

CERTAIN UNIFIED RECURRENCE RELATIONS ASSOCIATED WITH SPECIAL FUNCTIONS

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ABSTRACT. The principal aim of this paper is to investigate certain recurrence relations for \overline{H} -function. On account of general nature of \overline{H} -function, we have also obtained recurrence relations of H -function and Wright generalized hypergeometric function as special cases of our main findings.

1. INTRODUCTION AND PRELIMINARIES

In an attempt to achieve certain Feynman integrals in two different ways which arises in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions, Inayat-Hussain [1, 2] investigated \overline{H} -function, which is defined as

$$\begin{aligned}\overline{H}[z] &= \overline{H}_{p,q}^{m,n}[z] = \overline{H}_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q} \end{matrix} \right. \right] \\ &= \frac{1}{2\pi\omega} \int_L \chi(s) z^s ds\end{aligned}\tag{1}$$

where $\omega = \sqrt{-1}$ and

$$\chi(s) = \frac{\prod_{j=1}^m \Gamma(b_j - B_j s) \prod_{j=1}^n \Gamma(1 - a_j + A_j s)^{\alpha_j}}{\prod_{j=m+1}^q \Gamma(1 - b_j + B_j s)^{\beta_j} \prod_{j=n+1}^p \Gamma(a_j - A_j s)}\tag{2}$$

which contains fractional powers of some of the gamma functions. $L = L_{\omega\tau\infty}$ is a contour starting at the point $\tau - \omega\infty$, terminating at the point $\tau + \omega\infty$ with $\tau \in R = (-\infty, \infty)$. Here z may be real or complex but is not equal to zero and an empty product is interpreted as unity.

m, n, p, q are integers such that $1 \leq m \leq q$, $0 \leq n \leq p$; $A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$), a_j ($j = 1, \dots, p$) and b_j ($j = 1, \dots, q$) are complex numbers. The exponents α_j ($j = 1, \dots, n$) and β_j ($j = m + 1, \dots, q$) may take non-integer values. Also, from Inayat-Hussain [2], it follows that

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for small values of z

$$\overline{H}_{p,q}^{m,n} [z] = o(|z|^\mu), \tag{3}$$

where

$$\mu = \min_{1 \leq j \leq m} \operatorname{Re} \left\{ \left(\frac{b_j}{B_j} \right) \right\}, \tag{4}$$

and for large values of z

$$\overline{H}_{p,q}^{m,n} [z] = o(|z|^\nu), \tag{5}$$

where

$$\nu = \max_{1 \leq j \leq n} \operatorname{Re} \left\{ \alpha_j \left(\frac{a_j - 1}{A_j} \right) \right\}. \tag{6}$$

The sufficient condition for absolute convergence of the contour integral (1) is given by Buschman and Srivastava [3] as

$$\Omega = \sum_{j=1}^m |B_j| + \sum_{j=1}^n |\alpha_j A_j| - \sum_{j=m+1}^q |\beta_j B_j| - \sum_{j=n+1}^p |A_j| > 0 \tag{7}$$

and

$$|\arg z| < \frac{1}{2} \pi \Omega. \tag{8}$$

When the exponents $\alpha_j (j = 1, \dots, n) = \beta_j (m + 1, \dots, q) = 1$, the \overline{H} -function reduces to the familiar Fox's H -function.

The Fox's H -function is defined and represented in the following manner [4].

$$\begin{aligned} H [z] &= H_{p,q}^{m,n} [z] = H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_j, A_j)_{1,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q} \end{matrix} \right. \right], \\ &= \frac{1}{2\pi\omega} \int_L \chi(s) z^s ds, \end{aligned} \tag{9}$$

where $\omega = \sqrt{-1}$ and

$$\chi(s) = \frac{\prod_{j=1}^m \Gamma(b_j - B_j s) \prod_{j=1}^n \Gamma(1 - a_j + A_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + B_j s) \prod_{j=n+1}^p \Gamma(a_j - A_j s)}. \tag{10}$$

The nature of contour L of the integral (9), existence conditions of the H -function defined by (9) and other details can be found in the book Kilbas and Saigo [5]. The generalized Wright hypergeometric function [6, 7, 8] is defined by means of the series representation in the form

$$\begin{aligned} {}_p\Psi_q(z) &= {}_p\Psi_q \left[\begin{matrix} (a_p, A_p); \\ (b_q, B_q); \end{matrix} \left| z \right. \right] = {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_p, A_p); \\ (b_1, B_1), \dots, (b_q, B_q); \end{matrix} \left| z \right. \right], \\ &= \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(a_j + A_j k)}{\prod_{j=1}^q \Gamma(b_j + B_j k)} \frac{z^k}{k!}, \end{aligned} \tag{11}$$

where $z \in \mathbb{C}$, $a_j (j = 1, 2, \dots, p)$, $b_j (j = 1, 2, \dots, q) \in \mathbb{C}$, $A_j (j = 1, 2, \dots, p)$, $B_j (j = 1, 2, \dots, q)$ are positive real numbers such that

$$1 + \sum_{j=1}^q B_j - \sum_{j=1}^p A_j \geq 0.$$

2. RECURRENCE RELATIONS OF \overline{H} -FUNCTION

In earlier work, Jain [9], Anandani [10], Goyal [11] and Raina [12] have investigated various recurrence relations of Fox's H -function, Motivated by those avenues, we have investigated some recurrence relations of \overline{H} -function, Due to general nature of \overline{H} -function, some new recurrence relations of H -function and Wright generalized hypergeometric function have also been derived from the recurrence relations of \overline{H} -function as special cases from our main findings.

