrho z}{1-z},\ z\in E.$$

Further more, since

$$\omega(z) = \frac{\frac{B_\lambda(\alpha,\beta)f(z)}{z}-1}{\frac{B_\lambda(\alpha,\beta)f(z)}{z}-\varrho}, z\in E$$

and $|\omega(z)|<1,\ (z\in E)$, we conclude that

$$\Re\left\{\frac{B_\lambda(\alpha,\beta)f(z)}{z}\right\} > \frac{1-\varrho}{2}.$$

□

Taking $\varrho=0$ in the Theorem 2.1, we have the following corollary.

Corollary 2.2. *If $f\in A$ satisfies*

$$\Re\left\{\frac{z(B_\lambda(\alpha,\beta)f(z))'}{B_\lambda(\alpha,\beta)f(z)}\right\} > \frac{3}{2},\ z\in E$$

then

$$\frac{B_\lambda(\alpha,\beta)f(z}{z} \prec \frac{1}{1-z},\ z\in E$$

and

$$\Re\left\{\frac{B_\lambda(\alpha,\beta)f(z)}{z}\right\} > \frac{1}{2},\ z\in E.$$

Theorem 2.3. *If $f \in A$ satisfies*

$$\Re\left\{\frac{z(B_\lambda(\alpha,\beta)f(z))'}{B_\lambda(\alpha,\beta)f(z)}\right\} > \frac{3\varrho-1}{2(\varrho-1)},\ z \in E$$

for some $\varrho(-1 < \varrho \leq 0)$ then

$$\frac{z}{B_\lambda(\alpha,\beta)f(z)} \prec \frac{1+z}{1-z},\ z \in E$$

and

$$\left|\frac{B_\lambda(\alpha,\beta)f(z)}{z} - \frac{1}{1-\varrho}\right| < \frac{1}{1-\varrho},\ z \in E.$$

This implies that $\Re\left\{\frac{B_\lambda(\alpha,\beta)f(z)}{z}\right\} > 0,\ z \in E$.

Proof. Let us define the function $\omega(z)$ by

$$\frac{z}{B_\lambda(\alpha,\beta)f(z)} = \frac{1-\varrho\omega(z)}{1-\omega(z)},\ \omega(z) \neq 1. \tag{11}$$

Then, we have $\omega(z)$ is analytic in E and $\omega(0) = 0$. We want to prove that $|\omega(z)| < 1$ in E. Differenting equation (11), we obtain

$$\begin{aligned}\frac{z(B_\lambda(\alpha,\beta)f(z))'}{B_\lambda(\alpha,\beta)f(z)} &= \frac{-z\omega'(z)}{1-\omega(z)} + \frac{\varrho z\omega'(z)}{1-\varrho\omega(z)} + 1\\ \Rightarrow \Re\left\{\frac{z(B_\lambda(\alpha,\beta)f(z))'}{B_\lambda(\alpha,\beta)f(z)}\right\} &= \Re\left\{\frac{-z\omega'(z)}{1-\omega(z)} + \frac{\alpha z\omega'(z)}{1-\varrho\omega(z)} + 1\right\}\\ &> \frac{3\varrho-1}{2(\varrho-1)},\ z \in E,\end{aligned}$$

for $(-1 < \varrho \leq 0)$. If there exists a point $(z_0 \in E)$ such that Lemma 1.2, gives us that $\omega(z_0) = e^{i\theta}$ and $z_0\omega'(z_0) = k\omega(z_0), k \geq 1$. Thus we have

$$\begin{aligned}\frac{z_0(B_\lambda(\alpha,\beta)f(z_0))'}{B_\lambda(\alpha,\beta)f(z_0)} &= \frac{-z_0\omega'(z_0)}{1-\omega(z_0)} + \frac{\varrho z_0\omega'(z_0)}{1-\varrho\omega(z_0)} + 1\\ &= 1 - \frac{k}{1-e^{i\theta}} + \frac{k}{1-\varrho e^{i\theta}}.\end{aligned}$$

Therefore, we have

$$\Re\left\{\frac{z_0(B_\lambda(\alpha,\beta)f(z_0))'}{B_\lambda(\alpha,\beta)f(z_0)}\right\} = 1 + \frac{k(\varrho^2-1)}{2(1+\varrho^2-2\varrho cos\theta)}.$$

This implies that, for $-1 < \alpha \leq 0$,

$$\begin{aligned}\Re\left\{\frac{z_0(B_\lambda(\alpha,\beta)f(z_0))'}{B_\lambda(\alpha,\beta)f(z_0)}\right\} &= 1 - \frac{k(1-\alpha^2)}{2(1+\alpha^2-2\alpha cos\theta)}\\ &\leq \frac{3\alpha-1}{2(\alpha-1)}.\end{aligned}$$

This contradicts the condition in the theorem.

Hence, there is no $z_0 \in E$ such that $|\omega(z_0)| = 1$ for all $z \in E$, that is

$$\frac{z}{B_\lambda(\alpha,\beta)f(z)} \prec \frac{1+z}{1-z},\ z \in E.$$

Furthermore, since

$$\omega(z) = \frac{1 - \frac{B_\lambda(\alpha,\beta)f(z)}{z}}{1 - \frac{\varrho B_\lambda(\alpha,\beta)f(z)}{z}}, \ z \in E$$

and $|\omega(z)| < 1, (z \in E)$ we conclude that

$$\left| \frac{B_\lambda(\alpha,\beta)f(z)}{z} - \frac{1}{1-\varrho} \right| < \frac{1}{1-\varrho}, \ z \in E$$

which implies that

$$\Re\left\{ \frac{B_\lambda(\alpha,\beta)f(z)}{z} \right\} > 0, \ z \in E.$$

We complete the proof of the theorem. □

By setting $\varrho = 0$ in Theorem 2.3, we readily obtain the following.

