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SOME RESULTS ON GENERALIZED RELATIVE ORDER (α, β) AND GENERALIZED RELATIVE TYPE (α, β) OF MEROMORPHIC FUNCTIONS WITH RESPECT TO ENTIRE FUNCTIONS

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Abstract. Orders and types of entire and meromorphic functions have been actively investigated by many authors. In the present paper, we aim at investigating some basic properties in connection with sum and product of generalized relative order (α, β), generalized relative type (α, β) and generalized relative weak type (α, β) of meromorphic functions with respect to entire functions.

1. Introduction, Definitions and Notations

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [7, 13, 20]. We also use the standard notations and definitions of the theory of entire functions which are available in [19] and therefore we do not explain those in details. Let f be an entire function and $M_f(r) = \max\{|f(z)| : |z| = r\}$. A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds (see [1, 2]).When f is meromorphic, one may introduce another function $T_f(r)$, known as Nevanlinna's characteristic function of f (see [7, p.4]), playing the same role as $M_f(r)$, which is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r, a)$ $\left(\overline{N}_f(r, a)\right)$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as follows:

$$N_f(r, a) = \int_0^r \frac{n_f(t, a) - n_f(0, a)}{t} dt + n_f(0, a) \log r$$

$$\left(\overline{N}_f(r, a) = \int_0^r \frac{\overline{n}_f(t, a) - \overline{n}_f(0, a)}{t} dt + \overline{n}_f(0, a) \log r\right),$$

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in addition we represent by $n_f(r,a)$ $(\overline{n}_f(r,a))$ the number of a-points (distinct a-points) of f in $|z|\leq r$ and an ∞ -point is a pole of f. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are symbolized by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

On the other hand, the function $m_f(r,\infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of f is defined as:

$$m_f(r)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta,\quad \text{where}$$

$$\log^+ x=\max(\log x,0)\ \text{ for all } x\geq 0\ .$$

Also we may employ $m\left(r,\frac{1}{f-a}\right)$ by $m_f(r,a)$.

If f is entire, then the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r)=m_f(r)\,.$$

Moreover, if f is non-constant entire then $T_f(r)$ is also strictly increasing and continuous functions of r. Therefore its inverse $T_f^{-1}:(T_f(0),\infty)\to(0,\infty)$ exists and is such that $\lim\limits_{s\to\infty}T_f^{-1}(s)=\infty$. For $x\in[0,\infty)$ and $k\in\mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers, we define iterations of the exponential and logarithmic functions as $\exp^{[k]}x=\exp\left(\exp^{[k-1]}x\right)$ and $\log^{[k]}x=\log\left(\log^{[k-1]}x\right)$, with convention that $\log^{[0]}x=x$, $\log^{[-1]}x=\exp x$, $\exp^{[0]}x=x$, and $\exp^{[-1]}x=\log x$. Further we assume that p and q always denote positive integers. Now considering this, let us recall that Juneja et al. [8] defined the (p,q)-th order and (p,q)-th lower order of an entire function, respectively, as follows:

Definition 1. [8] *Let $p\geq q$. The (p,q)-th order denoted by $\rho^{(p,q)}(f)$ and (p,q)-th lower order denoted by $\lambda^{(p,q)}(f)$ of an entire function f are defined as:*

$$\rho^{(p,q)}(f)=\limsup_{r\to\infty}\frac{\log^{[p]}M_f(r)}{\log^{[q]}r}\ \text{ and }\ \lambda^{(p,q)}(f)=\liminf_{r\to\infty}\frac{\log^{[p]}M_f(r)}{\log^{[q]}r}.$$

If f is a meromorphic function, then

$$\rho^{(p,q)}(f)=\limsup_{r\to\infty}\frac{\log^{[p-1]}T_f(r)}{\log^{[q]}r}\ \text{ and }\ \lambda^{(p,q)}(f)=\liminf_{r\to\infty}\frac{\log^{[p-1]}T_f(r)}{\log^{[q]}r}.$$

For any entire function f, using the inequality $T_f(r)\leq\log M_f(r)\leq 3T_f(2r)$ $\{cf.$ [7]$\}$, one can easily verify that

$$\begin{aligned}\rho^{(p,q)}(f) &= \limsup_{r\to\infty}\frac{\log^{[p]}M_f(r)}{\log^{[q]}r}=\limsup_{r\to\infty}\frac{\log^{[p-1]}T_f(r)}{\log^{[q]}r}\\ \text{and }\lambda^{(p,q)}(f) &= \liminf_{r\to\infty}\frac{\log^{[p]}M_f(r)}{\log^{[q]}r}=\liminf_{r\to\infty}\frac{\log^{[p-1]}T_f(r)}{\log^{[q]}r},\end{aligned}$$

when $p\geq 2$.

The function f is said to be of regular (p,q) growth when (p,q)-th order and (p,q)-th lower order of f are the same. Functions which are not of regular (p,q) growth are said to be of irregular (p,q) growth.

Extending the notion (p,q)-th order, recently Shen et al. [9] introduced the new concept of $[p,q]$-φ order of entire and meromorphic function where $p\geq q$.

Later on, combining the definitions of (p,q)-order and $[p,q]$-φ order, Biswas (see, e.g., [3]) redefined the (p,q)-order of an entire and meromorphic function without restriction $p \geq q$.

However the above definition is very useful for measuring the growth of entire and meromorphic functions. If $p = l$ and $q = 1$ then we write $\rho^{(l,1)}(f) = \rho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ where $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of entire or meromorphic function f. For details about generalized order one may see [18]. Also for $p = 2$ and $q = 1$, we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire or meromorphic function f.

Now let L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$. For any $\alpha \in L$, we say that $\alpha \in L_1^0$, if $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$ and $\alpha \in L_2^0$, if $\alpha(\exp((1+o(1))x)) = (1+o(1))\alpha(\exp(x))$ as $x \to +\infty$. Finally for any $\alpha \in L$, we also say that $\alpha \in L_1$, if $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0,+\infty)$ and $\alpha \in L_2$, if $\alpha(\exp(cx)) = (1+o(1))\alpha(\exp(x))$ as $x_0 \leq x \to +\infty$ for each $c \in (0,+\infty)$. Clearly, $L_1 \subset L_1^0$, $L_2 \subset L_2^0$ and $L_2 \subset L_1$.

Considering the above, Sheremeta [17] introduced the concept of generalized order (α,β) of an entire function. For details about generalized order (α,β) one may see [17].

Now, Biswas et al. [5] have introduced the definition of the generalized order (α,β) of a meromorphic function which considerably extend the definition of φ-order introduced by Chyzhykov et al. [6]. In order to keep accordance with Definition 1, it has given a minor modification of the original definition of generalized order (α,β) of an entire function (e.g. see, [17]).

Definition 2. [5] *Let $\alpha,\beta \in L$. The generalized order (α,β) denoted by $\rho_{(\alpha,\beta)}[f]$ and generalized lower order (α,β) denoted by $\lambda_{(\alpha,\beta)}[f]$ of a meromorphic function f are defined as:*

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)}.$$

If f is an entire function, then

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to\infty} \frac{\alpha(M_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to\infty} \frac{\alpha(M_f(r))}{\beta(r)}.$$

Using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ $\{cf.\ [7]\}$, for an entire function f, one may easily verify that

$$\begin{aligned}\rho_{(\alpha,\beta)}[f] &= \limsup_{r\to\infty} \frac{\alpha(M_f(r))}{\beta(r)} = \limsup_{r\to\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)} \\ \text{and } \lambda_{(\alpha,\beta)}[f] &= \liminf_{r\to\infty} \frac{\alpha(M_f(r))}{\beta(r)} = \liminf_{r\to\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)},\end{aligned}$$

when $\alpha \in L_2$ and $\beta \in L_1$.

Definition 1 is a special case of Definition 2 for $\alpha(r) = \log^{[p]} r$ and $\beta(r) = \log^{[q]} r$.