Theorem 1. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0 (j = 1, \dots, p)$, $B_j > 0 (j = 1, \dots, q)$, $a_j (j = 1, \dots, p)$ and $b_j (j = 1, \dots, q)$ be complex numbers then there hold the following recurrence relation

$$\begin{aligned} & \mu \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} \left(\frac{1}{2} + \frac{\rho}{2} - \mu, \frac{1}{2}; 1\right), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-3}, \\ (\rho, 1), (b_j, B_j)_{2,m}, \\ \left(\frac{1}{2} + \frac{\rho}{2} + \mu, \frac{1}{2}\right), \left(\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}\right), \left(\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}\right) \\ (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] \\ &= \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} \left(\frac{1}{2} + \frac{\rho}{2} - \mu, \frac{1}{2}; 1\right), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-3}, \\ (\rho - 1, 1), (b_j, B_j)_{2,m}, \\ \left(-\frac{1}{2} + \frac{\rho}{2} + \mu, \frac{1}{2}\right), \left(\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}\right), \left(-\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}\right) \\ (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] \\ &+ \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} \left(-\frac{1}{2} + \frac{\rho}{2} - \mu, \frac{1}{2}; 1\right), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-3}, \\ (\rho - 1, 1), (b_j, B_j)_{2,m}, \\ \left(\frac{1}{2} + \frac{\rho}{2} + \mu, \frac{1}{2}\right), \left(\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}\right), \left(-\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}\right) \\ (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] \quad (12) \end{aligned}$$

where $1 \leq n \leq p - 3$ and $1 \leq m \leq q$.

Theorem 2. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0 (j = 1, \dots, p)$, $B_j > 0 (j = 1, \dots, q)$, $a_j (j = 1, \dots, p)$ and $b_j (j = 1, \dots, q)$ be complex numbers then there hold the following recurrence relation

$$\begin{aligned} & (2\mu + 1) \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} (-1, 1; 1), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p}, \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q-2}, \left(-1 - \frac{\mu}{2}, \frac{1}{2}; 1\right), \left(-\frac{1}{2} + \frac{\mu}{2}, \frac{1}{2}; 1\right) \end{array} \right. \right] \\ &= 2(\mu + 1) \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} (0, 1; 1), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p}, \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q-2}, \left(-1 - \frac{\mu}{2}, \frac{1}{2}; 1\right), \left(\frac{1}{2} + \frac{\mu}{2}, \frac{1}{2}; 1\right) \end{array} \right. \right] \\ &+ 2\mu \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} (0, 1; 1), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p}, \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q-2}, \left(-\frac{\mu}{2}, \frac{1}{2}; 1\right), \left(-\frac{1}{2} + \frac{\mu}{2}, \frac{1}{2}; 1\right) \end{array} \right. \right] \quad (13) \end{aligned}$$

where $1 \leq n \leq p$ and $1 \leq m \leq q - 2$.

Theorem 3. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$), a_j ($j = 1, \dots, p$) and b_j ($j = 1, \dots, q$) be complex numbers then there hold the following recurrence relation

$$\begin{aligned} & \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} (1 - \beta, 1; 1), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, (1 + \alpha - \beta, 1) \\ (1 + \mu + \alpha - \beta, 1), (b_j, B_j)_{2,m}, (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] \\ &= (2\mu + \alpha - 1) \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} (1 - \beta, 1; 1), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, (1 + \alpha - \beta, 1) \\ (\mu + \alpha - \beta, 1), (b_j, B_j)_{2,m}, (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] \\ &- (\mu - 1)(\mu + \alpha - 1) \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} (1 - \beta, 1; 1), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, (1 + \alpha - \beta, 1) \\ (-1 + \mu + \alpha - \beta, 1), (b_j, B_j)_{2,m}, (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] \\ &- \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} (-\beta, 1; 1), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, (\alpha - \beta, 1) \\ (-1 + \mu + \alpha - \beta, 1), (b_j, B_j)_{2,m}, (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] \end{aligned} \tag{14}$$

where $1 \leq n \leq p - 1$ and $1 \leq m \leq q$.

Theorem 4. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$), a_j ($j = 1, \dots, p$) and b_j ($j = 1, \dots, q$) be complex numbers then there hold the following recurrence relation

$$\begin{aligned} & \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} (-\mu - 2\rho, 1; 1), \left(\frac{1}{2} - \rho, \frac{1}{2}; 1\right), (a_j, A_j; \alpha_j)_{3,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q-2}, (-2\rho, 1; 1), \left(-\mu - \nu - \rho, \frac{1}{2}; 1\right) \end{array} \right. \right] \\ &= \frac{4(\mu + \nu - 1)}{(2\nu + \mu - 1)} \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} (-1 - \mu - 2\rho, 1; 1), \left(-\frac{1}{2} - \rho, \frac{1}{2}; 1\right), (a_j, A_j; \alpha_j)_{3,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q-2}, (-2 - 2\rho, 1; 1), \left(-\mu - \nu - \rho, \frac{1}{2}; 1\right) \end{array} \right. \right] \\ &- \frac{4(\mu - 1)}{(2\nu + \mu - 1)} \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} (-\mu - 2\rho, 1; 1), \left(-\frac{1}{2} - \rho, \frac{1}{2}; 1\right), (a_j, A_j; \alpha_j)_{3,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q-2}, (-2 - 2\rho, 1; 1), \left(1 - \mu - \nu - \rho, \frac{1}{2}; 1\right) \end{array} \right. \right] \end{aligned} \tag{15}$$

where $1 \leq n \leq p$ and $1 \leq m \leq q - 2$.

