

EXPONENTIAL GROWTH OF SOLUTION AND ASYMPTOTIC STABILITY RESULTS FOR HILFER FRACTIONAL WEIGHTED P-LAPLACIAN INITIAL VALUE PROBLEM WITH DUFFING-TYPE OSCILLATOR

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ABSTRACT. In this paper, we give global existence and asymptotic stability results of positive exponentially growing solutions of the initial value problem with nonautonomous and variable coefficients Duffing type equation by using fixed point arguments.

1. INTRODUCTION

In this paper, we consider the following initial value problem with Duffing type equation with variable coefficients

$$\begin{cases} -D_{0+}^{\alpha,\omega,\sigma}(\phi(h.u'))(t) + D_{0+}^{\beta,\omega,\sigma}(\delta.u)(t) + \eta(t).p(u(t)) + q(t)f(t, u(t)) = 0, & t > 0, \\ u(0) = 0, \end{cases} \quad (1)$$

where $\phi(x) = |x|^{a-2}.x$, $a > 1$, $p(u) = u^m$, $m \in \mathbb{N}^*$ is a polynomial function, $D_{0+}^{\mu,\omega,\sigma}$ is the σ -Hilfer fractional derivative of order $\mu \in \{\alpha, \beta\}$ and type $0 \leq \omega \leq 1$ with $0 < \beta < \alpha < 1$.

In the past two decades, there has been considerable attention devoted to the study of solutions of the Duffing equations, see [4, 7, 9, 10, 11, 12, 14, 15], and references therein. The Duffing oscillator is one of the prototype systems of nonlinear dynamics. It first became popular for studying anharmonic oscillations and, later, chaotic nonlinear dynamics in the wake of early studies by the engineer Georg Duffing (1861–1944), it is a non-linear second-order differential equation used to model certain damped and driven oscillators. The equation is given by $\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega t)$ where the (unknown) function $x(t)$ is the displacement at time t , the first derivative \dot{x} is the velocity, and the second time \ddot{x} derivative is acceleration.

The equation describes the motion of a damped oscillator with a more complex

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potential than in simple harmonic motion (which corresponds to the case $\delta = \beta = 0$), in physical terms, it models, for example, an elastic pendulum whose spring's stiffness does not exactly obey Hooke's law.

The Duffing equation is an example of a dynamical system that exhibits chaotic behavior. Moreover, the Duffing system presents in the frequency response the jump resonance phenomenon that is a sort of frequency hysteresis behaviour, where δ controls the amount of damping, α controls the linear stiffness, β controls the amount of non-linearity in the restoring force, γ is the amplitude of the periodic driving force and ω is the angular frequency of the periodic driving force. In most physical oscillation systems, the amplitude of excitation (force or moment) usually varies over time, and some external and internal excitation impulses can occur.

In [7] and [12], fractional Duffing's equations were studied in continuous and discrete cases and in the presence of both harmonic and nonharmonic external perturbations. In [14], the existence results for Duffing equations with a p-Laplacian operator \mathbb{C} .

Motivated by the cited papers, in the present article, we discuss the existence and asymptotic stability of positive and exponentially growing solutions for the problem (1).

Throughout the article, we assume that $\sigma \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ is increasing with $\sigma(0) = 0$ and $\sigma'(t) \neq 0$ for all $t \geq 0$, $p : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a two variable polynomial function and $f : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function such that $f(\cdot, 0)$ does not vanish identically on any subinterval of \mathbb{R}^+ , and there exists $k \in \mathbb{N}^*$, $\lambda, \epsilon > 0$ and $r > 0$ such that for all $x \in [0, r]$

$$f(\cdot, x) \geq 0 \text{ and } \sup \{f(t, e^{kt}x), t > 0\} \leq \lambda x + \epsilon \tag{2}$$