The function f is said to be of regular generalized growth (α,β) when generalized order (α,β) and generalized lower order (α,β) of f are the same. Functions which are not of regular generalized growth (α, β) are said to be of irregular generalized growth (α, β).

Mainly the growth investigation of entire and meromorphic functions has usually been done through their maximum moduli or Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire and meromorphic function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1, 2, 15]) will come. Now in order to make some progress in the study of relative order, Biswas et al. [5] have introduced the definitions of generalized relative order (α, β) and generalized relative lower order (α, β) of a meromorphic function with respect to another entire function in the following way:

Definition 3. [5] *Let $\alpha, \beta \in L$. The generalized relative order (α, β) denoted by $\rho_{(\alpha,\beta)}[f]_g$ and generalized relative lower order (α, β) denoted by $\lambda_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to an entire function g are defined as:*

$$\rho_{(\alpha,\beta)}[f]_g = \limsup_{r\to\infty} \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to\infty} \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)}.$$

The previous definitions are easily generated as particular cases, e.g. if $g = z$, then Definition 3 reduces to Definition 2. If $\alpha(r) = \beta(r) = \log r$, then we get the definition of relative order of meromorphic function f with respect to an entire function g introduced by Lahiri et al. [15] and if $g = \exp z$ and $\alpha(r) = \beta(r) = \log r$, then $\rho_{(\alpha,\beta)}[f]_g = \rho(f)$. And if $\alpha(r) = \log^{[p]} r$, $\beta(r) = \log^{[q]} r$ and $g = z$, then Definition 3 becomes the classical one given in [3].

Further if generalized relative order (α, β) and the generalized relative lower order (α, β) of a meromorphic function f with respect to an entire function g are the same, then f is called a function of regular generalized relative growth (α, β) with respect to g. Otherwise, f is said to be irregular generalized relative growth (α, β) with respect to g.

Now in order to refine the above growth scale, Biswas et al. [5] have introduced the definitions of other growth indicators, such as generalized relative type (α, β) and generalized relative lower type (α, β) of meromorphic function with respect to an entire function which are as follows:

Definition 4. [5] *Let $\alpha, \beta \in L$. The generalized relative type (α, β) denoted by $\sigma_{(\alpha,\beta)}[f]_g$ and generalized relative lower type (α, β) denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ of a meromorphic function f with respect to an entire function g having non-zero finite generalized relative order (α, β), $\rho_{(\alpha,\beta)}[f]_g$, are defined as :*

$$\begin{aligned} \sigma_{(\alpha,\beta)}[f]_g &= \limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}} \\ \text{and } \overline{\sigma}_{(\alpha,\beta)}[f]_g &= \liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}}. \end{aligned}$$

Analogously, to determine the relative growth of a meromorphic function f having same non zero finite generalized relative lower order (α, β) with respect to an entire function g, Biswas et al. [5] have introduced the definitions of generalized relative upper weak type (α, β) and generalized relative weak type (α, β) of f with respect to g of finite positive generalized relative lower order (α, β) in the following way:

Definition 5. [5] *Let $\alpha, \beta \in L$. The generalized relative upper weak type (α, β) denoted by $\overline{\tau}_{(\alpha,\beta)}\left[f\right]_g$ and generalized relative weak type (α, β) denoted by $\tau_{(\alpha,\beta)}\left[f\right]_g$ of a meromorphic function f with respect to an entire function g having non-zero finite generalized relative lower order (α, β), $\lambda_{(\alpha,\beta)}\left[f\right]_g$, are defined as :*

$$\overline{\tau}_{(\alpha,\beta)}\left[f\right]_g = \limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}\left(T_f\left(r\right)\right)))}{\left(\exp(\beta\left(r\right))\right)^{\lambda_{(\alpha,\beta)}[f]_g}}$$

$$\text{and } \tau_{(\alpha,\beta)}\left[f\right]_g = \liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}\left(T_f\left(r\right)\right)))}{\left(\exp(\beta\left(r\right))\right)^{\lambda_{(\alpha,\beta)}[f]_g}}.$$

During the past decades, several authors (see for example [11], [12], [16], [3], [3]; see also [17]) made close investigations on the properties of generalized order (α, β) in some different direction. Here, in this paper, we aim at investigating some basic properties of generalized relative order (α, β), generalized relative type (α, β) and generalized relative weak type (α, β) of a meromorphic function with respect to an entire function under somewhat different conditions which considerably extend some earlier results (see, e.g., [4], [10], [14]). Henceforth we assume that α, $\beta \in L_1$ and all the growth indicators are non-zero finite.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [1, 2] *Let f be an entire function which satisfies the Property (A) then for any positive integer n and for all sufficiently large r,*

$$\left[M_f\left(r\right)\right]^n \leq M_f\left(r^\delta\right)$$

holds where $\delta > 1$.

Lemma 2. [7, p. 18] *Let f be an entire function. Then for all sufficiently large values of r,*

$$T_f\left(r\right) \leq \log M_f\left(r\right) \leq 3T_f\left(2r\right).$$

3. Main Results

In this section we present the main results of the paper.

Theorem 1. *Let f_1, f_2 be meromorphic functions and g_1 be any entire function such that at least f_1 or f_2 is of regular generalized relative growth (α, β) with respect to g_1. Also let g_1 have the Property (A). Then we have*

$$\lambda_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1} \leq \max\left\{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1}, \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}\right\}.$$

The equality holds when any one of $\lambda_{(\alpha,\beta)}\left[f_i\right]_{g_1} > \lambda_{(\alpha,\beta)}\left[f_j\right]_{g_1}$ hold and at least any one of f_j is of regular generalized relative growth (α, β) with respect to g_1 where $i, j = 1, 2$ and $i \neq j$.

Proof. The result is obvious when $\lambda_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1} = 0$. So we suppose that $\lambda_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1} > 0$. We can clearly assume that $\lambda_{(\alpha,\beta)}\left[f_k\right]_{g_1}$ is finite for $k = 1, 2$. Now let us consider that $\max\left\{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1}, \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}\right\} = \Delta$ and f_2 be of regular generalized relative growth (α, β) with respect to g_1.

Now for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$, we have for a sequence values of r tending to infinity that

$$T_{f_1}(r) \leq T_{g_1}\left[\alpha^{-1}\left[\left(\lambda_{(\alpha,\beta)}[f_1]_{g_1} + \varepsilon\right)\beta(r)\right]\right]$$

$$i.e.,\ T_{f_1}(r) \leq T_{g_1}\left[\alpha^{-1}\left[(\Delta + \varepsilon)\beta(r)\right]\right]. \tag{1}$$

Also for any arbitrary $\varepsilon > 0$ from the definition of $\rho_{(\alpha,\beta)}[f_2]_{g_1}\left(= \lambda_{(\alpha,\beta)}[f_2]_{g_1}\right)$, we obtain for all sufficiently large values of r that

$$T_{f_2}(r) \leq T_{g_1}\left[\alpha^{-1}\left[\left(\lambda_{(\alpha,\beta)}[f_2]_{g_1} + \varepsilon\right)\beta(r)\right]\right] \tag{2}$$

$$i.e.,\ T_{f_2}(r) \leq T_{g_1}\left[\alpha^{-1}\left[(\Delta + \varepsilon)\beta(r)\right]\right]. \tag{3}$$

Since $T_{f_1 \pm f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r) + O(1)$ for all large r, in view of (1) , (3) and Lemma 2, we obtain for a sequence values of r tending to infinity that

$$T_{f_1 \pm f_2}(r) \leq 2\log M_{g_1}\left[\alpha^{-1}\left[(\Delta + \varepsilon)\beta(r)\right]\right] + O(1)$$

$$i.e.,\ T_{f_1 \pm f_2}(r) \leq 3\log M_{g_1}\left[\alpha^{-1}\left[(\Delta + \varepsilon)\beta(r)\right]\right]. \tag{4}$$