Theorem 5. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$), a_j ($j = 1, \dots, p$) and b_j ($j = 1, \dots, q$) be complex numbers then there hold the following recurrence relation

$$\begin{aligned} & \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} \left(\frac{2-\mu-\rho}{2}, \frac{1}{2}; 1\right), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, \left(\frac{2+\mu-\rho}{2}, \frac{1}{2}\right) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] \\ &= \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} \left(\frac{2-\mu-\rho}{2}, \frac{1}{2}; 1\right), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, \left(\frac{\mu-\rho}{2}, \frac{1}{2}\right) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] \\ &+ \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} \left(\frac{-\mu-\rho}{2}, \frac{1}{2}; 1\right), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, \left(\frac{2+\mu-\rho}{2}, \frac{1}{2}\right) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] \end{aligned} \tag{16}$$

where $1 \leq n \leq p - 1$ and $1 \leq m \leq q$.

Theorem 6. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$), a_j ($j = 1, \dots, p$) and b_j ($j = 1, \dots, q$) be complex numbers then there hold the following recurrence relation

$$\begin{aligned} & \mu \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{c} (1 - \rho, \frac{1}{2}; 1), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, (1 + \mu - \rho, \frac{1}{2}) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] \\ &= \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{c} (1 - \rho, \frac{1}{2}; 1), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, (\mu - \rho, \frac{1}{2}) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] \\ &+ \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{c} (-\rho, \frac{1}{2}; 1), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, (1 + \mu - \rho, \frac{1}{2}) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] \quad (17) \end{aligned}$$

where $1 \leq n \leq p - 1$ and $1 \leq m \leq q$.

Theorem 7. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$), a_j ($j = 1, \dots, p$) and b_j ($j = 1, \dots, q$) be complex numbers then there hold the following recurrence relation

$$\begin{aligned} & \mu \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{c} (0, 1; 1), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q-1}, (\frac{-1+\mu}{2}, \frac{1}{2}; 1) \end{array} \right. \right] \\ &= \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{c} (-1, 1; 1), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q-1}, (\frac{-1+\mu}{2}, \frac{1}{2}; 1) \end{array} \right. \right] \\ &- 2 \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{c} (0, 1; 1), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q-1}, (\frac{1+\mu}{2}, \frac{1}{2}; 1) \end{array} \right. \right] \quad (18) \end{aligned}$$

where $1 \leq n \leq p$ and $1 \leq m \leq q - 1$.

Theorem 8. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$), a_j ($j = 1, \dots, p$) and b_j ($j = 1, \dots, q$) be complex numbers then there hold the following recurrence relation

$$\begin{aligned} & (\mu - \rho + 1) \\ & \times \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{c} (\frac{1}{2}, \frac{1}{2}; 1), (0, \frac{1}{2}; 1), (a_j, A_j; \alpha_j)_{3,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q-2}, (\frac{1}{2} + \frac{\mu}{2} + \frac{\rho}{2}, \frac{1}{2}; 1), (-1 - \frac{\mu}{2} + \frac{\rho}{2}, \frac{1}{2}; 1) \end{array} \right. \right] \\ &= (2\mu + 1) \\ & \times \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{c} (0, \frac{1}{2}; 1), (-\frac{1}{2}, \frac{1}{2}; 1), (a_j, A_j; \alpha_j)_{3,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q-2}, (-\frac{1}{2} + \frac{\mu}{2} + \frac{\rho}{2}, \frac{1}{2}; 1), (-1 - \frac{\mu}{2} + \frac{\rho}{2}, \frac{1}{2}; 1) \end{array} \right. \right] \\ &- (\mu + \rho) \\ & \times \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{c} (\frac{1}{2}, \frac{1}{2}; 1), (0, \frac{1}{2}; 1), (a_j, A_j; \alpha_j)_{3,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q-2}, (-\frac{1}{2} + \frac{\mu}{2} + \frac{\rho}{2}, \frac{1}{2}; 1), (-\frac{\mu}{2} + \frac{\rho}{2}, \frac{1}{2}; 1) \end{array} \right. \right] \quad (19) \end{aligned}$$

where $1 \leq n \leq p$ and $1 \leq m \leq q - 2$.

Proof: To establish the recurrence relation (12), we have to prove

$$\begin{aligned}
 & \mu \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} \left(\frac{1}{2} + \frac{\rho}{2} - \mu, \frac{1}{2}; 1\right), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-3}, \\ (\rho, 1), (b_j, B_j)_{2,m}, \\ \left(\frac{1}{2} + \frac{\rho}{2} + \mu, \frac{1}{2}\right), \left(\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}\right), \left(\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}\right) \\ (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] \\
 & - \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} \left(\frac{1}{2} + \frac{\rho}{2} - \mu, \frac{1}{2}; 1\right), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-3}, \\ (\rho - 1, 1), (b_j, B_j)_{2,m}, \\ \left(-\frac{1}{2} + \frac{\rho}{2} + \mu, \frac{1}{2}\right), \left(\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}\right), \left(-\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}\right) \\ (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] \\
 & - \overline{H}_{p,q}^{m,n} \left[z \left| \begin{array}{l} \left(-\frac{1}{2} + \frac{\rho}{2} - \mu, \frac{1}{2}; 1\right), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-3}, \\ (\rho - 1, 1), (b_j, B_j)_{2,m}, \\ \left(\frac{1}{2} + \frac{\rho}{2} + \mu, \frac{1}{2}\right), \left(\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}\right), \left(-\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}\right) \\ (b_j, B_j; \beta_j)_{m+1,q} \end{array} \right. \right] = 0.
 \end{aligned} \tag{20}$$

