Journal of Fractional Calculus and Applications
Vol. 13(1) Jan. 2022, pp. 198-211.
ISSN: 2090-5858.
http://math-frac.org/Journals/JFCA/

NONLOCAL REACTION-DIFFUSION MODEL WITH SUBDIFFUSIVE KINETICS

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Abstract. We derive nonlocal reaction-diffusion system with subdiffusive kinetics from random walks using a probability measure on a n-multidimensional unit ball S^{n-1}. The system describes two particles moving with subdiffusive kinetics and then undergoing a chemical reaction which occurs if and only if a pair of the particles are in a distance less than R. We also study the existence and uniqueness of a mild solution to a fractional nonlinear Cauchy problem associated with the system by applying Banach's Fixed Point Theorem. The result shows that the mild solution to the problem exists uniquely under some Lipschitz continuity on the nonlinear part of the problem. Consequently, the system also has a unique mild solution since the reaction term of the system satisfies the Lipschitz continuity.

1. Introduction

Suzuki and Kavallaris in [4] studied the system

$$\begin{aligned}
&\frac{\partial q_A}{\partial t} = D_A \Delta q_A - k_A \int_{B(\cdot,R)\cap\Omega} q_B dy \cdot q_A, \quad \text{in } \Omega \times (0,T), \\
&\frac{\partial q_B}{\partial t} = D_B \Delta q_B - k_B \int_{B(\cdot,R)\cap\Omega} q_A dy \cdot q_B, \quad \text{in } \Omega \times (0,T), \\
&\frac{\partial q_A}{\partial n} = \frac{\partial q_B}{\partial n} = 0, \quad \text{on } \partial\Omega \times (0,T), \\
&q_A(\cdot,0) = q_{A0},\ q_B(\cdot,0) = q_{B0}, \quad \text{in } \Omega
\end{aligned} \tag{1}$$

with $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2 boundary. They derived the reaction-diffusion system arising as a mean field limit of a master equation using reaction radius in deterministic case. The system describes the chemical reaction

$$A + B \to C$$

that occurs if and only if a pair of $A-B$ molecules are in a distance less than R. A molecule of A reacts with a molecule of B to produce a molecule of C. They also studied the solution to the system.

2010 *Mathematics Subject Classification.* 34A08, 34A12.

Key words and phrases. nonlocal reaction-diffusion system, reaction radius, subdiffusive kinetics, random walk process, fractional nonlinear Cauchy problem, mild solution.

Submitted February 15, 2021. Revised July 18, 2021.

Here, we derive the similar system in stochastic case from a random walks process with a help of a probability measure on a n-multidimensional unit ball S^{n-1}. Differently from the system (1) as in [4], in this case, before the chemical reactions occurs, both molecules of A and B move with subdiffusive kinetics. We obtain the similar system involving α-order time derivative in Caputo sense with $0 < \alpha < 1$ as a fractional integro-differential equation system. We also study the existence and uniqueness of a mild solution to a fractional nonlinear Cauchy problem associated with the system by employing some properties of solution operators associated with the problem and applying Banach's Fixed Point Theorem. The result shows that the mild solution to the problem exists uniquely under some Lipschitz continuity on the nonlinear part of the problem. Consequently, the system also has a unique mild solution since the reaction term of the system satisfies the Lipschitz continuity.

This paper consists of five sections. In section 2, we provide briefly the fractional integration and derivation in Caputo sense and shows results concerning solution operators to a fractional Cauchy problem. In section 3, we derive a nonlocal reaction-diffusion system with subdiffusive kinetics from a random walks process using a probability measure on a n-multidimensional unit ball S^{n-1}. In section 4, we study the existence and uniqueness of a mild solution to a fractional nonlinear abstract Cauchy problem with some Lipschitz condition on the nonlinear part associated with the system obtained in section 3. In the last section, we discuss the existence and uniqueness of a mild solution to the system obtained in Section 3.

2. Preliminaries

2.1. **Fractional Time Derivative.** Let $0 < \alpha < 1$. The *fractional integral* of order α is defined by

$$J_t^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \quad f \in L^1(I),\ t > 0 \tag{2}$$

with $I = (0, \infty)$. We set $J_t^0 f(t) = f(t)$. The fractional integral operator (2) obeys the semigroup property

$$J_t^\alpha J_t^\beta = J_t^{\alpha+\beta}. \tag{3}$$

The *Caputo fractional derivative* d^α/dt^α of order α is defined by

$$\frac{d^\alpha}{dt^\alpha} f(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} f(s)ds,\ t > 0. \tag{4}$$

The operator d^α/dt^α is a left inverse of J_t^α, that is

$$\frac{d^\alpha}{dt^\alpha} J_t^\alpha f(t) = f(t), \quad t > 0, \tag{5}$$

but it is not a right inverse, that is

$$J_t^\alpha \frac{d^\alpha}{dt^\alpha} f(t) = f(t) - f(0), \quad t > 0. \tag{6}$$

We refer to Kilbas et al. [5] or Podlubny [10] for more details concerning the fractional integrals and derivatives.

2.2. **Analytic Solution Operators.** In this section, we provide briefly some results concerning solution operators for the fractional Cauchy problem

$$\begin{aligned} \frac{d^\alpha}{dt^\alpha}u(t) &= Au(t) + f(t),\ t > 0,\ 0 < \alpha < 1, \\ u(0) &= u_0. \end{aligned} \tag{7}$$

For more details concerning them, we refer to [2]. One can also refer to [1, 6, 7, 8, 11] regarding abstract fractional integro-differential equations.