Therefore in view of Lemma 1 and Lemma 2, we obtain from (4) for a sequence values of r tending to infinity and $\sigma > 1$ that

$$T_{f_1 \pm f_2}(r) \leq \frac{1}{3}\log\left[M_{g_1}\left[\alpha^{-1}\left[(\Delta + \varepsilon)\beta(r)\right]\right]\right]^9$$

$$i.e.,\ T_{f_1 \pm f_2}(r) \leq \frac{1}{3}\log M_{g_1}\left[\left[\alpha^{-1}\left[(\Delta + \varepsilon)\beta(r)\right]\right]^\sigma\right]$$

$$i.e.,\ T_{f_1 \pm f_2}(r) \leq T_{g_1}\left[2\left[\alpha^{-1}\left[(\Delta + \varepsilon)\beta(r)\right]\right]^\sigma\right].$$

Now we get from above by letting $\sigma \to 1^+$

$$i.e.,\ \liminf_{r\to\infty}\frac{\alpha\left(T_{g_1}^{-1}\left(T_{f_1 \pm f_2}(r)\right)\right)}{\beta(r)} < (\Delta + \varepsilon)\ .$$

Since $\varepsilon > 0$ is arbitrary,

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \Delta = \max\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\right\}.$$

Similarly, if we consider that f_1 is of regular generalized relative growth (α, β) with respect to g_1 or both f_1 and f_2 are of regular generalized relative growth (α, β) with respect to g_1, then one can easily verify that

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \Delta = \max\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\right\}. \tag{5}$$

Further without loss of any generality, let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ and $f = f_1 \pm f_2$. Then in view of (5) we get that $\lambda_{(\alpha,\beta)}[f]_{g_1} \leq \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. As, $f_2 = \pm(f - f_1)$ and in this case we obtain that $\lambda_{(\alpha,\beta)}[f_2]_{g_1} \leq \max\{\lambda_{(\alpha,\beta)}[f]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_1}\}$. As we assume that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$, therefore we have $\lambda_{(\alpha,\beta)}[f_2]_{g_1} \leq \lambda_{(\alpha,\beta)}[f]_{g_1}$ and hence $\lambda_{(\alpha,\beta)}[f]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \max\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\right\}$. Therefore, $\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1, 2$ provided $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \neq \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. Thus the theorem is established. □

Theorem 2. *Let f_1 and f_2 be any two meromorphic functions and g_1 be an entire function such that such that $\rho_{(\alpha,\beta)}[f_1]_{g_1}$ and $\rho_{(\alpha,\beta)}[f_2]_{g_1}$ exist. Also let g_1 have the Property (A). Then we have*

$$\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \max\left\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_1}\right\}.$$

The equality holds when $\rho_{(\alpha,\beta)}[f_1]_{g_1} \neq \rho_{(\alpha,\beta)}[f_2]_{g_1}$.

We omit the proof of Theorem 2 as it can easily be carried out in the line of Theorem 1.

Theorem 3. *Let f_1 be a meromorphic function and g_1, g_2 be any two entire functions such that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$ and $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ exist. Also let $g_1 \pm g_2$ have the Property (A). Then we have*

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \geq \min\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\right\}.$$

The equality holds when $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \neq \lambda_{(\alpha,\beta)}[f_1]_{g_2}$.

Proof. The result is obvious when $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \infty$. So we suppose that $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} < \infty$. We can clearly assume that $\lambda_{(\alpha,\beta)}[f_1]_{g_k}$ is finite for $k = 1, 2$. Further let $\Psi = \min\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\right\}$. Now for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_{(\alpha,\beta)}[f_1]_{g_k}$, we have for all sufficiently large values of r that

$$T_{g_k}\left[\alpha^{-1}\left[\left(\lambda_{(\alpha,\beta)}[f_1]_{g_k} - \varepsilon\right)\beta(r)\right]\right] \leq T_{f_1}(r) \quad \text{where } k = 1, 2 \tag{6}$$

$$i.e,\ T_{g_k}\left[\alpha^{-1}\left[(\Psi - \varepsilon)\beta(r)\right]\right] \leq T_{f_1}(r) \quad \text{where } k = 1, 2$$

Since $T_{g_1 \pm g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r) + O(1)$ for all large r, we obtain from above and Lemma 2 for all sufficiently large values of r that

$$T_{g_1 \pm g_2}\left[\alpha^{-1}\left[(\Psi - \varepsilon)\beta(r)\right]\right] \leq 2T_{f_1}(r) + O(1)$$

$$i.e.,\ T_{g_1 \pm g_2}\left[\alpha^{-1}\left[(\Psi - \varepsilon)\beta(r)\right]\right] < 3T_{f_1}(r).$$

Therefore in view of Lemma 1 and Lemma 2, we obtain from above for all sufficiently large values of r and any $\sigma > 1$ that

$$\frac{1}{9}\log M_{g_1 \pm g_2}\left[\frac{\alpha^{-1}\left[(\Psi - \varepsilon)\beta(r)\right]}{2}\right] < T_{f_1}(r)$$

$$i.e.,\ \log M_{g_1 \pm g_2}\left[\frac{\alpha^{-1}\left[(\Psi - \varepsilon)\beta(r)\right]}{2}\right]^{\frac{1}{9}} < T_{f_1}(r)$$

$$i.e.,\ \log M_{g_1 \pm g_2}\left[\left(\frac{\alpha^{-1}\left[(\Psi - \varepsilon)\beta(r)\right]}{2}\right)^{\frac{1}{\sigma}}\right] < T_{f_1}(r)$$

$$i.e.,\ T_{g_1 \pm g_2}\left[\left(\frac{\alpha^{-1}\left[(\Psi - \varepsilon)\beta(r)\right]}{2}\right)^{\frac{1}{\sigma}}\right] < T_{f_1}(r)$$

As $\varepsilon > 0$ is arbitrary, we get from above by letting $\sigma \to 1^+$

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \geq \Psi = \min\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\right\}. \tag{7}$$

Now without loss of any generality, we may consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $g = g_1 \pm g_2$. Then in view of (7) we get that $\lambda_{(\alpha,\beta)}[f_1]_g \geq$

$\lambda_{(\alpha,\beta)}[f_1]_{g_1}$. Further, $g_1 = (g \pm g_2)$ and in this case we obtain that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \geq \min\left\{\lambda_{(\alpha,\beta)}[f_1]_g, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\right\}$. As we assume that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, therefore we have $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \geq \lambda_{(\alpha,\beta)}[f_1]_g$ and hence $\lambda_{(\alpha,\beta)}[f_1]_g = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \min\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\right\}$. Therefore, $\lambda_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2$ provided $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \neq \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Thus the theorem follows. □

Theorem 4. *Let f_1 be a meromorphic function and g_1, g_2 be any two entire functions such that f_1 is of regular generalized relative growth (α,β) with respect to at least any one of g_1 and g_2. If $g_1 \pm g_2$ have the Property (A), then we have*

$$\rho_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} \geq \min\left\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\right\}.$$

The equality holds when any one of $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j}$ hold and at least f_1 is of regular generalized relative growth (α,β) with respect to any one of g_j where $i,j = 1,2$ and $i \neq j$.

We omit the proof of Theorem 4 as it can easily be carried out in the line of Theorem 3.

Theorem 5. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $g_1 \pm g_2$ have the Property (A). Then we have*

$$\begin{aligned}&\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1\pm g_2}\\ \leq\ & \max\left[\min\left\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\right\}, \min\left\{\rho_{(\alpha,\beta)}[f_2]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_2}\right\}\right]\end{aligned}$$

when the following two conditions holds:
(i) Any one of $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j}$ hold and at least f_1 is of regular generalized relative growth (α,β) with respect to any one of g_j for $i = 1, 2$, $j = 1,2$ and $i \neq j$; and
(ii) Any one of$\rho_{(\alpha,\beta)}[f_2]_{g_i} < \rho_{(\alpha,\beta)}[f_2]_{g_j}$ hold and at least f_2 is of regular generalized relative growth (α,β) with respect to any one of g_j for $i = 1, 2$, $j = 1,2$ and $i \neq j$.
The equality holds when $\rho_{(\alpha,\beta)}[f_i]_{g_1} < \rho_{(\alpha,\beta)}[f_j]_{g_1}$ and $\rho_{(\alpha,\beta)}[f_i]_{g_2} < \rho_{(\alpha,\beta)}[f_j]_{g_2}$ hold simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$.