Expressing \overline{H} -function occurring in L.H.S. of (20) in its well known contour integral form, we have

$$\begin{aligned}
 & \frac{\mu}{2\pi\omega} \int_L \frac{\prod_{j=2}^m \Gamma(b_j - B_j s) \prod_{j=2}^n \Gamma(1 - a_j + A_j s)^{\alpha_j}}{\prod_{j=m+1}^q \Gamma(1 - b_j + B_j s)^{\beta_j} \prod_{j=n+1}^{p-3} \Gamma(a_j - A_j s)} \\
 & \times \frac{\Gamma(\rho - s) \Gamma\{\frac{1}{2}(1 - \rho + 2\mu + s)\}}{\Gamma\{\frac{1}{2}(1 + \rho + 2\mu - s)\} \Gamma\{\frac{1}{2}(1 + \rho - s)\} \Gamma\{\frac{1}{2}(1 + \rho - s)\}} z^s ds \\
 & - \frac{1}{2\pi\omega} \int_L \frac{\prod_{j=2}^m \Gamma(b_j - B_j s) \prod_{j=2}^n \Gamma(1 - a_j + A_j s)^{\alpha_j}}{\prod_{j=m+1}^q \Gamma(1 - b_j + B_j s)^{\beta_j} \prod_{j=n+1}^{p-3} \Gamma(a_j - A_j s)} \\
 & \times \frac{\Gamma(\rho - 1 - s) \Gamma\{\frac{1}{2}(1 - \rho + 2\mu + s)\}}{\Gamma\{\frac{1}{2}(\rho - 1 + 2\mu - s)\} \Gamma\{\frac{1}{2}(1 + \rho - s)\} \Gamma\{\frac{1}{2}(\rho - 1 - s)\}} z^s ds \\
 & - \frac{1}{2\pi\omega} \int_L \frac{\prod_{j=2}^m \Gamma(b_j - B_j s) \prod_{j=2}^n \Gamma(1 - a_j + A_j s)^{\alpha_j}}{\prod_{j=m+1}^q \Gamma(1 - b_j + B_j s)^{\beta_j} \prod_{j=n+1}^{p-3} \Gamma(a_j - A_j s)} \\
 & \times \frac{\Gamma(\rho - 1 - s) \Gamma\{\frac{1}{2}(3 - \rho + 2\mu + s)\}}{\Gamma\{\frac{1}{2}(1 + \rho + 2\mu - s)\} \Gamma\{\frac{1}{2}(1 + \rho - s)\} \Gamma\{\frac{1}{2}(\rho - 1 - s)\}} z^s ds,
 \end{aligned}$$

now let

$$\chi(s) = \frac{\prod_{j=2}^m \Gamma(b_j - B_j s) \prod_{j=2}^n \Gamma(1 - a_j + A_j s)^{\alpha_j}}{\prod_{j=m+1}^q \Gamma(1 - b_j + B_j s)^{\beta_j} \prod_{j=n+1}^{p-3} \Gamma(a_j - A_j s)},$$

then the above expression can be written as

$$\begin{aligned} &= \frac{1}{2\pi\omega} \int_L \chi(s) \frac{\Gamma(\rho - s) \Gamma\{\frac{1}{2}(1 - \rho + 2\mu + s)\}}{\Gamma\{\frac{1}{2}(1 + \rho + 2\mu - s)\} \Gamma\{\frac{1}{2}(1 + \rho - s)\} \Gamma\{\frac{1}{2}(1 + \rho - s)\}} \\ &\times \left[\mu - \frac{(-1 + \rho + 2\mu - s)(-1 + \rho - s)}{4(\rho - 1 - s)} - \frac{(1 - \rho + 2\mu + s)(-1 + \rho - s)}{4(\rho - 1 - s)} \right] z^s ds \\ &= \frac{1}{2\pi\omega} \int_L \chi(s) \frac{\Gamma(\rho - s) \Gamma\{\frac{1}{2}(1 - \rho + 2\mu + s)\}}{\Gamma\{\frac{1}{2}(1 + \rho + 2\mu - s)\} \Gamma\{\frac{1}{2}(1 + \rho - s)\} \Gamma\{\frac{1}{2}(1 + \rho - s)\}} \times 0 \times z^s ds = 0. \end{aligned}$$

Hence the recurrence relation (12) is established.

On applying Gamma function formula $\Gamma(n + 1) = n\Gamma(n)$ and following similar lines of proof of recurrence relation (12), recurrence relations (13) to (19) can easily be established. We omit the details.

3. SPECIAL CASES

The following recurrence relations for the H -function are obtained if we take α_j and β_j to be equal to unity in recurrence relations (12) to (19) respectively.

Corollary 1. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$), a_j ($j = 1, \dots, p$) and b_j ($j = 1, \dots, q$) be complex numbers then there hold the following recurrence relation

$$\begin{aligned} &\mu H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (\frac{1}{2} + \frac{\rho}{2} - \mu, \frac{1}{2}), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p-3}, \\ (\rho, 1), (b_j, B_j)_{2,m}, \\ (\frac{1}{2} + \frac{\rho}{2} + \mu, \frac{1}{2}), (\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}), (\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}) \\ (b_j, B_j)_{m+1,q} \end{array} \right. \right] \\ &= H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (\frac{1}{2} + \frac{\rho}{2} - \mu, \frac{1}{2}), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p-3}, \\ (\rho - 1, 1), (b_j, B_j)_{2,m}, \\ (-\frac{1}{2} + \frac{\rho}{2} + \mu, \frac{1}{2}), (\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}), (-\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}) \\ (b_j, B_j)_{m+1,q} \end{array} \right. \right] \\ &+ H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (-\frac{1}{2} + \frac{\rho}{2} - \mu, \frac{1}{2}), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p-3}, \\ (\rho - 1, 1), (b_j, B_j)_{2,m}, \\ (\frac{1}{2} + \frac{\rho}{2} + \mu, \frac{1}{2}), (\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}), (-\frac{1}{2} + \frac{\rho}{2}, \frac{1}{2}) \\ (b_j, B_j)_{m+1,q} \end{array} \right. \right] \quad (21) \end{aligned}$$

where $1 \leq n \leq p - 3$ and $1 \leq m \leq q$.