$h, \delta, \eta, q : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are the measurable functions, where h and q do not vanish identically on any subinterval of \mathbb{R}^+ and there exists $k > 0$ such that

$$\int_0^{+\infty} \frac{ds}{h(s)} < \infty, \hat{\delta}_k \in L_{\sigma}^{\alpha-\beta}(\mathbb{R}^+, \mathbb{R}^+) \text{ and } \bar{\eta}_k, \check{q}_k \in L_{\sigma}^{\alpha}(\mathbb{R}^+, \mathbb{R}^+) \tag{3}$$

where $\bar{\eta}_k(s) = e^{-(a-m-1)ks}\eta(s)$, $\check{q}_k(s) = \phi(e^{-ks})q(s)$ and $\hat{\delta}_k(s) = e^{-k(a-2)s}\delta(s)$, and for $\mu > 0$

$$L_{\sigma}^{\mu}(\mathbb{R}^+, \mathbb{R}^+) = \left\{ u : \mathbb{R}^+ \rightarrow \mathbb{R}^+, \sup_{x \geq 0} \int_0^x \sigma'(s) (\sigma(x) - \sigma(s))^{\mu-1} u(s) ds < \infty \right\}.$$

In this paper, we present the existence results of nontrivial positive, global and exponentially growing solution and asymptotic stability of the initial value problem (1).

For sake of completeness let us recall some basic facts needed in this paper. Let E be a real Banach space equipped with its norm denoted $\|\cdot\|$. A nonempty closed convex subset P of E is said to be a cone if $P \cap (-P) = 0$ and $(tP) \subset P$ for all $t \geq 0$. It is well known that a cone P induces a partial order in the Banach space E . We write for all $x, y \in E$; $x \leq y$ if $y - x \in P$.

The mapping $L : E \rightarrow E$ is said to be positive in P if $L(P) \subset P$, and compact if it is continuous and $L(B)$ is relatively compact in E for all bounded subset B of E .

Definition 1. [13] Let $a \in \mathbb{R}^+$ and $\alpha > 0$. Also, let $\sigma(x)$ be an increasing and positive function having a continuous derivative $\sigma'(x)$ on $(a, +\infty)$. Then the left-sided fractional integral of a function u with respect to another function σ on \mathbb{R}^+ is defined by

$$I_{a^+}^{\alpha, \sigma} u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \sigma'(t) (\sigma(x) - \sigma(t))^{\alpha-1} u(t) dt.$$

In the case $\alpha = 0$, this integral is interpreted as the identity operator $I_{a^+}^{0, \sigma} u = u$.

Definition 2. [13] Let $\alpha \in (n-1, n)$ with $n \in \mathbb{N}$, and $u, \sigma \in C^n(\mathbb{R}^+, \mathbb{R})$ two functions such that σ is increasing and $\sigma'(t) \neq 0$, for all $t \in \mathbb{R}^+$. The σ -Hilfer fractional derivative $D_{a^+}^{\alpha, \omega, \sigma}$ of u of order $n-1 < \alpha < n$ and type $0 \leq \omega \leq 1$ is defined by

$$D_{a^+}^{\alpha, \omega, \sigma} u(x) = I_{a^+}^{\omega(n-\alpha), \sigma} \left(\frac{1}{\sigma'(x)} \frac{\partial}{\partial x} \right)^n I_{a^+}^{(1-\omega)(n-\alpha), \sigma} u(x).$$

Let's also recall the following important result [13]:

Theorem 1. If $u \in C^n(\mathbb{R}^+)$, $n-1 < \beta < \alpha < n$, $0 \leq \omega \leq 1$ and $\xi = \alpha + \omega(n-\alpha)$, then

$$I_{a^+}^{\alpha, \sigma} . D_{a^+}^{\alpha, \omega, \sigma} u(x) = u(x) - \sum_{k=1}^n \frac{(\sigma(x) - \sigma(a))^{\xi-k}}{\Gamma(\xi-k+1)} \left(\frac{1}{\sigma'(x)} \frac{\partial}{\partial x} \right)^{n-k} I_{a^+}^{(1-\omega)(n-\alpha), \sigma} u(a).$$

Moreover, $I_{a^+}^{\beta, \sigma} I_{a^+}^{\alpha-\beta, \sigma} (u) = I_{a^+}^{\alpha-\beta, \sigma} I_{a^+}^{\beta, \sigma} (u) = I_{a^+}^{\alpha, \sigma} (u)$ and ${}^H D_{a^+}^{\alpha, \omega, \sigma} I_{a^+}^{\alpha, \sigma} (u) = u$.