Henceforth, we assume that the linear operator $A : D(A) \subseteq X \to X$ satisfies the properties that there is a constant $\theta \in (\pi/2, \pi)$ such that

$$\rho(A) \supset S_\theta = \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \theta\}, \tag{8}$$

$$\|R(\lambda; A)\| \leq \frac{M}{|\lambda|},\ \lambda \in S_\theta, \tag{9}$$

with $R(\lambda; A) = (\lambda - A)^{-1}$ and $\rho(A)$ are the resolvent operator and resolvent set of A, respectively. We call A a sectorial operator. Observe that every sectorial operator is closed.

Definition 1. *For $r > 0$ and $\pi/2 < \omega < \theta$,*

$$\Gamma_{r,\omega} = \{\lambda \in \mathbb{C} : |\arg(\lambda)| = \omega, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \omega, |\lambda| = r\}.$$

The linear operator A generates solution operators for the problem (7), those are

$$S_\alpha(t) = \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} e^{\lambda t} \lambda^{\alpha - 1} R(\lambda^\alpha; A) d\lambda, \quad t > 0, \tag{10}$$

$$P_\alpha(t) = \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} e^{\lambda t} R(\lambda^\alpha; A) d\lambda, \quad t > 0, \tag{11}$$

with $\Gamma_{r,\omega}$ is oriented counterclockwise. By Cauchy's theorem, the integral form (10) and (11) are independent of $r > 0$ and $\omega \in (\pi/2, \theta)$.

Let $B(X)$ be the set of all bounded linear operators on X. The properties of the families $\{S_\alpha(t)\}_{t>0}$ and $\{P_\alpha(t)\}_{t>0}$ are given in the following theorems.

Theorem 1. *Let A be a sectorial operator and $S_\alpha(t)$ be the operator defined by* (10). *Then the following statements hold.*

(i) *$S_\alpha(t) \in B(X)$ and there exists a constant $C_1 = C_1(\alpha) > 0$ such that*

$$\|S_\alpha(t)\| \leq C_1, \quad t > 0,$$

(ii) *$S_\alpha(t) \in B(X; D(A))$ for $t > 0$, and if $x \in D(A)$ then $AS_\alpha(t)x = S_\alpha(t)Ax$. Moreover, there exists a constant $C_2 = C_2(\alpha) > 0$ such that*

$$\|AS_\alpha(t)\| \leq C_2 t^{-\alpha}, \quad t > 0,$$

(iii) *The function $t \mapsto S_\alpha(t)$ belongs to $C^\infty((0, \infty); B(X))$ and it holds that*

$$S_\alpha^{(n)}(t) = \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} e^{t\lambda} \lambda^{\alpha + n - 1} R(\lambda^\alpha; A) d\lambda,\ n = 1, 2, \ldots$$

and there exist constants $M_n = M_n(\alpha) > 0, n = 1, 2, \ldots$ such that

$$\|S_\alpha^{(n)}(t)\| \leq M_n t^{-n}, \quad t > 0,$$

Moreover, it has an analytic continuation $S_\alpha(z)$ to the sector $S_{\theta-\pi/2}$ and, for $z \in S_{\theta-\pi/2}$, $\eta \in (\pi/2, \theta)$, it holds that

$$S_\alpha(z) = \frac{1}{2\pi i}\int_{\Gamma_{r,\eta}} e^{\lambda z}\lambda^{\alpha-1}R(\lambda^\alpha; A)d\lambda.$$

Theorem 2. *Let A be a sectorial operator and $P_\alpha(t)$ be the operator defined by* (11)*. Then the following statements hold.*

(i) *$P_\alpha(t) \in B(X)$ and there exists a constant $L_1 = L_1(\alpha) > 0$ such that*

$$\|P_\alpha(t)\| \le L_1 t^{\alpha-1}, \quad t > 0,$$

(ii) *$P_\alpha(t) \in B(X; D(A))$ for all $t > 0$, and if $x \in D(A)$ then $AP_\alpha(t)x = P_\alpha(t)Ax$. Moreover, there exists a constant $L_2 = L_2(\alpha) > 0$ such that*

$$\|AP_\alpha(t)\| \le L_2 t^{-1}, \quad t > 0,$$

(iii) *The function $t \mapsto P_\alpha(t)$ belongs to $C^\infty((0,\infty); B(X))$ and it holds that*

$$P_\alpha^{(n)}(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{t\lambda}\lambda^n R(\lambda^\alpha; A)d\lambda,\ n = 1, 2, \dots$$

and there exist constants $K_n = K_n(\alpha) > 0, n = 1, 2, \dots$ such that

$$\|P_\alpha^{(n)}(t)\| \le K_n t^{\alpha-n-1}, \quad t > 0,$$

Moreover, it has an analytic continuation $P_\alpha(z)$ to the sector $S_{\theta-\pi/2}$ and, for $z \in S_{\theta-\pi/2}$, $\eta \in (\pi/2, \theta)$, it holds that

$$P_\alpha(z) = \frac{1}{2\pi i}\int_{\Gamma_{r,\eta}} e^{\lambda z}R(\lambda^\alpha; A)d\lambda.$$

The following theorem states some identities concerning the operators $S_\alpha(t)$ and $P_\alpha(t)$ including the semigroup-like property.