Corollary 2. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$), a_j ($j = 1, \dots, p$) and b_j ($j = 1, \dots, q$) be complex numbers then there hold the following recurrence relation

$$\begin{aligned} & (2\mu + 1)H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (-1, 1), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q-2}, (-1 - \frac{\mu}{2}, \frac{1}{2}), (-\frac{1}{2} + \frac{\mu}{2}, \frac{1}{2}) \end{array} \right. \right] \\ &= 2(\mu + 1)H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (0, 1), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q-2}, (-1 - \frac{\mu}{2}, \frac{1}{2}), (\frac{1}{2} + \frac{\mu}{2}, \frac{1}{2}) \end{array} \right. \right] \\ &+ 2\mu H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (0, 1), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q-2}, (-\frac{\mu}{2}, \frac{1}{2}), (-\frac{1}{2} + \frac{\mu}{2}, \frac{1}{2}) \end{array} \right. \right] \end{aligned} \tag{22}$$

where $1 \leq n \leq p$ and $1 \leq m \leq q - 2$.

Corollary 3. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$), a_j ($j = 1, \dots, p$) and b_j ($j = 1, \dots, q$) be complex numbers then there hold the following recurrence relation

$$\begin{aligned} & H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (1 - \beta, 1), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, (1 + \alpha - \beta, 1) \\ (1 + \mu + \alpha - \beta, 1), (b_j, B_j)_{2,m}, (b_j, B_j)_{m+1,q} \end{array} \right. \right] \\ &= (2\mu + \alpha - 1) \\ &\times H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (1 - \beta, 1), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, (1 + \alpha - \beta, 1) \\ (\mu + \alpha - \beta, 1), (b_j, B_j)_{2,m}, (b_j, B_j)_{m+1,q} \end{array} \right. \right] \\ &- (\mu - 1)(\mu + \alpha - 1) \\ &\times H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (1 - \beta, 1), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, (1 + \alpha - \beta, 1) \\ (-1 + \mu + \alpha - \beta, 1), (b_j, B_j)_{2,m}, (b_j, B_j)_{m+1,q} \end{array} \right. \right] \\ &- H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (-\beta, 1), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, (\alpha - \beta, 1) \\ (-1 + \mu + \alpha - \beta, 1), (b_j, B_j)_{2,m}, (b_j, B_j)_{m+1,q} \end{array} \right. \right] \end{aligned} \tag{23}$$

where $1 \leq n \leq p - 1$ and $1 \leq m \leq q$.

Corollary 4. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$), a_j ($j = 1, \dots, p$) and b_j ($j = 1, \dots, q$) be complex numbers then there hold the following recurrence relation

$$\begin{aligned} & H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (-\mu - 2\rho, 1), (\frac{1}{2} - \rho, \frac{1}{2}), (a_j, A_j)_{3,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q-2}, (-2\rho, 1), (-\mu - \nu - \rho, \frac{1}{2}) \end{array} \right. \right] \\ &= \frac{4(\mu + \nu - 1)}{(2\nu + \mu - 1)} H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (-1 - \mu - 2\rho, 1), (-\frac{1}{2} - \rho, \frac{1}{2}), (a_j, A_j)_{3,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q-2}, (-2 - 2\rho, 1), (-\mu - \nu - \rho, \frac{1}{2}) \end{array} \right. \right] \\ &- \frac{4(\mu - 1)}{(2\nu + \mu - 1)} H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (-\mu - 2\rho, 1), (-\frac{1}{2} - \rho, \frac{1}{2}), (a_j, A_j)_{3,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q-2}, (-2 - 2\rho, 1), (1 - \mu - \nu - \rho, \frac{1}{2}) \end{array} \right. \right] \end{aligned} \tag{24}$$

where $1 \leq n \leq p$ and $1 \leq m \leq q - 2$.

Corollary 5. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$), a_j ($j = 1, \dots, p$) and b_j ($j = 1, \dots, q$) be complex numbers then there hold the following recurrence relation

$$\begin{aligned} & \mu H_{p,q}^{m,n} \left[z \left| \begin{array}{c} \left(\frac{2-\mu-\rho}{2}, \frac{1}{2} \right), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, \left(\frac{2+\mu-\rho}{2}, \frac{1}{2} \right) \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q} \end{array} \right. \right] \\ &= H_{p,q}^{m,n} \left[z \left| \begin{array}{c} \left(\frac{2-\mu-\rho}{2}, \frac{1}{2} \right), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, \left(\frac{\mu-\rho}{2}, \frac{1}{2} \right) \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q} \end{array} \right. \right] \\ &+ H_{p,q}^{m,n} \left[z \left| \begin{array}{c} \left(\frac{-\mu-\rho}{2}, \frac{1}{2} \right), (a_j, A_j; \alpha_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, \left(\frac{2+\mu-\rho}{2}, \frac{1}{2} \right) \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q} \end{array} \right. \right] \end{aligned} \quad (25)$$

where $1 \leq n \leq p-1$ and $1 \leq m \leq q$.

Corollary 6. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$), a_j ($j = 1, \dots, p$) and b_j ($j = 1, \dots, q$) be complex numbers then there hold the following recurrence relation

$$\begin{aligned} & \mu H_{p,q}^{m,n} \left[z \left| \begin{array}{c} \left(1-\rho, \frac{1}{2} \right), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, \left(1+\mu-\rho, \frac{1}{2} \right) \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q} \end{array} \right. \right] \\ &= H_{p,q}^{m,n} \left[z \left| \begin{array}{c} \left(1-\rho, \frac{1}{2} \right), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, \left(\mu-\rho, \frac{1}{2} \right) \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q} \end{array} \right. \right] \\ &+ H_{p,q}^{m,n} \left[z \left| \begin{array}{c} \left(-\rho, \frac{1}{2} \right), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p-1}, \left(1+\mu-\rho, \frac{1}{2} \right) \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q} \end{array} \right. \right] \end{aligned} \quad (26)$$

where $1 \leq n \leq p-1$ and $1 \leq m \leq q$.