Proof. Let the conditions (i) and (ii) of the theorem hold. Therefore in view of Theorem 2 and Theorem 4 we get that

$$\begin{aligned}&\max\left[\min\left\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\right\}, \min\left\{\rho_{(\alpha,\beta)}[f_2]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_2}\right\}\right]\\ &= \max\left[\rho_{(\alpha,\beta)}[f_1]_{g_1\pm g_2}, \rho_{(\alpha,\beta)}[f_2]_{g_1\pm g_2}\right]\\ &\geq \rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1\pm g_2}\ .\end{aligned} \tag{8}$$

Since $\rho_{(\alpha,\beta)}[f_i]_{g_1} < \rho_{(\alpha,\beta)}[f_j]_{g_1}$ and $\rho_{(\alpha,\beta)}[f_i]_{g_2} < \rho_{(\alpha,\beta)}[f_j]_{g_2}$ hold simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$, we obtain that

$$\text{either } \min\left\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\right\} > \min\left\{\rho_{(\alpha,\beta)}[f_2]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_2}\right\} \text{ or}$$

$$\min\left\{\rho_{(\alpha,\beta)}[f_2]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_2}\right\} > \min\left\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\right\} \text{ holds.}$$

Now in view of the conditions (i) and (ii) of the theorem, it follows from above that

$$\text{either } \rho_{(\alpha,\beta)}\left[f_1\right]_{g_1\pm g_2} > \rho_{(\alpha,\beta)}\left[f_2\right]_{g_1\pm g_2} \text{ or } \rho_{(\alpha,\beta)}\left[f_2\right]_{g_1\pm g_2} > \rho_{(\alpha,\beta)}\left[f_1\right]_{g_1\pm g_2}$$

which is the condition for holding equality in (8).

Hence the theorem follows. □

Theorem 6. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let g_1, g_2 and $g_1 \pm g_2$ satisfy the Property (A). Then we have*

$$\begin{aligned}&\lambda_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1\pm g_2}\\ &\geq \min\left[\max\left\{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1}, \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}\right\}, \max\left\{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2}, \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_2}\right\}\right]\end{aligned}$$

when the following two conditions holds:
(i) Any one of $\lambda_{(\alpha,\beta)}\left[f_i\right]_{g_1} > \lambda_{(\alpha,\beta)}\left[f_j\right]_{g_1}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_1 for $i = 1, 2$, $j = 1,2$ and $i \neq j$; and
(ii) Any one of $\lambda_{(\alpha,\beta)}\left[f_i\right]_{g_2} > \lambda_{(\alpha,\beta)}\left[f_j\right]_{g_2}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_2 for $i = 1, 2$, $j = 1,2$ and $i \neq j$.
The equality holds when $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_i} < \lambda_{(\alpha,\beta)}\left[f_1\right]_{g_j}$ and $\lambda_{(\alpha,\beta)}\left[f_2\right]_{g_i} < \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_j}$ hold simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$.

Proof. Suppose that the conditions (i) and (ii) of the theorem holds. Therefore in view of Theorem 1 and Theorem 3, we obtain that

$$\begin{aligned}&\min\left[\max\left\{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1}, \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}\right\}, \max\left\{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2}, \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_2}\right\}\right]\\ &= \min\left[\lambda_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1}, \lambda_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_2}\right]\\ &\geq \lambda_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1\pm g_2}. \qquad (9)\end{aligned}$$

Since $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_i} < \lambda_{(\alpha,\beta)}\left[f_1\right]_{g_j}$ and $\lambda_{(\alpha,\beta)}\left[f_2\right]_{g_i} < \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_j}$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$, we get that

$$\text{either } \max\left\{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1}, \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}\right\} < \max\left\{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2}, \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_2}\right\} \text{ or}$$

$$\max\left\{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2}, \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_2}\right\} < \max\left\{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1}, \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}\right\} \text{ holds.}$$

Since condition (i) and (ii) of the theorem holds, it follows from above that

$$\text{either } \lambda_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1} < \lambda_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_2} \text{ or } \lambda_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_2} < \lambda_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1}$$

which is the condition for holding equality in (9).

Hence the theorem follows. □

Theorem 7. *Let f_1, f_2 be any two meromorphic functions and g_1 be any entire function such that at least f_1 or f_2 is of regular generalized relative growth (α,β) with respect to g_1. Also let g_1 satisfy the Property (A). Then we have*

$$\lambda_{(\alpha,\beta)}\left[f_1 \cdot f_2\right]_{g_1} \leq \max\left\{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1}, \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}\right\}.$$

The equality holds when any one of $\lambda_{(\alpha,\beta)}\left[f_i\right]_{g_1} > \lambda_{(\alpha,\beta)}\left[f_j\right]_{g_1}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_1 where $i, j = 1,2$ and $i \neq j$.

Proof. Since $T_{f_1 \cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$ for all large r, applying the same procedure as adopted in Theorem 1 we get that

$$\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \leq \max\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\right\}.$$

Now without loss of any generality, let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ and $f = f_1 \cdot f_2$. Then $\lambda_{(\alpha,\beta)}[f]_{g_1} \leq \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. Further, $f_2 = \frac{f}{f_1}$ and $T_{f_1}(r) = T_{\frac{1}{f_1}}(r) + O(1)$. Therefore $T_{f_2}(r) \leq T_f(r) + T_{f_1}(r) + O(1)$ and in this case we obtain that $\lambda_{(\alpha,\beta)}[f_2]_{g_1} \leq \max\left\{\lambda_{(\alpha,\beta)}[f]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_1}\right\}$. As we assume that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$, therefore we have $\lambda_{(\alpha,\beta)}[f_2]_{g_1} \leq \lambda_{(\alpha,\beta)}[f]_{g_1}$ and hence $\lambda_{(\alpha,\beta)}[f]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}$. Therefore, $\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1, 2$ provided $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \neq \lambda_{(\alpha,\beta)}[f_2]_{g_1}$.

Hence the theorem follows. □

Next we prove the result for the quotient $\frac{f_1}{f_2}$, provided $\frac{f_1}{f_2}$ is meromorphic.

Theorem 8. *Let f_1, f_2 be any two meromorphic functions and g_1 be any entire function such that at least f_1 or f_2 is of regular generalized relative growth (α, β) with respect to g_1. Also let g_1 satisfy the Property (A). Then we have*

$$\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} \leq \max\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\right\},$$

provided $\frac{f_1}{f_2}$ is meromorphic. The equality holds when at least f_2 is of regular generalized relative growth (α, β) with respect to g_1 and $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \neq \lambda_{(\alpha,\beta)}[f_2]_{g_1}$.

Proof. Since $T_{f_2}(r) = T_{\frac{1}{f_2}}(r) + O(1)$ and $T_{\frac{f_1}{f_2}}(r) \leq T_{f_1}(r) + T_{\frac{1}{f_2}}(r)$, we get in view of Theorem 1 that

$$\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} \leq \max\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\right\}. \tag{10}$$

Now in order to prove the equality conditions, we discuss the following two cases:

Case I. Suppose $\frac{f_1}{f_2}\,(= h)$ satisfies the following condition

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1},$$

and f_2 is of regular generalized relative growth (α, β) with respect to g_1.

Now if possible, let $\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. Therefore from $f_1 = h \cdot f_2$ we get that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ which is a contradiction. Therefore $\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} \geq \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ and in view of (10), we get that

$$\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}.$$

Case II. Suppose $\frac{f_1}{f_2}\,(= h)$ satisfies the following condition

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1},$$

and f_2 is of regular generalized relative growth (α, β) with respect to g_1.

Now from $f_1 = h \cdot f_2$ we get that either $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \leq \lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1}$ or $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \leq \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. But according to our assumption $\lambda_{(\alpha,\beta)}[f_1]_{g_1}\, \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. Therefore $\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} \geq \lambda_{(\alpha,\beta)}[f_1]_{g_1}$ and in view of (10), we get that

$$\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_1}.$$

Hence the theorem follows. □

Now we state the following theorem which can easily be carried out in the line of Theorem 7 and Theorem 8 and therefore its proof is omitted.