Remark 1.

In this paper, we assume that $\sigma(x)$ is increasing and positive with $\sigma(0) = 0$, having a continuous derivative $\sigma'(x)$ on \mathbb{R}^+ and $\sigma'(x) \neq 0$, for all $x \in \mathbb{R}^+$. If $\alpha \in (0, 1)$, then $n = 1$ and

$$I_{0^+}^{\alpha, \sigma} . D_{0^+}^{\alpha, \omega, \sigma} u(x) = u(x) - \frac{(\sigma(x))^{\xi-1}}{\Gamma(\xi)} \left(I_{0^+}^{(1-\omega)(1-\alpha), \sigma} u \right) (0).$$

Moreover, if $u : \mathbb{R}^+ \rightarrow \mathbb{R}$ is continuous then

$$\lim_{x \rightarrow 0^+} \frac{(\sigma(x))^{\xi-1}}{\Gamma(\xi)} \left(I_{0^+}^{(1-\omega)(1-\alpha), \sigma} u \right) (x) = 0$$

and so $I_{0^+}^{\alpha, \sigma} . {}^H D_{0^+}^{\alpha, \omega, \sigma} u(x) = u(x)$.

Definition 3. A positive solution u of problem (1) is said to be exponentially growing solution, if there exists the constants $c_1, c_2 > 0$ and a positive and increasing function v such that

$$u(x) \geq c_1 \exp(v(x)) \text{ for all } x \geq c_2.$$

For $k \in \mathbb{N}^*$ given in (3), let E be a real Banach space defined as

$$E = \left\{ u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{|t| \rightarrow \infty} e^{-kt} u(t) < +\infty \right\}$$

equipped with the norm $\|\cdot\|$, where for $u \in E$, $\|u\| = \sup_{t \in \mathbb{R}^+} (e^{-kt} |u(t)|)$,

$$K = \{u \in E : u(t) \geq 0 \text{ for all } t \in \mathbb{R}^+\}$$

be the cone of E .

Definition 4. Positive solutions of ivp (1) are locally attractive in K if there exists a nonempty bounded convex and open subset Ω of E such that, for any solutions $u, v \in K \cap \Omega$ of ivp (1), we can write

$$\lim_{x \rightarrow +\infty} e^{-kx} (u(x) - v(x)) = 0. \tag{4}$$

If limit (4) is uniform with respect to $K \cap \Omega$, then the solutions of ivp (1) are said to be locally asymptotically stable. Moreover, if (4) is verified for all solutions u, v , (1) is said to be asymptotically stable.

Lemma 2. [3] A non empty subset M of E is relatively compact if the following conditions hold :

- (1) M is bounded in E ,
- (2) The set $\{e^{-kt}u, u \in M\}$ is locally equicontinuous on $[0, +\infty)$, and
- (3) The set $\{e^{-kt}u, u \in M\}$ is equiconvergent, that is, for any given $\epsilon > 0$, there exists $A > 0$ such that $\left| e^{-kx}u(x) - \lim_{y \rightarrow +\infty} e^{-ky}u(y) \right| < \epsilon$, for any $x > A$, $u \in M$.

In what follows, we use of the following Schauder Fixed-Point Theorem [6] :

Theorem 3. Let E be a Banach space, let C be a nonempty bounded convex and closed subset of E , and let $T : C \rightarrow C$ be a compact and continuous map. Then T has at least one fixed point in C .

2. EXISTENCE AND ASYMPTOTIC STABILITY

We consider the operator $T : E \rightarrow C^1(\mathbb{R}^+)$ defined by

$$Tu(x) = \int_0^x \frac{1}{h(t)} \psi \left(I_{0^+}^{\alpha-\beta, \sigma} (\delta.u) + I_{0^+}^{\alpha, \sigma} (\eta.p(u) + q.f(., u)) \right) (t) dt.$$

Set

$$\Lambda_r = \sup_{x \geq 0} \psi \left(I_{0^+}^{\alpha-\beta, \sigma} (\hat{\delta}) + I_{0^+}^{\alpha, \sigma} \left(\bar{\eta}_k.r^{m-1} + \left(\lambda + \frac{\epsilon}{r} \right) .\check{q}_k \right) \right),$$

where $\bar{\eta}_k(s) = e^{-k(a-m-1)s} \eta(s)$, $\check{q}_k(s) = \phi(e^{-ks}) q(s)$ and $\hat{\delta}_k(s) = e^{-k(a-2)s} \delta(s)$, for $s > 0$.