Corollary 7. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$), a_j ($j = 1, \dots, p$) and b_j ($j = 1, \dots, q$) be complex numbers then there hold the following recurrence relation

$$\begin{aligned} & \mu H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (0, 1), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q-1}, \left(\frac{-1+\mu}{2}, \frac{1}{2} \right) \end{array} \right. \right] \\ &= H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (-1, 1), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q-1}, \left(\frac{-1+\mu}{2}, \frac{1}{2} \right) \end{array} \right. \right] \\ &- 2H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (0, 1), (a_j, A_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q-1}, \left(\frac{1+\mu}{2}, \frac{1}{2} \right) \end{array} \right. \right] \end{aligned} \quad (27)$$

where $1 \leq n \leq p$ and $1 \leq m \leq q-1$.

Corollary 8. If $z \in \mathbb{C}$, m, n, p, q be non-negative integers, $A_j > 0$ ($j = 1, \dots, p$), $B_j > 0$ ($j = 1, \dots, q$), a_j ($j = 1, \dots, p$) and b_j ($j = 1, \dots, q$) be complex numbers then there hold the following recurrence relation

$$\begin{aligned} & (\mu - \rho + 1) H_{p,q}^{m,n} \left[z \left| \begin{array}{c} \left(\frac{1}{2}, \frac{1}{2} \right), \left(0, \frac{1}{2} \right), (a_j, A_j)_{3,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q-2}, \left(\frac{1}{2} + \frac{\mu}{2} + \frac{\rho}{2}, \frac{1}{2} \right), \left(-1 - \frac{\mu}{2} + \frac{\rho}{2}, \frac{1}{2} \right) \end{array} \right. \right] \\ &= (2\mu + 1) H_{p,q}^{m,n} \left[z \left| \begin{array}{c} \left(0, \frac{1}{2} \right), \left(-\frac{1}{2}, \frac{1}{2} \right), (a_j, A_j)_{3,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q-2}, \left(-\frac{1}{2} + \frac{\mu}{2} + \frac{\rho}{2}, \frac{1}{2} \right), \left(-1 - \frac{\mu}{2} + \frac{\rho}{2}, \frac{1}{2} \right) \end{array} \right. \right] \end{aligned}$$

$$\begin{aligned}
 & -(\mu + \rho)H_{p,q}^{m,n} \left[z \left| \begin{matrix} (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2}), (a_j, A_j; \alpha_j)_{3,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j)_{m+1,q-2}, (-\frac{1}{2} + \frac{\mu}{2} + \frac{\rho}{2}, \frac{1}{2}), (-\frac{\mu}{2} + \frac{\rho}{2}, \frac{1}{2}) \end{matrix} \right. \right] \\
 & \hspace{15em} (28)
 \end{aligned}$$

where $1 \leq n \leq p$ and $1 \leq m \leq q - 2$.

Further on reducing H -function to Wright generalized hypergeometric function due to the relation

$$\begin{aligned}
 {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} ; z \right] &= \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(a_j + A_j k)}{q^k \prod_{j=1}^q \Gamma(b_j + B_j k)} \frac{z^k}{k!}, \\
 &= H_{p,q+1}^{1,p} \left[-z \left| \begin{matrix} (1 - a_1, A_1), \dots, (1 - a_p, A_p) \\ (0, 1), (1 - b_1, B_1), \dots, (1 - b_q, B_q) \end{matrix} \right. \right]
 \end{aligned}$$

We easily obtain recurrence relations of Wright generalized hypergeometric function ${}_p\Psi_q$.

Corollary 9. If $z \in \mathbb{C}$, $a_j(j = 1, 2, \dots, p), b_j(j = 1, 2, \dots, q) \in \mathbb{C}$, $A_j(j = 1, 2, \dots, p), B_j(j = 1, 2, \dots, q)$ be positive real numbers then there hold the following recurrence relation

$$\begin{aligned}
 & \mu {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-2}, A_{p-2}), (\rho, -1), (\frac{1-\rho+2\mu}{2}, \frac{1}{2}) \\ (b_1, B_1), \dots, (b_{q-3}, B_{q-3}), (\frac{1+\rho+2\mu}{2}, -\frac{1}{2}), (\frac{1+\rho}{2}, -\frac{1}{2}), (\frac{1+\rho}{2}, -\frac{1}{2}) \end{matrix} ; z \right] \\
 &= {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-2}, A_{p-2}), (\rho - 1, -1), (\frac{1-\rho+2\mu}{2}, \frac{1}{2}) \\ (b_1, B_1), \dots, (b_{q-3}, B_{q-3}), (\frac{-1+\rho+2\mu}{2}, -\frac{1}{2}), (\frac{1+\rho}{2}, -\frac{1}{2}), (\frac{-1+\rho}{2}, -\frac{1}{2}) \end{matrix} ; z \right] \\
 &+ {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-2}, A_{p-2}), (\rho - 1, -1), (\frac{3-\rho+2\mu}{2}, \frac{1}{2}) \\ (b_1, B_1), \dots, (b_{q-3}, B_{q-3}), (\frac{1+\rho+2\mu}{2}, -\frac{1}{2}), (\frac{1+\rho}{2}, -\frac{1}{2}), (\frac{-1+\rho}{2}, -\frac{1}{2}) \end{matrix} ; z \right]. \\
 & \hspace{15em} (29)
 \end{aligned}$$