Theorem 3. *Let A be a sectorial operator, $S_\alpha(t)$ and $P_\alpha(t)$ be the operators defined by* (10) *and* (11)*, respectively. Then the following statements hold.*

(i) *For $x \in X$ and $t > 0$,*

$$S_\alpha(t)x = J_t^{1-\alpha}P_\alpha(t)x, \quad D_t S_\alpha(t)x = AP_\alpha(t)x,$$

(ii) *For $x \in D(A)$ and $s, t > 0$,*

$$D_t^\alpha S_\alpha(t)x = AS_\alpha(t)x,$$

$$S_\alpha(t+s)x = S_\alpha(t)S_\alpha(s)x - A\int_0^t\int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)}P_\alpha(\tau)P_\alpha(r)x\,dr\,d\tau.$$

Next theorem shows us the behavior of the operator $S_\alpha(t)$ at t close to 0^+.

Theorem 4. *Let A be a sectorial operator and $S_\alpha(t)$ be the operator defined by* (10)*. Then the following statements hold.*

(i) *If $x \in \overline{D(A)}$ then $\lim_{t\to 0^+} S_\alpha(t)x = x$,*
(ii) *For every $x \in D(A)$ and $t > 0$,*

$$\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}S_\alpha(\tau)x\,d\tau \in D(A),$$

$$\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}AS_\alpha(\tau)x\,d\tau = S_\alpha(t)x - x,$$

(iii) *If* $x \in D(A)$ *and* $Ax \in \overline{D(A)}$ *then*

$$\lim_{t \mapsto 0^+} \frac{S_\alpha(t)x - x}{t^\alpha} = \frac{1}{\Gamma(\alpha+1)} Ax.$$

The representation of the solution to (7) in term of $S_\alpha(t)$ and $P_\alpha(t)$ is given in the following theorem.

Theorem 5. *Let* $u \in C^1((0,\infty);X) \cap L^1((0,\infty);X)$, $u(t) \in D(A)$ *for* $t \in [0,\infty)$, $Au \in L^1((0,\infty);X)$, $f \in L^1((0,\infty);D(A))$, *and* $Af \in L^1((0,\infty);X)$. *If* u *is a solution to the problem* (7) *then*

$$u(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s)f(s)ds, \quad t > 0. \tag{12}$$

3. Main Results

3.1. **Mathematical Modeling.** We consider a particle undergoing a sequence of jumps on $\mathbb{R}^n$ in a random walk process. Let $\phi(t), t > 0$ be the probability of the particle to jump after waiting a time t and $T(x;\omega)$ be the probability of the particle to jump from position $x \in \mathbb{R}^n$ in a direction $\omega \in S^{n-1}$ satisfying

$$\int_0^\infty \phi(t)dt = 1; \int_{S^{n-1}} T(x;\omega)d\omega = 1,$$

respectively, where

$$S^{n-1} = \{\omega \in \mathbb{R}^n : |\omega| = 1\}.$$

We assume that the particle jumps with a constant jump lenght Δx and suppose that $Q_k(x,t)$ is the conditional probability of the particle to arrive at a posistion x and at a time t after k steps. Following [9], we have that

$$Q_k(x,t) = \int_0^t \int_{S^{n-1}} \phi(t-\tau)T(x-\omega\Delta x;\omega)Q_{k-1}(x-\omega\Delta x,\tau)d\omega d\tau, \ x > 0, t > 0.$$

Let $Q(x,t)$ be the probability density function of the particle to arrive at the position x and the time t. Therefore, we have

$$\begin{aligned} Q(x,t) &= \sum_{k=0}^\infty Q_k(x,t) \\ &= Q_0(x,t) + \int_0^t \int_{S^{n-1}} \phi(t-\tau)T(x-\omega\Delta x;\omega)Q(x-\omega\Delta x,\tau)d\omega d\tau \\ &= \delta(x)\delta(t) + \int_0^t \int_{S^{n-1}} \phi(t-\tau)T(x-\omega\Delta x;\omega)Q(x-\omega\Delta x,\tau)d\omega d\tau. \end{aligned}$$

with $Q_0(x,t) = \delta(x)\delta(t)$.

We next suppose that $q(x,t)$ is the probability of the particle to be at the posistion x and at the time t from the initial position $x = 0$ and the initial time $t = 0$. Then, we have

$$\begin{aligned} q(x,t) &= \int_0^t \Phi(t,\tau;x)Q(x,\tau)d\tau \\ &= \Phi(t)\delta(x) + \int_0^t \int_{S^{n-1}} \phi(t-\tau)T(x-\omega\Delta x;\omega)q(x-\omega\Delta x,\tau)d\omega d\tau \end{aligned} \tag{13}$$

with $\Phi(t,\tau;x)=\Phi(t-\tau)$ is the probability of the particle to arrive at the position x and at the time $\tau<t$ and not to jump during time interval $t-\tau$ where

$$\Phi(t)=\int_t^\infty \phi(s)ds=1-\int_0^t \phi(s)ds.$$

We now consider two molecules of A and B moving with subdiffusive kinetics and then undergoing a chemical reaction

$$A+B\to C$$

to produce a molecule of C which occurs if and only if the pair of $A-B$ molecules are in a distance less than R. Then, from (13), we have

$$\begin{aligned} q_A(x,t)=\Phi(t)\delta(x)+\int_0^t\int_{S^{n-1}}\phi(t-\tau)T(x-\omega\Delta x;\omega)q_A(x-\omega\Delta x,\tau)d\omega d\tau\\ -\int_0^t\Phi(t-\tau)\left[\frac{k^{A\to B}}{v}\int_{B(x,R)\cap\Omega}q_B(y,\tau)dy\cdot q_A(x,\tau)\right]d\tau \end{aligned} \tag{14}$$