Hypothesis (3) gives that that $\Lambda_r < \infty$.

Lemma 4. $u \in C^1(\mathbb{R}^+)$ is solution of ivp (1) if and only if u is fixed point of T (i.e $Tu = u$).

Proof. Let $u \in E$ be a fixed point of T , then $u \in C^1(\mathbb{R}^+)$, $u(0) = 0$ and

$$\phi(h.u')(t) = I_{0^+}^{\alpha-\beta, \sigma} (\delta.u) + I_{0^+}^{\alpha, \sigma} (\eta.p(u) + q.f(., u)) (t)$$

then it follows from Theorem (1) that

$$D_{0+}^{\alpha,\omega,\sigma} \phi(h.u')(t) = D_{0+}^{\alpha,\omega,\sigma} I_{0+}^{\alpha-\beta,\sigma} (\delta.u)(t) + \eta.p(u) + q.f(.,u)(t) \quad (5)$$

The boundeness of the function $t \mapsto \delta(t).u(t)$ gives that

$$\lim_{t \rightarrow 0+} \left(I_{0+}^{(1-\omega)(1-\beta),\sigma} (\delta.u) \right) (t) = 0$$

and so

$$\begin{aligned} D_{0+}^{\alpha,\omega,\sigma} I_{0+}^{\alpha-\beta,\sigma} (\delta.u) &= D_{0+}^{\alpha,\omega,\sigma} I_{0+}^{\alpha-\beta,\sigma} \left(I_{0+}^{\beta,\sigma} D_{0+}^{\beta,\omega,\sigma} (\delta.u) + \frac{(\sigma(x))^{\xi-1}}{\Gamma(\xi)} \left(I_{0+}^{(1-\omega)(1-\alpha),\sigma} (\delta.u) \right) (0) \right) \\ &= D_{0+}^{\alpha,\omega,\sigma} I_{0+}^{\alpha-\beta,\sigma} \left(I_{0+}^{\beta,\sigma} D_{0+}^{\beta,\omega,\sigma} (\delta.u) \right) = D_{0+}^{\alpha,\omega,\sigma} I_{0+}^{\alpha,\sigma} \left(D_{0+}^{\beta,\omega,\sigma} (\delta.u) \right) \\ &= D_{0+}^{\beta,\omega,\sigma} (\delta.u) \end{aligned}$$

then equation (5) means that u is solution of ivp (1).

Conversely, it is easily to show, by a direct calculation, that the solution u of the ivp (1) satisfies the equation $u = Tu$. This completes the proof. \square

Lemma 5. Assume that Hypothesis (2) and (3) hold true.

Then the operator $T : K \cap \bar{B}(0, r) \rightarrow K$ is compact, where r is the constant given in (2).

Proof. Let $M_r = T(\Omega_r)$, where $\Omega_r = K \cap \bar{B}(0, r)$.

It's clear that the continuity of the functions f and p and the hypothesis (3) make the operator $T : \Omega_r \rightarrow E$ continuous.

Now, we show that M_r is relatively compact.

In first, we show that the set $M_r = T(\Omega_r)$ is a subset of E . For $u \in \Omega_r$, with $\phi(e^{-kt}) = (e^{-kt})^{p-1}$

$$\begin{aligned} e^{-kx} Tu(x) &= e^{-kx} \int_0^x \frac{1}{h(t)} \psi \left(I_{0+}^{\alpha-\beta,\sigma} (\delta.u) + I_{0+}^{\alpha,\sigma} (\eta.p(u) + q.f(.,u)) \right) (t) dt \\ &\leq \int_0^x \frac{e^{-kt}}{h(t)} \psi \left(I_{0+}^{\alpha-\beta,\sigma} (\delta.u) + I_{0+}^{\alpha,\sigma} (\eta.p(u) + q.f(.,u)) \right) (t) dt. \end{aligned}$$