Theorem 9. *Let f_1 and f_2 be any two meromorphic functions and g_1 be any entire function such that such that $\rho_{(\alpha,\beta)}[f_1]_{g_1}$ and $\rho_{(\alpha,\beta)}[f_2]_{g_1}$ exist. Also let g_1 satisfy the Property (A). Then we have*

$$\rho_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \leq \max\left\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_1}\right\}.$$

The equality holds when $\rho_{(\alpha,\beta)}[f_1]_{g_1} \neq \rho_{(\alpha,\beta)}[f_2]_{g_1}$. Similar results hold for the quotient $\frac{f_1}{f_2}$, provided $\frac{f_1}{f_2}$ is meromorphic.

Theorem 10. *Let f_1 be a meromorphic function and g_1, g_2 be any two entire functions such that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$ and $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ exist. Also let $g_1 \cdot g_2$ satisfy the Property (A). Then we have*

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \geq \min\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\right\}.$$

The equality holds when any one of $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j}$ hold where $i, j = 1, 2$ and $i \neq j$ and g_i satisfy the Property (A). Similar results hold for the quotient $\frac{g_1}{g_2}$, provided $\frac{g_1}{g_2}$ is entire and satisfies the Property (A). The equality holds when $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \neq \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and g_1 satisfy the Property (A).

Proof. Since $T_{g_1 \cdot g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r)$ for all large r, applying the same procedure as adopted in Theorem 3 we get that

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \geq \min\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\right\}.$$

Now without loss of any generality, we may consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $g = g_1 \cdot g_2$. Then $\lambda_{(\alpha,\beta)}[f_1]_g \geq \lambda_{(\alpha,\beta)}[f_1]_{g_1}$. Further, $g_1 = \frac{g}{g_2}$ and and $T_{g_2}(r) = T_{\frac{1}{g_2}}(r) + O(1)$. Therefore $T_{g_1}(r) \leq T_g(r) + T_{g_2}(r) + O(1)$ and in this case we obtain that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \geq \min\left\{\lambda_{(\alpha,\beta)}[f_1]_g, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\right\}$. As we assume that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, so we have $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \geq \lambda_{(\alpha,\beta)}[f_1]_g$ and hence $\lambda_{(\alpha,\beta)}[f_1]_g = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \min\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\right\}$. Therefore, $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1, 2$ provided $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and g_1 satisfy the Property (A). Hence the first part of the theorem follows.

Now we prove our results for the quotient $\frac{g_1}{g_2}$, provided $\frac{g_1}{g_2}$ is entire and $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \neq \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Since $T_{g_2}(r) = T_{\frac{1}{g_2}}(r) + O(1)$ and $T_{\frac{g_1}{g_2}}(r) \leq T_{g_1}(r) + T_{\frac{1}{g_2}}(r)$, we get in view of Theorem 3 that

$$\lambda_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} \geq \min\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\right\}. \tag{11}$$

Now in order to prove the equality conditions, we discuss the following two cases:

Case I. Suppose $\frac{g_1}{g_2}\,(=h)$ satisfies the following condition

$$\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1} > \lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2}.$$

Now if possible, let $\lambda_{(\alpha,\beta)}\left[f_1\right]_{\frac{g_1}{g_2}} > \lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2}$. Therefore from $g_1 = h \cdot g_2$ we get that $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1} = \lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2}$, which is a contradiction. Therefore $\lambda_{(\alpha,\beta)}\left[f_1\right]_{\frac{g_1}{g_2}} \le \lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2}$ and in view of (11), we get that

$$\lambda_{(\alpha,\beta)}\left[f_1\right]_{\frac{g_1}{g_2}} = \lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2}.$$

Case II. Suppose that $\frac{g_1}{g_2}\,(=h)$ satisfies the following condition

$$\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1} < \lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2}.$$

Therefore from $g_1 = h \cdot g_2$, we get that either $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1} \ge \lambda_{(\alpha,\beta)}\left[f_1\right]_{\frac{g_1}{g_2}}$ or $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1} \ge \lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2}$. But according to our assumption $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1} \lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2}$. Therefore $\lambda_{(\alpha,\beta)}\left[f_1\right]_{\frac{g_1}{g_2}} \le \lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1}$ and in view of (11), we get that

$$\lambda_{(\alpha,\beta)}\left[f_1\right]_{\frac{g_1}{g_2}} = \lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1}.$$

Hence the theorem follows. □

Theorem 11. *Let f_1 be any meromorphic function and g_1, g_2 be any two entire functions such that $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}$ and $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}$ exist. Further let f_1 be of regular generalized relative growth (α, β) with respect to at least any one of g_1 and g_2. Also let $g_1 \cdot g_2$ satisfies the Property (A). Then we have*

$$\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1 \cdot g_2} \ge \min\left\{\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}, \rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}\right\}.$$

The equality holds when any one of $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_i} < \rho_{(\alpha,\beta)}\left[f_1\right]_{g_j}$ hold and at least f_1 is of regular generalized relative growth (α, β) with respect to any one of g_j where $i, j = 1, 2$ and $i \neq j$ and g_i satisfies the Property (A).

Theorem 12. *Let f_1 be any meromorphic function and g_1, g_2 be any two entire functions such that $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}$ and $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}$ exist. Further let f_1 be of regular generalized relative growth (α, β) with respect to at least any one of g_1 or g_2. Then we have*

$$\rho_{(\alpha,\beta)}\left[f_1\right]_{\frac{g_1}{g_2}} \ge \min\left\{\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}, \rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}\right\},$$

provided $\frac{g_1}{g_2}$ is entire and satisfies the Property (A). The equality holds when at least f_1 is of regular generalized relative growth (α, β) with respect to g_2, $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1} \neq \rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}$ and g_1 satisfies the Property (A).

We omit the proof of Theorem 11 and Theorem 12 as those can easily be carried out in the line of Theorem 10.

Now we state the following four theorems without their proofs as those can easily be carried out in the line of Theorem 5 and Theorem 6 respectively.

Theorem 13. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $g_1 \cdot g_2$ satisfy the Property (A). Then we have*

$$\begin{aligned} & \rho_{(\alpha,\beta)} [f_1 \cdot f_2]_{g_1 \cdot g_2} \\ \leq \quad & \max\left[\min\left\{\rho_{(\alpha,\beta)} [f_1]_{g_1}, \rho_{(\alpha,\beta)} [f_1]_{g_2}\right\}, \min\left\{\rho_{(\alpha,\beta)} [f_2]_{g_1}, \rho_{(\alpha,\beta)} [f_2]_{g_2}\right\}\right], \end{aligned}$$

when the following two conditions holds:
(i) *Any one of $\rho_{(\alpha,\beta)} [f_1]_{g_i} < \rho_{(\alpha,\beta)} [f_1]_{g_j}$ hold and at least f_1 is of regular generalized relative growth (α, β) with respect to any one of g_j and g_i satisfy the Property (A) for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and*
(ii) *Any one of $\rho_{(\alpha,\beta)} [f_2]_{g_i} < \rho_{(\alpha,\beta)} [f_2]_{g_j}$ hold and at least f_2 is of regular generalized relative growth (α, β) with respect to any one of g_j and g_i satisfy the Property (A) for $i = 1, 2$, $j = 1, 2$ and $i \neq j$.*
The quality holds when $\rho_{(\alpha,\beta)} [f_i]_{g_1} < \rho_{(\alpha,\beta)} [f_j]_{g_1}$ and $\rho_{(\alpha,\beta)} [f_i]_{g_2} < \rho_{(\alpha,\beta)} [f_j]_{g_2}$ holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Theorem 14. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $g_1 \cdot g_2$, g_1 and g_2 satisfy the Property (A). Then we have*