Corollary 10. If $z \in \mathbb{C}$, $a_j(j = 1, 2, \dots, p), b_j(j = 1, 2, \dots, q) \in \mathbb{C}$, $A_j(j = 1, 2, \dots, p), B_j(j = 1, 2, \dots, q)$ be positive real numbers then there hold the following recurrence relation

$$\begin{aligned}
 & (2\mu + 1) {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-1}, A_{p-1}), (2, 1) \\ (b_1, B_1), \dots, (b_{q-2}, B_{q-2}), (\frac{4+\mu}{2}, \frac{1}{2}), (\frac{3-\mu}{2}, \frac{1}{2}) \end{matrix} ; z \right] \\
 &= 2(\mu + 1) {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-1}, A_{p-1}), (1, 1) \\ (b_1, B_1), \dots, (b_{q-2}, B_{q-2}), (\frac{4+\mu}{2}, \frac{1}{2}), (\frac{1-\mu}{2}, \frac{1}{2}) \end{matrix} ; z \right] \\
 &+ 2\mu {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-1}, A_{p-1}), (1, 1) \\ (b_1, B_1), \dots, (b_{q-2}, B_{q-2}), (\frac{2+\mu}{2}, \frac{1}{2}), (\frac{3-\mu}{2}, \frac{1}{2}) \end{matrix} ; z \right] \\
 & \hspace{15em} (30)
 \end{aligned}$$

Corollary 11. If $z \in \mathbb{C}$, $a_j(j = 1, 2, \dots, p), b_j(j = 1, 2, \dots, q) \in \mathbb{C}$, $A_j(j = 1, 2, \dots, p), B_j(j = 1, 2, \dots, q)$ be positive real numbers then there hold the following

recurrence relation

$$\begin{aligned}
& {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-2}, A_{p-2}), (\beta, 1), (1 + \alpha - \beta + \mu, -1) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}), (1 + \alpha - \beta, -1) \end{matrix} \middle| z \right] \\
&= (2\mu + \alpha - 1) {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-2}, A_{p-2}), (\beta, 1), (\alpha - \beta + \mu, -1) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}), (1 + \alpha - \beta, -1) \end{matrix} \middle| z \right] \\
&- (\mu - 1)(\mu + \alpha - 1) {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-2}, A_{p-2}), (\beta, 1), (\alpha - \beta + \mu - 1, -1) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}), (1 + \alpha - \beta, -1) \end{matrix} \middle| z \right] \\
&- {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-2}, A_{p-2}), (\beta + 1, 1), (\alpha - \beta + \mu - 1, -1) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}), (\alpha - \beta, -1) \end{matrix} \middle| z \right]. \quad (31)
\end{aligned}$$

Corollary 12. If $z \in \mathbb{C}$, $a_j(j = 1, 2, \dots, p), b_j(j = 1, 2, \dots, q) \in \mathbb{C}$, $A_j(j = 1, 2, \dots, p), B_j(j = 1, 2, \dots, q)$ be positive real numbers then there hold the following recurrence relation

$$\begin{aligned}
& {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-2}, A_{p-2}), (1 + \mu + 2\rho, 1), \left(\frac{1+2\rho}{2}, \frac{1}{2}\right) \\ (b_1, B_1), \dots, (b_{q-2}, B_{q-2}), (1 + 2\rho, 1), \left(1 + \mu + \nu + \rho, \frac{1}{2}\right) \end{matrix} \middle| z \right] \\
&= \frac{4(\mu + \nu - 1)}{(2\nu + \mu - 1)} {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-2}, A_{p-2}), (2 + \mu + 2\rho, 1), \left(\frac{3+2\rho}{2}, \frac{1}{2}\right) \\ (b_1, B_1), \dots, (b_{q-2}, B_{q-2}), (3 + 2\rho, 1), \left(1 + \mu + \nu + \rho, \frac{1}{2}\right) \end{matrix} \middle| z \right] \\
&- \frac{4(\mu - 1)}{(2\nu + \mu - 1)} {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-2}, A_{p-2}), (1 + \mu + 2\rho, 1), \left(\frac{3+2\rho}{2}, \frac{1}{2}\right) \\ (b_1, B_1), \dots, (b_{q-2}, B_{q-2}), (3 + 2\rho, 1), \left(\mu + \nu + \rho, \frac{1}{2}\right) \end{matrix} \middle| z \right]. \quad (32)
\end{aligned}$$

Corollary 13. If $z \in \mathbb{C}$, $a_j(j = 1, 2, \dots, p), b_j(j = 1, 2, \dots, q) \in \mathbb{C}$, $A_j(j = 1, 2, \dots, p), B_j(j = 1, 2, \dots, q)$ be positive real numbers then there hold the following recurrence relation

$$\begin{aligned}
& \mu {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-1}, A_{p-1}), \left(\frac{\mu+\rho}{2}, \frac{1}{2}\right) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}), \left(\frac{2+\mu-\rho}{2}, -\frac{1}{2}\right) \end{matrix} \middle| z \right] \\
&= {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-1}, A_{p-1}), \left(\frac{\mu+\rho}{2}, \frac{1}{2}\right) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}), \left(\frac{\mu-\rho}{2}, -\frac{1}{2}\right) \end{matrix} \middle| z \right] \\
&+ {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-1}, A_{p-1}), \left(\frac{2+\mu+\rho}{2}, \frac{1}{2}\right) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}), \left(\frac{2+\mu-\rho}{2}, -\frac{1}{2}\right) \end{matrix} \middle| z \right]. \quad (33)
\end{aligned}$$