Hypothesis (2) leads

$$e^{-kx} Tu(x) \leq \int_0^x \frac{1}{h(t)} \psi \left(I_{0+}^{\alpha-\beta,\sigma} (\hat{\delta}_k \cdot \tilde{u}_k) + I_{0+}^{\alpha,\sigma} [(\tilde{\eta}_k (\tilde{u}_k)^m + \lambda \tilde{q}_k \tilde{u}_k + \epsilon \tilde{q}_k)] \right) (t) dt$$

this is for all $x \geq 0$, where $\tilde{u}_k(s) = e^{-ks} u(s) \in [0, r]$. Then

$$\|Tu\| \leq \psi(r) \Lambda_r \cdot \int_0^{+\infty} \frac{dt}{h(t)}$$

proving the boundeness of M_r .

Let $b_1 \leq t_1 < t_2 \leq b_2$, $b_1, b_2 \in \mathbb{R}^+$ and set $w(t) = e^{-kt}$. For all $u \in \Omega_r$ we have

$$\begin{aligned} |w(t_2)Tu(t_2) - w(t_1)Tu(t_1)| &\leq w(t_2) |Tu(t_2) - Tu(t_1)| \\ &\quad + Tu(t_1) |w(t_2) - w(t_1)| \end{aligned}$$

with

$$\begin{aligned} w(t_2) |Tu(t_2) - Tu(t_1)| &= w(t_2) \int_{t_1}^{t_2} \frac{1}{h(t)} \psi \left(I_{0+}^{\alpha-\beta, \sigma} (\delta \cdot u) + I_{0+}^{\alpha, \sigma} (\eta \cdot p(u) + q \cdot f(\cdot, u)) \right) (t) dt \\ &\leq \int_{t_1}^{t_2} \frac{1}{h(t)} \psi \left(I_{0+}^{\alpha-\beta, \sigma} (\hat{\delta} \cdot \tilde{u}_k) + I_{0+}^{\alpha, \sigma} [(\bar{\eta}_k \cdot (\tilde{u}_k)^m + \lambda \check{q}_k \tilde{u}_k + \epsilon \check{q}_k)] \right) (t) dt \\ &\leq \psi(r) \Lambda_r \cdot \int_{t_1}^{t_2} \frac{dt}{h(t)}. \end{aligned}$$

Because that w and $x \rightarrow \int_0^x \frac{dt}{h(t)}$ are uniformly continuous on compact intervals, the above estimates prove that $\{e^{-kt}u, u \in M_r\}$ is locally equicontinuous on $[0, +\infty)$.

Now, let $u \in \Omega_r, x \in \mathbb{R}^+$. For $y > x$

$$\begin{aligned} |e^{-kx}T(u)(x) - e^{-ky}T(u)(y)| &\leq e^{-ky} |Tu(x) - Tu(y)| \\ &\quad + Tu(x) |e^{-kx} - e^{-ky}| \\ &\leq \psi(r) \Lambda_r \cdot \int_x^y \frac{dt}{h(t)} \\ &\quad + \psi(r) \Lambda_r \cdot \int_0^{+\infty} \frac{dt}{h(t)} \cdot |e^{-kx} - e^{-ky}| \end{aligned}$$

then

$$\begin{aligned} \left| e^{-kt}T(u)(x) - \lim_{y \rightarrow +\infty} e^{-ky}T(u)(y) \right| &\leq \psi(r) \Lambda_r \cdot \int_x^{+\infty} \frac{dt}{h(t)} \\ &\quad + e^{-kx} \cdot \psi(r) \Lambda_r \cdot \int_0^{+\infty} \frac{dt}{h(t)} \end{aligned}$$

with

$$\lim_{x \rightarrow +\infty} \int_x^{+\infty} \frac{dt}{h(t)} = \lim_{x \rightarrow +\infty} e^{-kx} = 0,$$

so, the equiconvergence of $\{e^{-kt}u, u \in M_r\}$ holds. By Lemma (2), we deduce that M_r is relatively compact.