with $k^{A\to B}$ denotes the rate by which A molecule hits B molecule to cause the chemical reaction per unit time and v is the volume of a n-dimensional ball $B(x,R)$ with radius R and center x. The equation (14) means that the existence of A molecule at the position x and the time t depends on the existences of A molecule at the position x and the time $t=0$ which have not yet jumped until the time t (the first term on the right hand side), A molecule at a position $x-\omega\Delta x$ and a time $\tau<t$ that then jumps at the time t after waiting a time interval $t-\tau$ in the direction $-\omega$ with the jump length Δx (the second term on the right hand side), and A molecule which undergoes the chemical reaction at the position x and the time $\tau<t$ (with B molecule at the position $y\in B(x,R)$ and the time $\tau<t$) that then jumps at the time t after waiting the time interval $t-\tau$ (the third term on the right hand side). By using Laplace transform

$$\tilde{h}(s)=\int_0^\infty e^{-st}h(t)dt$$

applied to (14), we get

$$\begin{aligned} \tilde{q}_A(x,s)=\frac{1-\tilde{\phi}(s)}{s}\delta(x)+\tilde{\phi}(s)\int_{S^{n-1}}T(x-\omega\Delta x;\omega)\tilde{q}_A(x-\omega\Delta x,s)d\omega\\ -\frac{1-\tilde{\phi}(s)}{s}\tilde{F}_{A\to B}(x,s) \end{aligned}$$

with

$$F_{A\to B}(x,t)=\frac{k^{A\to B}}{v}\int_{B(x,R)\cap\Omega}q_B(y,t)dy\cdot q_A(x,t).$$

It follows that

$$\begin{aligned} &\tilde{q}_A(x,s)-\frac{1}{s}\delta(x)\\ &\quad=\tilde{H}(s)\left[-\tilde{q}_A(x,s)+\int_{S^{n-1}}T(x-\omega\Delta x;\omega)\tilde{q}_A(x-\omega\Delta x,s)d\omega\right]\\ &\quad-\frac{1}{s}\tilde{F}_{A\to B}(x,s) \end{aligned} \tag{15}$$

with

$$\tilde{H}(s) = \frac{\tilde{\phi}(s)}{1 - \tilde{\phi}(s)}.$$

Consequently, using the inverse of Laplace transform applied to (15), we obtain

$$\begin{aligned} &q_A(x,t) - q_A(x,0) \\ &\quad = \int_0^t H(t-\tau)\left[-q_A(x,\tau) + \int_{S^{n-1}} T(x-\omega\Delta x;\omega) q_A(x-\omega\Delta x,t)d\omega\right]d\tau \\ &\quad\quad - \frac{k^{A\to B}}{v}\int_0^t \int_{B(x,R)\cap\Omega} q_B(y,\tau)dy \cdot q_A(x,\tau)d\tau. \end{aligned} \tag{16}$$

If the waiting time is Poissonian,

$$\phi(t) = \frac{1}{\lambda}e^{-\frac{t}{\lambda}},\ t>0,\ \lambda>0, \tag{17}$$

then $H(t) = 1/\lambda$. Then, the equation (16) is reduced to

$$\begin{aligned} &q_A(x,t) - q_A(x,0) \\ &\quad = \frac{1}{\lambda}\int_0^t \left[-q_A(x,\tau) + \int_{S^{n-1}} T(x-\omega\Delta x;\omega) q_A(x-\omega\Delta x,\tau)d\omega\right]d\tau \\ &\quad\quad - \frac{k^{A\to B}}{v}\int_0^t \int_{B(x,R)\cap\Omega} q_B(y,\tau)dy \cdot q_A(x,\tau)d\tau. \end{aligned} \tag{18}$$

If the waiting time is nonpoissonian,

$$\phi_\alpha(t) = \frac{t^{\alpha-1}}{\lambda^\alpha}E_{\alpha,\alpha}\left(-\left(\frac{t}{\lambda}\right)\right),\ 0<\alpha<1,\ t>0,\ \lambda>0 \tag{19}$$

with

$$E_{\alpha,\beta}(t) = \sum_{i=0}^{\infty}\frac{t^n}{\Gamma(\alpha n+\beta)},\ \alpha,\beta>0,$$

then

$$H(t) = \frac{1}{\lambda}\cdot\frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

Thus, the equation (16) is reduced to

$$\begin{aligned} &q_A(x,t) - q_A(x,0) \\ &= \frac{1}{\lambda^\alpha}\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left[-q_A(x,\tau) + \int_{S^{n-1}} T(x-\omega\Delta x;\omega) q_A(x-\omega\Delta x,\tau)d\omega\right]d\tau \\ &\quad - \frac{k^{B\to A}}{v}\int_0^t \int_{B(x,R)\cap\Omega} q_B(y,\tau)dy \cdot q_A(x,\tau)d\tau. \end{aligned} \tag{20}$$

Note that for $\alpha = 1$ we have the Equation (18). Observe that the equation (18) is similar to the master equation (1.6) for the particle movement in deterministic case as studied in [4].

We now assume that $T(x;\omega)$ is a constant or does not depend on the position x and the direction ω. It means that the particle in the process moves in a homogeneous medium or the particle movement is not influenced by any external force field. Then, we have $T_\omega(x) = 1/|S^{n-1}|$, where

$$|S^{n-1}| = \int_{S^{n-1}} d\omega = \frac{2^{n/2}}{\Gamma(n/2)}.$$

We next consider the following fact [3].