$$\begin{aligned} & \lambda_{(\alpha,\beta)} [f_1 \cdot f_2]_{g_1 \cdot g_2} \\ \geq \quad & \min\left[\max\left\{\lambda_{(\alpha,\beta)} [f_1]_{g_1}, \lambda_{(\alpha,\beta)} [f_2]_{g_1}\right\}, \max\left\{\lambda_{(\alpha,\beta)} [f_1]_{g_2}, \lambda_{(\alpha,\beta)} [f_2]_{g_2}\right\}\right] \end{aligned}$$

when the following two conditions holds:
(i) *Any one of $\lambda_{(\alpha,\beta)} [f_i]_{g_1} > \lambda_{(\alpha,\beta)} [f_j]_{g_1}$ hold and at least any one of f_j is of regular generalized relative growth (α, β) with respect to g_1 for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and*
(ii) *Any one of $\lambda_{(\alpha,\beta)} [f_i]_{g_2} > \lambda_{(\alpha,\beta)} [f_j]_{g_2}$ hold and at least any one of f_j is of regular generalized relative growth (α, β) with respect to g_2 for $i = 1, 2$, $j = 1, 2$ and $i \neq j$.*
The equality holds when $\lambda_{(\alpha,\beta)} [f_1]_{g_i} < \lambda_{(\alpha,\beta)} [f_1]_{g_j}$ and $\lambda_{(\alpha,\beta)} [f_2]_{g_i} < \lambda_{(\alpha,\beta)} [f_2]_{g_j}$ holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Theorem 15. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions such that $\frac{f_1}{f_2}$ is meromorphic and $\frac{g_1}{g_2}$ is entire. Also let $\frac{g_1}{g_2}$ satisfy the Property (A). Then we have*

$$\begin{aligned} & \rho_{(\alpha,\beta)} \left[\frac{f_1}{f_2}\right]_{\frac{g_1}{g_2}} \\ \leq \quad & \max\left[\min\left\{\rho_{(\alpha,\beta)} [f_1]_{g_1}, \rho_{(\alpha,\beta)} [f_1]_{g_2}\right\}, \min\left\{\rho_{(\alpha,\beta)} [f_2]_{g_1}, \rho_{(\alpha,\beta)} [f_2]_{g_2}\right\}\right] \end{aligned}$$

when the following two conditions holds:
(i) *At least f_1 is of regular generalized relative growth (α, β) with respect to g_2 and $\rho_{(\alpha,\beta)} [f_1]_{g_1} \neq \rho_{(\alpha,\beta)} [f_1]_{g_2}$; and*
(ii) *At least f_2 is of regular generalized relative growth (α, β) with respect to g_2 and $\rho_{(\alpha,\beta)} [f_2]_{g_1} \neq \rho_{(\alpha,\beta)} [f_2]_{g_2}$.*
The equality holds when $\rho_{(\alpha,\beta)} [f_i]_{g_1} < \rho_{(\alpha,\beta)} [f_j]_{g_1}$ and $\rho_{(\alpha,\beta)} [f_i]_{g_2} < \rho_{(\alpha,\beta)} [f_j]_{g_2}$ holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Theorem 16. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions such that $\frac{f_1}{f_2}$ is meromorphic and $\frac{g_1}{g_2}$ is entire. Also let $\frac{g_1}{g_2}$, g_1 and*

g_2 satisfy the Property (A). Then we have

$$\lambda_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{\frac{g_1}{g_2}}$$
$$\geq \min\left[\max\left\{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1}, \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}\right\}, \max\left\{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2}, \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_2}\right\}\right]$$

when the following two conditions hold:
(i) At least f_2 is of regular generalized relative growth (α, β) with respect to g_1 and $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1} \neq \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}$; and
(ii) At least f_2 is of regular generalized relative growth (α, β) with respect to g_2 and $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2} \neq \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_2}$.
The equality holds when $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_i} < \lambda_{(\alpha,\beta)}\left[f_1\right]_{g_j}$ and $\lambda_{(\alpha,\beta)}\left[f_2\right]_{g_i} < \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_j}$ holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Next we intend to find out the sum and product theorems of generalized relative type (α, β) (respectively generalized relative lower type (α, β)) and generalized relative weak type (α, β) of meromorphic function with respect to an entire function taking into consideration of the above theorems.

Theorem 17. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}$, $\rho_{(\alpha,\beta)}\left[f_2\right]_{g_1}$, $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}$ and$\rho_{(\alpha,\beta)}\left[f_2\right]_{g_2}$ be all non zero and finite.*
(A) *If any one of $\rho_{(\alpha,\beta)}\left[f_i\right]_{g_1} > \rho_{(\alpha,\beta)}\left[f_j\right]_{g_1}$ hold for i, $j = 1, 2$; $i \neq j$, and g_1 has the Property (A), then*

$$\sigma_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1} = \sigma_{(\alpha,\beta)}\left[f_i\right]_{g_1} \text{ and } \overline{\sigma}_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}\left[f_i\right]_{g_1} \mid i = 1, 2.$$

(B) *If any one of $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_i} < \rho_{(\alpha,\beta)}\left[f_1\right]_{g_j}$ hold and at least f_1 is of regular generalized relative growth (α, β) with respect to any one of g_j for i, $j = 1, 2$; $i \neq j$ and $g_1 \pm g_2$ has the Property (A), then*

$$\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1 \pm g_2} = \sigma_{(\alpha,\beta)}\left[f_1\right]_{g_i} \text{ and } \overline{\sigma}_{(\alpha,\beta)}\left[f_1\right]_{g_1 \pm g_2} = \overline{\sigma}_{(\alpha,\beta)}\left[f_1\right]_{g_i} \mid i = 1, 2.$$

(C) *Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*
(i) Any one of $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_i} < \rho_{(\alpha,\beta)}\left[f_1\right]_{g_j}$ hold and at least f_1 is of regular generalized relative growth (α, β) with respect to any one of g_j for $i = 1$, 2, $j = 1, 2$ and $i \neq j$;
(ii) Any one of $\rho_{(\alpha,\beta)}\left[f_2\right]_{g_i} < \rho_{(\alpha,\beta)}\left[f_2\right]_{g_j}$ hold and at least f_2 is of regular generalized relative growth (α, β) with respect to any one of g_j for $i = 1$, 2, $j = 1, 2$ and $i \neq j$;
(iii) $\rho_{(\alpha,\beta)}\left[f_i\right]_{g_1} > \rho_{(\alpha,\beta)}\left[f_j\right]_{g_1}$ and $\rho_{(\alpha,\beta)}\left[f_i\right]_{g_2} > \rho_{(\alpha,\beta)}\left[f_j\right]_{g_2}$ holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$;
(iv) $\rho_{(\alpha,\beta)}\left[f_l\right]_{g_m} =$
$\max\left[\min\left\{\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}, \rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}\right\}, \min\left\{\rho_{(\alpha,\beta)}\left[f_2\right]_{g_1}, \rho_{(\alpha,\beta)}\left[f_2\right]_{g_2}\right\}\right] \mid l, m = 1, 2,$
and $g_1 \pm g_2$ has the Property (A);
then

$$\sigma_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1 \pm g_2} = \sigma_{(\alpha,\beta)}\left[f_l\right]_{g_m} \mid l, m = 1, 2$$

and

$$\overline{\sigma}_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1 \pm g_2} = \overline{\sigma}_{(\alpha,\beta)}\left[f_l\right]_{g_m} \mid l, m = 1, 2.$$

Proof. From the definition of generalized relative type (α,β) and generalized relative lower type (α,β) of meromorphic function with respect to an entire function, we have for all sufficiently large values of r that

$$T_{f_k}(r) \leq T_{g_l}\left[\alpha^{-1}\left(\log\left\{\left(\sigma_{(\alpha,\beta)}[f_k]_{g_l}+\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_k]_{g_l}}\right\}\right)\right], \tag{12}$$

$$T_{f_k}(r) \geq T_{g_l}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\sigma}_{(\alpha,\beta)}[f_k]_{g_l}-\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_k]_{g_l}}\right\}\right)\right] \tag{13}$$

and for a sequence of values of r tending to infinity, we obtain that

$$T_{f_k}(r) \geq T_{g_l}\left[\alpha^{-1}\left(\log\left\{\left(\sigma_{(\alpha,\beta)}[f_k]_{g_l}-\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_k]_{g_l}}\right\}\right)\right], \tag{14}$$

and

$$T_{f_k}(r) \leq T_{g_l}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\sigma}_{(\alpha,\beta)}[f_k]_{g_l}+\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_k]_{g_l}}\right\}\right)\right], \tag{15}$$

where $\varepsilon > 0$ is any arbitrary positive number $k = 1,\, 2$ and $l = 1, 2$.