Finally, we have from hypothesis (2) and (3) that for $u \in \Omega_r$ the functions $q \cdot f(\cdot, u), \eta \cdot p(u)$ and $\delta \cdot u$ are positive, and so $T(K \cap \bar{B}(0, r)) \subset K$.

Proving our claim. □

$$\text{Set } H(x) = \frac{e^{-kx}}{h(x)\psi(\Gamma(\alpha-\beta))} \psi \left(\int_A^x \sigma'(t) (\sigma(x) - \sigma(t))^{\alpha-\beta-1} \delta(t) e^{kt} dt \right).$$

Theorem 6. Assume that Hypothesis (2) and (3) hold true.

If

$$\Lambda_r \leq r^{\frac{a-2}{a-1}} \left(\int_0^{+\infty} \frac{dt}{h(t)} \right)^{-1}, \tag{6}$$

then ivp (1) admits at least one positive solution growing exponentially.

Proof. Let $u \in K \cap \bar{B}(0, r)$, for $x > 0$

$$\begin{aligned} e^{-kx}Tu(x) &= e^{-kx} \int_0^x \frac{1}{h(t)} \psi \left(I_{0^+}^{\alpha-\beta, \sigma} (\delta.u) + I_{0^+}^{\alpha, \sigma} (\eta.p(u) + q.f(., u)) \right) (t) dt \\ &\leq \int_0^x \frac{1}{h(t)} \psi \left(I_{0^+}^{\alpha-\beta, \sigma} (\hat{\delta}.\tilde{u}_k) + I_{0^+}^{\alpha, \sigma} [(\tilde{\eta}_k. (\tilde{u}_k)^m + \lambda\tilde{q}_k\tilde{u}_k + \epsilon\tilde{q}_k)] \right) (t) dt \\ &\leq \psi(r) \Lambda_r. \int_0^{+\infty} \frac{dt}{h(t)} \leq r \end{aligned}$$

then

$$\|Tu\| \leq \|u\|.$$

We have that the compact operator T maps the closed bounded convex set $K \cap \bar{B}(0, r)$ into itself. So, Schauder's fixed point theorem guarantees existence of a fixed point u of T , which is a positive solution of ivp (1). Moreover, u is nontrivial since $f(., 0)$ does not vanish identically on any subinterval of \mathbb{R}^+ .

Now, we have to prove that the solution u grows exponentially at ∞ . We distinguish two cases:

Case 1. $\lim_{x \rightarrow +\infty} e^{-kx}u(x) > 0$. As $u > 0$ on $(0, \infty)$, then there exists $w, L > 0$ such that for all $x \geq w$, $u(x) \geq L.e^{kx}$.

Case 2. $\lim_{x \rightarrow +\infty} e^{-kx}u(x) = 0$. Then there exists $A > 0$ such that the function $e^{-kx}u(x)$ is nonincreasing on $[A, +\infty)$ and such that for $x > A$, $u(x) \geq 1$. Then

$$\begin{aligned} u'(x) &= (Tu)'(x) = \frac{1}{h(x)} \psi \left(I_{0^+}^{\alpha-\beta, \sigma} (\delta.u) + I_{0^+}^{\alpha, \sigma} (\eta.p(-u) + q.f(., u)) \right) (x) \\ &\geq \frac{1}{h(x)} \psi \left(I_{0^+}^{\alpha-\beta, \sigma} (\delta.u) \right) (x) \\ &\geq \frac{1}{h(x)\psi(\Gamma(\alpha-\beta))} \psi \left(\int_A^x \sigma'(t) (\sigma(x) - \sigma(t))^{\alpha-\beta-1} \delta(t) u(t) dt \right) \\ &\geq H(x) u(x) \end{aligned}$$

where

$$H(x) = \frac{e^{-kx}}{h(x)\psi(\Gamma(\alpha-\beta))} \psi \left(\int_A^x \sigma'(t) (\sigma(x) - \sigma(t))^{\alpha-\beta-1} \delta(t) e^{kt} dt \right)$$