Lemma 1. *If $d\omega$ is isotropic, that is*

$$\int_{S^{n-1}} \omega_i d\omega = 0,$$

$$\int_{S^{n-1}} \omega_i \omega_j d\omega = \frac{\delta_{ij}}{n}|S^{n-1}|, \quad i,j = 1,2...,n,$$

then, for $f = f(x) \in C^2(\mathbb{R}^n)$,

$$\int_{S^{n-1}} f(x+\omega\Delta x) - f(x)d\omega = \frac{1}{2n}|S^{n-1}|(\Delta x)^2\Delta f(x) + o\big((\Delta x)^2\big)$$

with δ_{ij} is the kronecker delta function.

By Lemma 1, the Equations (18) and (20) are reduced to

$$\begin{aligned} q_A(x,t) - q_A(x,0) = {} & \frac{(\Delta x)^2}{2n\lambda}\int_0^t \Delta q_A(x,\tau)d\tau \\ & - \frac{k^{A\to B}}{v}\int_0^t \int_{B(x,R)\cap\Omega} q_B(y,\tau)dy \cdot q_A(x,\tau)d\tau + o\big((\Delta x)^2\big) \end{aligned} \tag{21}$$

and

$$\begin{aligned} q_A(x,t) - q_A(x,0) = {} & \frac{(\Delta x)^2}{2n\lambda^\alpha}\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Delta q_A(x,\tau)d\tau \\ & - \frac{k^{A\to B}}{v}\int_0^t \int_{B(x,R)\cap\Omega} q_B(y,\tau)dy \cdot q_A(x,\tau)d\tau + o\big((\Delta x)^2\big), \end{aligned} \tag{22}$$

respectively. Thus, taking $\Delta x \to 0, \lambda \to 0$ such that $(\Delta x)^2/\lambda$ is still finite and then differentiating both sides of (21), we obtain

$$\frac{\partial}{\partial t}q_A(x,t) = D_A\Delta q_A(x,t) - k_A\int_{B(x,R)\cap\Omega} q_B(y,t)dy \cdot q_A(x,t) \tag{23}$$

with

$$D_A = \frac{(\Delta x)^2}{2n\lambda}, \quad k_A = \frac{k^{A\to B}}{v}.$$

Similarly, taking $\Delta x \to 0, \lambda \to 0$ such that $(\Delta x)^2/\lambda^\alpha$ is still finite and then differentiating both sides of (22) in fractional α-order, we obtain

$$\frac{\partial^\alpha}{\partial t^\alpha}q_A(x,t) = D_A\Delta q_A(x,t) - k_A\int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\int_{B(x,R)\cap\Omega} q_B(y,\tau)dy \cdot q_A(x,\tau)d\tau \tag{24}$$

with

$$D_A = \frac{(\Delta x)^2}{2n\lambda^\alpha}, \quad k_A = \frac{k^{A\to B}}{v}.$$

Note that, for $\alpha = 1$, the Equation (24) is reduced to the Equation (23).

If B molecule hits A molecule, by the similar way as before, for Poissonian waiting time (17) and nonpoissonian waiting time (19), we have

$$\frac{\partial}{\partial t} q_B(x,t) = D_B \Delta q_B(x,t) - k_B \int_{B(x,R)\cap\Omega} q_A(y,t)dy \cdot q_B(x,t) \tag{25}$$

with

$$D_B = \frac{(\Delta x)^2}{2n\lambda}, \quad k_B = \frac{k^{B\to A}}{v}$$

and

$$\frac{\partial^\alpha}{\partial t^\alpha} q_B(x,t) = D_B \Delta q_B(x,t) - k_B \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \int_{B(x,R)\cap\Omega} q_A(y,\tau)dy \cdot q_B(x,\tau)d\tau \tag{26}$$

with

$$D_B = \frac{(\Delta x)^2}{2n\lambda^\alpha}, \quad k_B = \frac{k^{B\to A}}{v},$$

respectively. Thus we obtain the system

$$\begin{aligned} &\frac{\partial^\alpha q_A}{\partial t^\alpha} = D_A \Delta q_A - k_A J_t^{1-\alpha} \int_{B(\cdot,R)\cap\Omega} q_B dy \cdot q_A, \quad \text{in } \Omega\times(0,T), \\ &\frac{\partial^\alpha q_B}{\partial t^\alpha} = D_B \Delta q_B - k_B J_t^{1-\alpha} \int_{B(\cdot,R)\cap\Omega} q_A dy \cdot q_B, \quad \text{in } \Omega\times(0,T), \\ &q_A(\cdot,0) = q_{A0},\ q_B(\cdot,0) = q_{B0}, \quad \text{in } \Omega \end{aligned} \tag{27}$$

with $0<\alpha<1$ describing two molecules of A and B moving with subdiffusive kinetics and then undergoing the chemical reaction

$$A + B \to C$$

to produce the molecule C which occurs if and only if a pair of $A-B$ molecules are in a distance less than R. Observe that, for $\alpha = 1$, the system (27) is reduced to the system (1) as discussed in [4].

3.2. **Fractional Nonlinear Cauchy Problem.** Consider the problem

$$\begin{aligned} &\frac{d^\alpha u}{dt^\alpha} = Au + f(t,u),\ 0<t\le T,\ 0<\alpha<1, \\ &u(0) = u_0, \end{aligned} \tag{28}$$

with X is a Banach space, $A : D(A) \to X$ is a sectorial linear operator, $u_0 \in X$, and $f : (0,T]\times X \to X$.