Corollary 14. If $z \in \mathbb{C}$, $a_j(j = 1, 2, \dots, p), b_j(j = 1, 2, \dots, q) \in \mathbb{C}$, $A_j(j = 1, 2, \dots, p), B_j(j = 1, 2, \dots, q)$ be positive real numbers then there hold the following recurrence relation

$$\begin{aligned}
& \mu {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-1}, A_{p-1}), \left(\rho, \frac{1}{2}\right) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}), \left(1 - \rho + \mu, -\frac{1}{2}\right) \end{matrix} \middle| z \right] \\
&= {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-1}, A_{p-1}), \left(\rho, \frac{1}{2}\right) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}), \left(\mu - \rho, -\frac{1}{2}\right) \end{matrix} \middle| z \right] \\
&+ {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-1}, A_{p-1}), \left(1 + \rho, \frac{1}{2}\right) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}), \left(1 + \mu - \rho, -\frac{1}{2}\right) \end{matrix} \middle| z \right]. \quad (34)
\end{aligned}$$

Corollary 15. If $z \in \mathbb{C}$, $a_j(j = 1, 2, \dots, p), b_j(j = 1, 2, \dots, q) \in \mathbb{C}$, $A_j(j = 1, 2, \dots, p), B_j(j = 1, 2, \dots, q)$ be positive real numbers then there hold the following

recurrence relation

$$\begin{aligned} & \mu {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-1}, A_{p-1}), (1, 1) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}), \left(\frac{3-\mu}{2}, \frac{1}{2}\right) \end{matrix} \middle| z \right] \\ &= {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-1}, A_{p-1}), (2, 1) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}), \left(\frac{3-\mu}{2}, \frac{1}{2}\right) \end{matrix} \middle| z \right] \\ &- 2 {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-1}, A_{p-1}), (1, 1) \\ (b_1, B_1), \dots, (b_{q-1}, B_{q-1}), \left(\frac{1-\mu}{2}, \frac{1}{2}\right) \end{matrix} \middle| z \right]. \end{aligned} \quad (35)$$

Corollary 16. If $z \in \mathbb{C}$, $a_j (j = 1, 2, \dots, p), b_j (j = 1, 2, \dots, q) \in \mathbb{C}$, $A_j (j = 1, 2, \dots, p), B_j (j = 1, 2, \dots, q)$ be positive real numbers then there hold the following recurrence relation

$$\begin{aligned} & (\mu - \rho + 1) {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-2}, A_{p-2}), \left(\frac{1}{2}, \frac{1}{2}\right), \left(1, \frac{1}{2}\right) \\ (b_1, B_1), \dots, (b_{q-2}, B_{q-2}), \left(\frac{1-\mu-\rho}{2}, \frac{1}{2}\right), \left(\frac{4+\mu-\rho}{2}, \frac{1}{2}\right) \end{matrix} \middle| z \right] \\ &= (2\mu + 1) {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-2}, A_{p-2}), \left(1, \frac{1}{2}\right), \left(\frac{3}{2}, \frac{1}{2}\right) \\ (b_1, B_1), \dots, (b_{q-2}, B_{q-2}), \left(\frac{3-\mu-\rho}{2}, \frac{1}{2}\right), \left(\frac{4+\mu-\rho}{2}, \frac{1}{2}\right) \end{matrix} \middle| z \right] \\ &- (\mu + \rho) {}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_{p-2}, A_{p-2}), \left(\frac{1}{2}, \frac{1}{2}\right), \left(1, \frac{1}{2}\right) \\ (b_1, B_1), \dots, (b_{q-2}, B_{q-2}), \left(\frac{3-\mu-\rho}{2}, \frac{1}{2}\right), \left(\frac{2+\mu-\rho}{2}, \frac{1}{2}\right) \end{matrix} \middle| z \right]. \end{aligned} \quad (36)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

REFERENCES

- [1] A.A. Inayat-Hussain, New properties of hypergeometric series derivable from Feynman integrals. I. Transformation and reduction formulae, J. Phys. A, 20, no. 13, 4109-4117, 1987.
- [2] A.A. Inayat-Hussain, New properties of hypergeometric series derivable from Feynman integrals: II. A generalization of the H -function, J. Phys., A, 20, no. 13, 4119-4128, 1987.
- [3] R.G. Buschman and H.M. Srivastava, The \overline{H} -function associated with certain class of Feynman integrals, J. Phys. A . Math. Gen. 23, 4707-4710, 1990.
- [4] A.M. Mathai, R.K. Saxena and H. J. Haubold, The H -function : Theory and Applications, Springer, New York, 2010.
- [5] A.A. Kilbas and M. Saigo , H -Transforms, Theory and Applications, Chapman & Hall/CRC Press, Boca Raton, FL., 2004.
- [6] E.M. Wright, The asymptotic expansion of the generalized hypergeometric function. Proc. London Math. Soc. 46, no. 2, 389-408, 1940.
- [7] E.M. Wright, The asymptotic expansion of integral functions defined by Taylor series , Philos. Trans. Roy. Soc. London, Ser. A. 238, 423-451, 1940.
- [8] E.M. Wright, The asymptotic expansion of the generalized hypergeometric function II, Philos. Trans. Roy. Soc. London, Ser. A. 239, 217-232, 1941.
- [9] U.C. Jain, Certain recurrence relations for Fox's H -function, Proc. Nat. Inst. Sci. India Part A, 33, 19-24, 1967.
- [10] P. Anandani, On some recurrence formulae for the H -function, Ann. Polon. Math., 21, 113-117, 1969c.
- [11] S.P. Goyal, On certain recurrence relations for Fox's H -function, Proc. Indian Acad. Sci. Sect. A, 41, 201-208, 1971a.
- [12] R.K. Raina, Some recurrence relations for the H -function, Math. Education Sect. A. 10, 45-49, 1976.

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