It follows that there exists $c > 0$ such that for all $x \geq A$,

$$u(x) \geq c \exp \left(\int_A^x H(t) dt \right)$$

which means that u is an exponentially growing solution. \square

Now, we consider the following hypothesis

$$\left\{ \begin{array}{l} \text{There exists } \theta \geq k \text{ and a function } \rho : \mathbb{R}^+ \rightarrow \mathbb{R}, \text{ such that for all } t > 0, \text{ if } x, y \in [0, r] \text{ then} \\ q(t)(f(t, e^{kt}x) - f(t, e^{kt}y)) \leq \rho(t) + \eta(t) \cdot (e^{kt}y)^m \text{ and} \\ \lim_{x \rightarrow +\infty} e^{-kx} \int_0^x \frac{1}{h(t)} \left[(N_r(t))^{\frac{2-a}{a-1}} \left(I_{0+}^{\alpha-\beta, \sigma} (2\check{\delta}_k r) + I_{0+}^{\alpha, \sigma} (\check{\eta}_k \cdot r^m) \right) + (M_r(t))^{\frac{2-a}{a-1}} I_{0+}^{\alpha, \sigma} (\rho)(t) \right] (t) dt = 0, \end{array} \right. \tag{7}$$

where r is the constant given in hypothesis (2) and $M_r(t) = \begin{cases} N_r(t) & \text{if } I_{0+}^{\alpha, \sigma} (\rho)(t) \geq 0 \\ L_r(t) & \text{if } I_{0+}^{\alpha, \sigma} (\rho)(t) \leq 0 \end{cases}$

with

$$N_r(t) = \begin{cases} I_{0+}^{\alpha-\beta, \sigma} (\check{\delta}_k \cdot r)(t) + I_{0+}^{\alpha, \sigma} (\check{\eta}_k \cdot r^m + (\lambda \cdot r + \epsilon) \cdot q)(t), & \text{if } 1 < a \leq 2 \\ \frac{\epsilon}{\Gamma(\alpha - \beta)} \int_0^t \sigma'(s) (\sigma(t) - \sigma(s)) q(s) ds & \text{if } a > 2 \end{cases}$$

$$L_r(t) = \begin{cases} I_{0+}^{\alpha-\beta, \sigma} (\check{\delta}_k \cdot r)(t) + I_{0+}^{\alpha, \sigma} (\check{\eta}_k \cdot r^m + (\lambda \cdot r + \epsilon) \cdot q)(t), & \text{if } a > 2 \\ \frac{\epsilon}{\Gamma(\alpha - \beta)} \int_0^t \sigma'(s) (\sigma(t) - \sigma(s)) q(s) ds & \text{if } 1 < a \leq 2. \end{cases}$$

and $\check{\delta}_k(x) = e^{kx}\delta(x)$ and $\check{\eta}_k(s) = e^{kms}\eta(s)$.

Theorem 7. Assume that Hypothesis (2), (3) and (7) hold true.

If $\Lambda_r \leq r \frac{a-2}{a-1} \left(\int_0^{+\infty} \frac{dt}{h(t)} \right)^{-1}$, then the positive solutions of problem (1) are locally asymptotically stable.

Proof. We have from theorem (6) that T admits a fixed point in $K \cap \bar{B}(0, r)$, which is a solution of ivp (1) in $\bar{B}(0, r)$.

Now, we show that the solutions are locally asymptotically stable. We assume that $u, v \in K \cap B(0, r)$ are solutions of ivp (1). For $x > 0$, we have

$$|u(x) - v(x)| = |Tu(x) - Tv(x)| = \left| \int_0^x \frac{1}{h(t)} (\psi(Bu) - \psi(Bv))(t) dt \right|$$

where

$$Bu(t) = I_{0+}^{\alpha-\beta, \sigma} (\delta \cdot u) + I_{0+}^{\alpha, \sigma} (\eta \cdot (u)^m + q \cdot f(t, u)).$$

Then there exists $\chi \in [\min(Bu, Bv), \max(Bu, Bv)]$ such that for all $x \geq 0$

$$(u - v)(x) = \frac{1}{a-1} \int_0^x \frac{1}{h(t)} \left[(\chi(t))^{\frac{1}{a-1}-1} \left(I_{0+}^{\alpha-\beta, \sigma} (\delta \cdot (u - v)) + I_{0+}^{\alpha, \sigma} (\eta \cdot (p(u) - p(v))) \right) (t) \right] dt$$

$$+ \frac{1}{a-1} \int_0^x \frac{1}{h(t)} \left[(\chi(t))^{\frac{1}{a-1}-1} I_{0+}^{\alpha, \sigma} (q \cdot (f(\cdot, u) - f(\cdot, v))) (t) \right] dt.$$