Using the solution operator families $\{S_\alpha(t)\}_t > 0$ and $\{P_\alpha(t)\}_t > 0$ as discussed in [2], a mild solution to the problem (28) is defined as follows.

Definition 2. *A continuous function $u : [0,T] \to X$ is a mild solution to the problem* (28) *if it satisfies*

$$u(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s)f(s,u(s))ds, \quad 0<t\le T.$$

We next suppose that $BC((0,T];X)$ is the space of bounded continuous functions on $(0,T]$ with values in X. The following theorem shows us the existence and uniqueness of the mild solution to (28) assuming the Lipschitz continuity on f.

Theorem 6. *Let $f:(0,T]\times X\to X$ and there exist $K(t),L(t)\in L^1((0,T];X)$ such that*

$$\|f(t,u)-f(t,v)\|\le K(t)\|u-v\|_Y,\ 0<t\le T,\ u,v\in X, \tag{29}$$

$$\|f(t,u)\|\le L(t)\|u\|_Y,\ 0<t\le T,\ u\in X, \tag{30}$$

with $\|u\|\le c$ and $\|v\|\le c$ where $Y=BC((0,T];X)$. If $u_0\in\overline{D(A)}$ then there exists $T_0>0$ such that the problem (28) has a unique mild solution $u\in C([0,T_0];X)$.

Proof. Given $\varepsilon>0$. Since $u_0\in\overline{D(A)}$, we have $\|S_\alpha(t)u_0-u_0\|\to 0$ as $t\to 0^+$ by Theorem 4(i). It means that there exists $0<\tau\le T$ such that

$$\|S_\alpha(t)u_0-u_0\|\le\varepsilon/2,\quad 0<t\le\tau.$$

We next suppose $Z=BC((0,T_0];X)$ and its subset $W=\{u\in Z:\|u-u_0\|_Z\le\varepsilon\}$ with

$$T_0=\inf\left\{\tau,\left(\frac{\alpha\varepsilon}{2L_1c\|L\|_{L^1((0,T];X)}}\right)^{1/\alpha},\left(\frac{\alpha}{2L_1\|K\|_{L^1((0,T];X)}}\right)^{1/\alpha}\right\}. \tag{31}$$

We now define a mapping F on W by

$$F(u_0):=u_0;\ Fu(t):=S_\alpha(t)u_0+\int_0^t P_\alpha(t-s)f(s,u(s))ds,\quad 0<t\le T_0.$$

We first prove that $F:W\to W$. Note that, by Theorem 2(i) and (30), for $0<t\le T_0$,

$$\begin{aligned}\|Fu(t)-u_0\|&\le\|Fu(t)-S_\alpha u_0\|+\|S_\alpha(t)u_0-u_0\|\\&\le\int_0^t\|P_\alpha(t-s)\|\|f(s,u(s))\|ds+\|S_\alpha(t)u_0-u_0\|\\&\le\|u\|_Y\int_0^t\|P_\alpha(t-s)\|L(s)\|ds+\varepsilon/2\\&\le\frac{L_1}{\alpha}\|u\|_YT_0^\alpha\|L\|_{L^1((0,T];X)}+\varepsilon/2\\&\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

implying

$$\|Fu-u_0\|_Z\le\varepsilon.$$

We now prove the continuity of Fu in $t \in [0, T_0]$. Observe that, By Theorem 2(i) and (29), for $0 < t \leq T_0$,

$$\begin{aligned}
&\|Fu(t+h) - Fu(t)\| \\
&\leq \|S(t+h)u_0 - S(t)u_0\| + \|\int_0^{t+h} P(t+h-s)f(s,u(s))ds \\
&\quad - \int_0^t P(t-s)f(s,u(s))ds\| \\
&\leq \|S(t+h)u_0 - S(t)u_0\| + \int_0^t \|P(t+h-s) - P(t-s)\|\|f(s,u(s))\|ds \\
&\quad + \int_t^{t+h} \|P(t+h-s)\|\|f(s,u(s))\|ds \\
&\leq \|S(t+h)u_0 - S(t)u_0\| + \|u\|_Y \int_0^t \|P(t+h-s) - P(t-s)\|L(s)ds \\
&\quad + \|u\|_Y \int_t^{t+h} \|P(t+h-s)\|L(s)ds \\
&\leq \|S(t+h)u_0 - S(t)u_0\| + \|u\|_Y \int_0^t \|P(t+h-s) - P(t-s)\|L(s)ds \\
&\quad + \|u\|_Y \|L\|_{L^1((0,T];X)} h^\alpha
\end{aligned}$$

Next, note that, by Theorem 2(i),

$$\|P(t+h) - P(t)\| \leq L_1 t^{\alpha-1} \in L^1((0,T];X)$$

and, by Theorem 2(iii), $P(t)$ is continuous at $t \in (0, +\infty)$. It implies, by the Dominated Convergence Theorem,

$$\int_0^t \|P(t+h-s) - P(t-s)\|L(s)ds \to 0, \quad \text{as } h \to 0.$$

Therefore, $\|Fu(t+h) - Fu(t)\| \to 0$ as $h \to 0$ for $0 < t \leq T_0$. For $t = 0$, consider that

$$\begin{aligned}
\|Fu(h) - Fu_0\| &\leq \|S(h) - u_0\| + \int_0^h \|P(h-s)f(s,u(s))\|ds \\
&\leq \|S(h) - u_0\| + \frac{L_1}{\alpha}\|L\|_{L^1((0,T];X)}\|u\|_Y h^\alpha
\end{aligned}$$

Then, $\|Fu(h) - Fu_0\| \to 0$ as $h \to 0$. Therefore $Fu \in Z$. Thus, we can conclude that $F : W \to W$.