Corollary 2.4. *If $f \in A$ satisfies*

$$\Re\left\{ \frac{z\left(B_\lambda(\alpha,\beta)f(z)\right)'}{B_\lambda(\alpha,\beta)f(z)} \right\} > \frac{1}{2}, z \in E$$

then

$$\frac{z}{B_\lambda(\alpha,\beta)f(z)} \prec \frac{1+z}{1-z}, \ z \in E$$

and

$$\left| \frac{B_\lambda(\alpha,\beta)f(z)}{z} - 1 \right| < 1, \ z \in E.$$

Theorem 2.5. *If $f \in A$ satisfies*

$$\Re\left\{ \frac{z\left(B_\lambda(\alpha,\beta)f(z)\right)'}{B_\lambda(\alpha,\beta)f(z)} \right\} < \frac{\varrho(2-\gamma)-(2+\gamma)}{2(\varrho-1)}, z \in E$$

for some ϱ $(-1 < \varrho \le 0)$ and $0 < \gamma \le 1$ then

$$\left(\frac{B_\lambda(\alpha,\beta)f(z)}{z} \right)^{\frac{1}{\gamma}} \prec \frac{1+\varrho z}{1-z}, \ z \in E.$$

Then implies that

$$\Re\left(\left(\frac{B_\lambda(\alpha,\beta)f(z)}{z} \right)^{\frac{1}{\gamma}} \right) > \frac{1-\varrho}{2}, \ z \in E.$$

Proof. Let us define the function $\omega(z)$ by

$$\frac{B_\lambda(\alpha,\beta)f(z)}{z} = \left(\frac{1-\varrho\omega(z)}{1-\omega(z)} \right)^{\gamma}, \ \omega(z) \ne 1.$$

Clearly, $\omega(z)$ is analytic in E and $\omega(0) = 0$. We want to prove that $|\omega(z)| < 1$ in E. Since

$$\frac{z\left(B_\lambda(\alpha,\beta)f(z)\right)'}{B_\lambda(\alpha,\beta)f(z)} = \gamma\left(\frac{z\omega'(z)}{1-\omega(z)} - \frac{\varrho z\omega'(z)}{1-\varrho\omega(z)} \right) + 1.$$

We see that

$$\Re\left\{ \frac{z\left(B_\lambda(\alpha,\beta)f(z)\right)'}{B_\lambda(\alpha,\beta)f(z)} \right\} = \Re\left\{ \gamma\left(\frac{z\omega'(z)}{1-\omega(z)} - \frac{\varrho z\omega'(z)}{1-\varrho\omega(z)} \right) + 1 \right\}$$
$$< \frac{\varrho(2-\gamma)-(2+\gamma)}{2(\varrho-1)}, \ z \in E,$$

for $\varrho(-1 < \varrho \leq 0)$ and $0 < \gamma \leq 1$. If there exists a point $(z_0 \in E)$ such that

$$\max_{|z|<|z_0|} |\omega(z)| = |\omega(z_0)| = 1$$

then by Lemma 1.2, gives us that $\omega(z_0) = e^{i\theta}$ and $z_0\omega'(z_0) = k\omega(z_0),\ k \geq 1$.

Thus we have

$$\begin{aligned} \frac{z_0 \left(B_\lambda(\alpha,\beta)f(z_0)\right)'}{B_\lambda(\alpha,\beta)f(z_0)} &= \gamma\left(\frac{z_0\omega'(z_0)}{1-\omega(z_0)} - \frac{\varrho z_0\omega'(z_0)}{1-\varrho\omega(z_0)}\right) + 1 \\ &= 1 + \frac{k}{1-e^{i\theta}} - \frac{k}{1-\varrho e^{i\theta}}. \end{aligned}$$

Therefore, we have

$$\Re\left\{\frac{z_0 \left(B_\lambda(\alpha,\beta)f(z_0)\right)'}{B_\lambda(\alpha,\beta)f(z_0)}\right\} = 1 + \frac{\gamma k(1-\varrho^2)}{2(1+\varrho^2-2\varrho cos\theta)}.$$

Thus implies that, for $\varrho(-1 < \varrho \leq 0)$ and $0 < \gamma \leq 1$

$$\Re\left\{\frac{z_0 \left(B_\lambda(\alpha,\beta)f(z_0)\right)'}{B_\lambda(\alpha,\beta)f(z_0)}\right\} \geq \frac{\varrho(2-\gamma)-(2+\gamma)}{2(\varrho-1)}.$$

This contradicts the condition in the theorem.

Hence, there is no $z_0 \in E$ such that $|\omega(z_0)| = 1$ for all $z \in E$, that is

$$\left(\frac{B_\lambda(\alpha,\beta)f(z)}{z}\right)^{\frac{1}{\gamma}} \prec \frac{1-\varrho z}{1-z},\ z \in E.$$

Furthermore, since

$$\omega(z) = \frac{\left(\frac{B_\lambda(\alpha,\beta)f(z)}{z}\right)^{\frac{1}{\gamma}} - 1}{\left(\frac{B_\lambda(\alpha,\beta)f(z)}{z}\right)^{\frac{1}{\gamma}} - \varrho}$$

and $|\omega(z)| < 1,\ (z \in E)$, we conclude that

$$\Re\left(\frac{B_\lambda(\alpha,\beta)f(z)}{z}\right)^{\frac{1}{\gamma}} > \frac{1-\varrho}{2},\ z \in E,$$

we complete the proof of the theorem. □

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