For $w \in \{u, v\}$ and for $t \in [0, x]$

$$\begin{aligned} Bw(t) &= I_{0+}^{\alpha-\beta, \sigma}(\delta \cdot w) + I_{0+}^{\alpha, \sigma}(\eta \cdot (w)^m + q \cdot f(t, w)) \\ &\leq I_{0+}^{\alpha-\beta, \sigma}(\check{\delta}_k \cdot \tilde{w}) + I_{0+}^{\alpha, \sigma}(\check{\eta}_k \cdot (\tilde{w})^m + \lambda q \cdot \tilde{w} + q\epsilon) \\ &\leq I_{0+}^{\alpha-\beta, \sigma}(\check{\delta}_k \cdot r) + I_{0+}^{\alpha, \sigma}(\check{\eta}_k \cdot (r)^m + q(\lambda \cdot r + \epsilon)) \end{aligned}$$

and

$$\begin{aligned} Bw(t) &\geq I_{0+}^{\alpha, \sigma}(q \cdot f(t, w)) \\ &\geq \frac{1}{\Gamma(\alpha - \beta)} \int_0^t \sigma'(s) (\sigma(t) - \sigma(s)) q(s) \cdot \epsilon ds, \end{aligned}$$

where $\check{\eta}_k(s) = e^{kms} \eta(s)$, $\check{\delta}_k(s) = e^{ks} \delta(s)$ and $\tilde{w}(s) = e^{-ks} w(s) \in [0, r]$.

From the inequality of hypothesis (7) we obtain for x_0

$$\begin{aligned} (u - v)(x) &\leq \frac{1}{a-1} \int_0^x \frac{1}{h(t)} \left[(N_r(t))^{\frac{1}{a-1}-1} \left(I_{0+}^{\alpha-\beta, \sigma}(\delta \cdot |u - v|) + I_{0+}^{\alpha, \sigma}(\eta \cdot p(u)) \right) (t) \right] dt \\ &\quad + \frac{1}{a-1} \int_0^x \frac{1}{h(t)} \left[(Bw(t))^{\frac{1}{a-1}-1} I_{0+}^{\alpha, \sigma}(q \cdot (f(\cdot, u) - f(\cdot, v)) - (\eta \cdot v^m)) (t) \right] dt \\ &\leq \frac{1}{a-1} \int_0^x \frac{1}{h(t)} \left[(N_r(t))^{\frac{2-a}{a-1}} \left(I_{0+}^{\alpha-\beta, \sigma}(\check{\delta}_k \cdot |\tilde{u} - \tilde{v}|) + I_{0+}^{\alpha, \sigma}(\check{\eta}_k \cdot \tilde{u}^m) \right) (t) \right] dt \\ &\quad + \frac{1}{a-1} \int_0^x \frac{1}{h(t)} \left[(M_r(t))^{\frac{2-a}{a-1}} I_{0+}^{\alpha, \sigma}(\rho) (t) \right] dt \\ &\leq \frac{1}{a-1} \int_0^x \frac{1}{h(t)} (N_r(t))^{\frac{2-a}{a-1}} \left[I_{0+}^{\alpha-\beta, \sigma}(\check{\delta}_k \cdot r) + I_{0+}^{\alpha, \sigma}(\check{\eta}_k \cdot r^m) \right] (t) dt \\ &\quad + \frac{1}{a-1} \int_0^x \frac{1}{h(t)} (M_r(t))^{\frac{2-a}{a-1}} I_{0+}^{\alpha, \sigma}(\rho) (t) dt \end{aligned}$$

and from (7) we conclude that $\lim_{x \rightarrow \infty} e^{-kx} |u - v|(x) = 0$. \square

Remark 2.

Assume that (2) and (3) hold true and the hypothesis (6) and (7) are satisfied for all $r > 0$ then ivp (1) is asymptotically stable.

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