We next prove that $F : W \to W$ is a contraction. Observe that, again by Theorem 2(i) and (29), for $u, v \in W$,

$$\begin{aligned}
\|Fu(t) - Fv(t)\| &\leq \int_0^t \|P_\alpha(t-s)\|\|f(s,u(s)) - f(s,v(s))\|ds \\
&\leq \|u - v\|_Z \int_0^t \|P_\alpha(t-s)\|K(s)ds \\
&\leq \frac{L_1}{\alpha}\|K\|_{L^1((0,T];X)} T_0^\alpha \|u - v\|_Z.
\end{aligned}$$

Consequently,

$$\|Fu - Fv\|_Z \leq \frac{L_1}{\alpha}\|K\|_{L^1((0,T];X)} T_0^\alpha \|u - v\|_Z.$$

Next, observe that

$$0 < \frac{L_1}{\alpha}\|K\|_{L^1((0,T];X)} T_0^\alpha \leq \frac{1}{2}.$$

Therefore $F : W \to W$ is a contraction, and, by the Contractive Mapping Theorem, F has a unique fixed point $u \in W$. Here, we have

$$u(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s)f(s, u(s))ds, \quad 0 < t \leq T_0.$$

Note that, since $u_0 \in \overline{D(A)}$, then by Theorem 2(i), Theorem 4(i), and (30), we get $\|u(t) - u_0\| \to 0$ as $t \to 0^+$. It follows that $u \in C([0, T_0]; X)$. □

3.3. **Existence and Uniqueness of the Solution to the System.** In this section, we study the existence and uniqueness of the mild solution to the system (27) with Neumann boundary condition

$$\begin{aligned}
&\frac{\partial^\alpha q_A}{\partial t^\alpha} = D_A \Delta q_A - k_A J_t^{1-\alpha} \int_{B(\cdot,R)\cap\Omega} q_B dy \cdot q_A, \quad \text{in } \Omega \times (0,T),\\
&\frac{\partial^\alpha q_B}{\partial t^\alpha} = D_B \Delta q_B - k_B J_t^{1-\alpha} \int_{B(\cdot,R)\cap\Omega} q_A dy \cdot q_B, \quad \text{in } \Omega \times (0,T),\\
&\frac{\partial q_A}{\partial n} = \frac{\partial q_B}{\partial n} = 0, \quad \text{on } \partial\Omega \times (0,T),\\
&q_A(\cdot, 0) = q_{A0},\ q_B(\cdot, 0) = q_{B0}, \quad \text{in } \Omega
\end{aligned}$$

with $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2 boundary. The abstract formulation of the problem (27) is

$$\begin{aligned}
&\frac{d^\alpha U}{dt^\alpha} = AU + F(t, U), \quad 0 < t \leq T,\\
&U(0) = U_0
\end{aligned}$$

in

$$\left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u, v \in L^2(\Omega) \right\}$$

with

$$A = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix},\ F(t,U) = \begin{pmatrix} J_t^{1-\alpha} u\overline{v} \\ J_t^{1-\alpha} v\overline{u} \end{pmatrix},\ U = \begin{pmatrix} u \\ v \end{pmatrix},\ U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

$$\overline{v} = \int_{B(\cdot,R)\cap\Omega} \chi(\cdot, y)v(y)dy,$$

and $\chi(x,y) = \chi_{B(x,R)\cap\Omega}(y)$ denotes the caractheristic function of $B(x,R)\cap\Omega$. Here,

$$0 \leq u \leq \|u_0\|_\infty,\ 0 \leq v \leq \|v_0\|_\infty \text{ in } \Omega \times (0,T] \tag{32}$$

since the reaction terms in (27) are negative. We next set

$$D(A) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u, v \in H_N^2\{\Omega\} \right\}$$

with

$$H_N^2(\Omega) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

The operator Δ is dissipative and self adjoint implying that Δ is sectorial in X. Moreover, for any $\lambda \in S_\theta$ with $\theta \in (\pi/2, \pi)$, we get

$$(\lambda - A)^{-1} = \begin{pmatrix} (\lambda-\Delta)^{-1} & 0 \\ 0 & (\lambda-\Delta)^{-1} \end{pmatrix}.$$

Thus, there exists $M > 0$ such that $\|(\lambda - A)^{-1}\| \le M/|\lambda|$ for all $\lambda \in S_\theta$.