Case I. Suppose that $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$ hold. Also let $\varepsilon\,(>0)$ be arbitrary. Since $T_{f_1\pm f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r) + O(1)$ for all large r, so in view of (12), we get for all sufficiently large values of r that

$$T_{f_1\pm f_2}(r) \leq$$
$$T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right](1+A). \tag{16}$$

where $A = \frac{T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\sigma_{(\alpha,\beta)}[f_2]_{g_1}+\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_2]_{g_1}}\right\}\right)\right]+O(1)}{T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right]}$, and in view of $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$, and for all sufficiently large values of r, we can make the term A sufficiently small . Hence for any $\delta = 1+\varepsilon_1$, it follows from (16) for all sufficiently large values of r that

$$T_{f_1\pm f_2}(r) \leq T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right]\cdot(1+\varepsilon_1)$$

$$i.e.,\ T_{f_1\pm f_2}(r) \leq T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right]\cdot\delta.$$

Hence making $\delta \to 1+$, we get in view of Theorem 2, $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$ and above for all sufficiently large values of r that

$$\limsup_{r\to\infty}\frac{\exp\left(\alpha\left(T_{g_1}^{-1}\left(T_{f_1\pm f_2}(r)\right)\right)\right)}{[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1\pm f_2]_{g_1}}} \leq \sigma_{(\alpha,\beta)}[f_1]_{g_1}$$

$$i.e.,\ \sigma_{(\alpha,\beta)}[f_1\pm f_2]_{g_1} \leq \sigma_{(\alpha,\beta)}[f_1]_{g_1}\ . \tag{17}$$

Now we may consider that $f = f_1 \pm f_2$. Since $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$ hold. Then $\sigma_{(\alpha,\beta)}[f]_{g_1} = \sigma_{(\alpha,\beta)}[f_1\pm f_2]_{g_1} \leq \sigma_{(\alpha,\beta)}[f_1]_{g_1}$. Further, let $f_1 = (f \pm f_2)$. Therefore in view of Theorem 2 and $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$, we obtain that $\rho_{(\alpha,\beta)}[f]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$ holds. Hence in view of (17) $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \leq \sigma_{(\alpha,\beta)}[f]_{g_1} = \sigma_{(\alpha,\beta)}[f_1\pm f_2]_{g_1}$. Therefore $\sigma_{(\alpha,\beta)}[f]_{g_1} = \sigma_{(\alpha,\beta)}[f_1]_{g_1} \Rightarrow \sigma_{(\alpha,\beta)}[f_1\pm f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_1]_{g_1}$.

Similarly, if we consider $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_2]_{g_1}$, then one can easily verify that $\sigma_{(\alpha,\beta)}[f_1\pm f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_2]_{g_1}$.

Case II. Let us consider that $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_2]_{g_1}$ hold. Also let $\varepsilon\,(>0)$ are arbitrary. Since $T_{f_1\pm f_2}(r)\leq T_{f_1}(r)+T_{f_2}(r)+O(1)$ for all large r, from (12) and (15), we get for a sequence of values of r tending to infinity that

$$T_{f_1\pm f_2}(r)\leq$$
$$T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right](1+B). \quad (18)$$

where $B=\frac{T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\sigma_{(\alpha,\beta)}[f_2]_{g_1}+\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_2]_{g_1}}\right\}\right)\right]+O(1)}{T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right]}$, and in view of $\rho_{(\alpha,\beta)}[f_1]_{g_1}>\rho_{(\alpha,\beta)}[f_2]_{g_1}$, we can make the term B sufficiently small by taking r sufficiently large and therefore using the similar technique for as executed in the proof of Case I we get from (18) that $\overline{\sigma}_{(\alpha,\beta)}[f_1\pm f_2]_{g_1}=\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}$ when $\rho_{(\alpha,\beta)}[f_1]_{g_1}>\rho_{(\alpha,\beta)}[f_2]_{g_1}$ hold. Likewise, if we consider $\rho_{(\alpha,\beta)}[f_1]_{g_1}<\rho_{(\alpha,\beta)}[f_2]_{g_1}$, then one can easily verify that $\overline{\sigma}_{(\alpha,\beta)}[f_1\pm f_2]_{g_1}=\overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$.

Thus combining Case I and Case II, we obtain the first part of the theorem.

Case III. Let us consider that $\rho_{(\alpha,\beta)}[f_1]_{g_1}<\rho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_2. We can make the term

$$C=\frac{T_{g_2}\left[\alpha^{-1}\left(\log\left\{\left(\sigma_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right]+O(1)}{T_{g_2}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}-\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_2}}\right\}\right)\right]}$$

sufficiently small by taking r sufficiently large, since $\rho_{(\alpha,\beta)}[f_1]_{g_1}<\rho_{(\alpha,\beta)}[f_1]_{g_2}$. Hence $C<\varepsilon_1$.

As $T_{g_1\pm g_2}(r)\leq T_{g_1}(r)+T_{g_2}(r)+O(1)$ for all large r, we get that

$$T_{g_1\pm g_2}\left(\alpha^{-1}\left(\log\left\{\left(\sigma_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right)\leq$$
$$T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\sigma_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right]+$$
$$T_{g_2}\left[\alpha^{-1}\left(\log\left\{\left(\sigma_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right]+O(1).$$

Therefore for any $\delta=1+\varepsilon_1$, we obtain in view of $C<\varepsilon_1$, (13) and (14) for a sequence of values of r tending to infinity that

$$T_{g_1\pm g_2}\left(\alpha^{-1}\left(\log\left\{\left(\sigma_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right)\leq\delta T_{f_1}(r)$$

Now making $\delta\to 1+$, we obtain from above for a sequence of values of r tending to infinity that

$$\left(\sigma_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1\pm g_2}}<\exp\left(\alpha\left(T_{g_1\pm g_2}^{-1}\left(T_{f_1}(r)\right)\right)\right)$$

Since $\varepsilon>0$ is arbitrary, we find that

$$\sigma_{(\alpha,\beta)}[f_1]_{g_1\pm g_2}\geq\sigma_{(\alpha,\beta)}[f_1]_{g_1}. \quad (19)$$

Now we may consider that $g=g_1\pm g_2$. Also $\rho_{(\alpha,\beta)}[f_1]_{g_1}<\rho_{(\alpha,\beta)}[f_1]_{g_2}$ and at least f_1 is of regular generalized relative growth (α,β) with respect to g_2. Then $\sigma_{(\alpha,\beta)}[f_1]_g=\sigma_{(\alpha,\beta)}[f_1]_{g_1\pm g_2}\geq\sigma_{(\alpha,\beta)}[f_1]_{g_1}$. Further let $g_1=(g\pm g_2)$. Therefore in view of Theorem 4 and $\rho_{(\alpha,\beta)}[f_1]_{g_1}<\rho_{(\alpha,\beta)}[f_1]_{g_2}$, we obtain that $\rho_{(\alpha,\beta)}[f_1]_g<\rho_{(\alpha,\beta)}[f_1]_{g_2}$ as at least f_1 is of regular generalized relative growth (α,β) with respect to g_2. Hence in view of (19), $\sigma_{(\alpha,\beta)}[f_1]_{g_1}\geq\sigma_{(\alpha,\beta)}[f_1]_g=\sigma_{(\alpha,\beta)}[f_1]_{g_1\pm g_2}$. Therefore $\sigma_{(\alpha,\beta)}[f_1]_g=\sigma_{(\alpha,\beta)}[f_1]_{g_1}\Rightarrow\sigma_{(\alpha,\beta)}[f_1]_{g_1\pm g_2}=\sigma_{(\alpha,\beta)}[f_1]_{g_1}$.