Observe that

$$\|\overline{u}\|_2 \le |\Omega| \|u\|_2.$$

Therefore, if

$$U = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \in Y, \quad V = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \in Y, \quad \text{where } Y = BC((0,T]; D(A)),$$

then

$$\begin{aligned}
&\|F(t,U) - F(t,V)\|_2^2 \\
&\quad = \left\| \begin{pmatrix} J_t^{1-\alpha}[u_1\overline{v}_1] - J_t^{1-\alpha}[u_2\overline{v}_2] \\ J_t^{1-\alpha}[v_1\overline{u}_1] - J_t^{1-\alpha}[v_2\overline{u}_2] \end{pmatrix} \right\|_2^2 \\
&\quad = \|J_t^{1-\alpha}[u_1\overline{v}_1 - u_2\overline{v}_2]\|_2^2 + \|J_t^{1-\alpha}[v_1\overline{u}_1 - v_2\overline{u}_2]\|_2^2 \\
&\quad \le (J_t^{1-\alpha}[\|(u_1-u_2)\overline{v}_1 + u_2(\overline{v}_1 - \overline{v}_2)\|_2])^2 \\
&\qquad + (J_t^{1-\alpha}[\|(v_1-v_2)\overline{u}_1 + v_2(\overline{u}_1 - \overline{u}_2)\|_2])^2 \\
&\quad \le (J_t^{1-\alpha}[\|u_1-u_2\|_2\|\overline{v}_1\|_2 + \|u_2\|_2\|\overline{v}_1 - \overline{v}_2\|_2])^2 \\
&\qquad + (J_t^{1-\alpha}[\|v_1-v_2\|_2\|\overline{u}_1\|_2 + \|v_2\|_2\|\overline{u}_1 - \overline{u}_2\|_2])^2 \\
&\quad \le |\Omega|^2(J_t^{1-\alpha}[\|u_1-u_2\|_2\|v_1\|_2 + \|u_2\|_2\|v_1 - v_2\|_2])^2 \\
&\qquad + |\Omega|^4(J_t^{1-\alpha}[\|v_1-v_2\|_2\|u_1\|_2 + \|v_2\|_2\|u_1 - u_2\|_2])^2 \\
&\quad \le |\Omega|^2 \left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} [\|u_1-u_2\|_Y\|v_1\|_Y + \|u_2\|_Y\|v_1-v_2\|_Y] \right)^2 \\
&\qquad + |\Omega|^2 \left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} [\|v_1-v_2\|_Y\|u_1\|_Y + \|v_2\|_Y\|u_1-u_2\|_Y] \right)^2 \\
&\quad \le \frac{|\Omega|^2}{\Gamma^2(2-\alpha)} t^{2(1-\alpha)} [\|u_1-u_2\|_Y + \|v_1-v_2\|_Y]^2 \\
&\qquad \times [(\|v_1\|_Y + \|u_2\|_Y)^2 + (\|u_1\|_Y + \|v_2\|_Y)^2] \\
&\quad \le \frac{4|\Omega|^2}{\Gamma^2(2-\alpha)} t^{2(1-\alpha)} (\|u_1-u_2\|_Y^2 + \|v_1-v_2\|_Y^2) \\
&\qquad \times (\|u_1\|_Y^2 + \|u_2\|_Y^2 + \|v_1\|_Y^2 + \|v_2\|_Y^2) \\
&\quad = \frac{4|\Omega|^2}{\Gamma^2(2-\alpha)} t^{2(1-\alpha)} \|U-V\|_Y^2 (\|U\|_Y^2 + \|V\|_Y^2).
\end{aligned} \tag{33}$$

Consequently, by (32), we have

$$\|F(t,U) - F(t,V)\| \le K(t)\|U - V\|_Y \tag{34}$$

with

$$K(t) = C_K t^{1-\alpha}.$$

for some constant $C_K > 0$. By the similar way, we also obtain

$$\|F(t,U)\| \le L(t)\|U\|_Y \tag{35}$$

with

$$L(t) = C_L t^{1-\alpha}.$$

for some constant $C_L > 0$. Then, note that $K(t), L(t) \in L^1((0,T];X)$. Thus, by Theorem 6, if $U_0 \in \overline{D(A)}$, the System (27) has a unique mild solution $u \in C([0,T_0];D(A))$ for some $T_0 > 0$.

References

[1] E. Bazhlekova, *Fractional Evolution Equations in Banach Spaces*, Ph.D. Thesis, Eindhoven University of Technology, Eindhoven, 2001.

[2] B. H. Guswanto, *On the Properties of Solution Operators of Fractional Evolution Equations*, Journal of Fractional Calculus and Applications, 6 (1) (2015), 131-159.

[3] Ichikawa, K., Rouzimaimaiti, M., Suzuki, T., *Reaction Diffusion Equation with Non-local Term Arises as A Mean Field Limit of the Master Equation*, Discrete and Continuous Dynamical Systems, 5 (1) (2012), 115126.

[4] Kavalaris, N. I.; Suzuki, T.; *Non-local Reaction-Dffusion System Involved by Reaction Radius I*, IMA Journal of Applied Mathematics, (2012), 1-19.

[5] A. A. Kilbas, H. M. Srivastava, J. J. Trujilo, *Theory and Application of Fractional Differential Equations*, North Holland Mathematics Studies, Elsevier, 2006.

[6] M. Kostić, (a,k)-*Regularized C-Resolvent Families : Regularity and Local Properties*, Abstr. Appl. Anal., vol. 2009 (2009), Article ID 858242, 27 pages.

[7] M. Kostić, *Abstract Volterra Integro-Differential Equations*, CRC Press, Boca Raton, 2015.

[8] M. Li, C. Chen, F.-B. Li, *On Fractional Powers of Generators of Fractional Resolvent Families*, J. Funct Anal. 259 (2010), 27022726.

[9] H. G. Othmer, S. R. Dunbar, W. Alt, *Models of Dispersal in Bilogical System*, J. Math. Bio., 26 (1998), 263-298.

[10] I. Podlubny, *Fractional Differential Equations*, Academic Press 198, 1999.

[11] J. Prüss, *Evolutionary Integral Equations and Applications*, Monograph in Mathematics, Vol. 87, Birkhäuser, Basel, 1993.

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