Similarly if we consider $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1, then $\sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_2}$.

Case IV. In this case suppose that $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_2. we can also make the term $D = \frac{T_{g_2}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right]+O(1)}{T_{g_2}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}-\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_2}}\right\}\right)\right]}$ sufficiently small by taking r sufficiently large as $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$. So $D < \varepsilon_1$ for sufficiently large r. As $T_{g_1 \pm g_2}(r) \le T_{g_1}(r) + T_{g_2}(r) + O(1)$ for all large r, therefore from (13), we get for all sufficiently large values of r that

$$T_{g_1\pm g_2}\left(\alpha^{-1}\left(\log\left\{\left(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right) \le$$
$$T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right] +$$
$$T_{g_2}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right] + O(1)$$
$$i.e.,\ T_{g_1\pm g_2}\left(\alpha^{-1}\left(\log\left\{\left(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\rho_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right)$$
$$\le (1+\varepsilon_1)\, T_{f_1}(r), \quad (20)$$

and therefore using the similar technique for as executed in the proof of Case III we get from (20) that $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}$ where $\rho_{(\alpha,\beta)}[f_1]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_2}$ and at least f_1 is of regular generalized relative growth (α,β) with respect to g_2.

Likewise if we consider $\rho_{(\alpha,\beta)}[f_1]_{g_1} > \rho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1, then $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The third part of the theorem is a natural consequence of Theorem 5 and the first part and second part of the theorem. Hence its proof is omitted. □

Theorem 18. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$, $\lambda_{(\alpha,\beta)}[f_2]_{g_1}$, $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $\lambda_{(\alpha,\beta)}[f_2]_{g_2}$ be all nonzero and finite.*

(A) *Any one of $\lambda_{(\alpha,\beta)}[f_i]_{g_1} > \lambda_{(\alpha,\beta)}[f_j]_{g_1}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_1 for i, $j = 1,2$; $i \ne j$, and g_1 has the Property (A), then*

$$\tau_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_i]_{g_1} \ \text{ and } \ \overline{\tau}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1,2.$$

(B) *Any one of $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j}$ hold for i, $j = 1,2$; $i \ne j$ and $g_1 \pm g_2$ has the Property (A), then*

$$\tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_i} \ \text{ and } \ \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2.$$

(C) *Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*
(i) Any one of $\rho_{(\alpha,\beta)}[f_i]_{g_1} > \rho_{(\alpha,\beta)}[f_j]_{g_1}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_1 for i, $j = 1,2$ and $i \ne j$;
(ii) Any one of $\rho_{(\alpha,\beta)}[f_i]_{g_2} > \rho_{(\alpha,\beta)}[f_j]_{g_2}$ hold and at least any one of f_j is of regular generalized relative growth (α,β) with respect to g_2 for i, $j = 1,2$ and $i \ne j$;
(iii) $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j}$ and $\rho_{(\alpha,\beta)}[f_2]_{g_i} < \rho_{(\alpha,\beta)}[f_2]_{g_j}$ holds simultaneously

for $i,\ j = 1, 2$ *and* $i \neq j$;
(iv) $\lambda_{(\alpha,\beta)}\left[f_l\right]_{g_m} =$
$\min\left[\max\left\{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1}, \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}\right\}, \max\left\{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2}, \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_2}\right\}\right] \mid l, m = 1, 2$
and $g_1 \pm g_2$ *has the Property (A)*
then we have

$$\tau_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1 \pm g_2} = \tau_{(\alpha,\beta)}\left[f_l\right]_{g_m} \mid l, m = 1, 2$$

and

$$\overline{\tau}_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1 \pm g_2} = \overline{\tau}_{(\alpha,\beta)}\left[f_l\right]_{g_m} \mid l, m = 1, 2.$$

Proof. For any arbitrary positive number $\varepsilon(> 0)$, we have for all sufficiently large values of r that

$$T_{f_k}(r) \leq T_{g_l}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\tau}_{(\alpha,\beta)}\left[f_k\right]_{g_l} + \varepsilon\right)\left[\exp\beta(r)\right]^{\lambda_{(\alpha,\beta)}\left[f_k\right]_{g_l}}\right\}\right)\right], \tag{21}$$

$$T_{f_k}(r) \geq T_{g_l}\left[\alpha^{-1}\left(\log\left\{\left(\tau_{(\alpha,\beta)}\left[f_k\right]_{g_l} - \varepsilon\right)\left[\exp\beta(r)\right]^{\lambda_{(\alpha,\beta)}\left[f_k\right]_{g_l}}\right\}\right)\right], \tag{22}$$

and for a sequence of values of r tending to infinity we obtain that

$$T_{f_k}(r) \geq T_{g_l}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\tau}_{(\alpha,\beta)}\left[f_k\right]_{g_l} - \varepsilon\right)\left[\exp\beta(r)\right]^{\lambda_{(\alpha,\beta)}\left[f_k\right]_{g_l}}\right\}\right)\right] \tag{23}$$

and

$$T_{f_k}(r) \leq T_{g_l}\left[\alpha^{-1}\left(\log\left\{\left(\tau_{(\alpha,\beta)}\left[f_k\right]_{g_l} + \varepsilon\right)\left[\exp\beta(r)\right]^{\lambda_{(\alpha,\beta)}\left[f_k\right]_{g_l}}\right\}\right)\right], \tag{24}$$

where $k = 1, 2$ and $l = 1, 2$.

Case I. Let $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1} > \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}$ with at least f_2 is of regular generalized relative growth (α, β) with respect to g_1. Also let $\varepsilon(> 0)$ be arbitrary. Since $T_{f_1 \pm f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r) + O(1)$ for all large r, we get from (21) and (24), for a sequence of values of r tending to infinity that

$$T_{f_1 \pm f_2}(r) \leq$$
$$T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\tau_{(\alpha,\beta)}\left[f_1\right]_{g_1} + \varepsilon\right)\left[\exp\beta(r)\right]^{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1}}\right\}\right)\right](1 + E). \tag{25}$$

where $E = \frac{T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\tau}_{(\alpha,\beta)}\left[f_2\right]_{g_1} + \varepsilon\right)\left[\exp\beta(r)\right]^{\lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}}\right\}\right)\right] + O(1)}{T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\tau_{(\alpha,\beta)}\left[f_1\right]_{g_1} + \varepsilon\right)\left[\exp\beta(r)\right]^{\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1}}\right\}\right)\right]}$ and in view of $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1} > \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}$, we can make the term E sufficiently small by taking r sufficiently large. Now with the help of Theorem 1 and using the similar technique of Case I of Theorem 17, we get from (25) that

$$\tau_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1} \leq \tau_{(\alpha,\beta)}\left[f_1\right]_{g_1}. \tag{26}$$

Further, we may consider that $f = f_1 \pm f_2$. Also suppose that $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1} > \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}$ and at least f_2 is of regular generalized relative growth (α, β) with respect to g_1. Then $\tau_{(\alpha,\beta)}\left[f\right]_{g_1} = \tau_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1} \leq \tau_{(\alpha,\beta)}\left[f_1\right]_{g_1}$. Now let $f_1 = (f \pm f_2)$. Therefore in view of Theorem 1, $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1} > \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}$ and at least f_2 is of regular generalized relative growth (α, β) with respect to g_1, we obtain that $\lambda_{(\alpha,\beta)}\left[f\right]_{g_1} > \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}$ holds. Hence in view of (26), $\tau_{(\alpha,\beta)}\left[f_1\right]_{g_1} \leq \tau_{(\alpha,\beta)}\left[f\right]_{g_1} = \tau_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1}$. Therefore $\tau_{(\alpha,\beta)}\left[f\right]_{g_1} = \tau_{(\alpha,\beta)}\left[f_1\right]_{g_1} \Rightarrow \tau_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1} = \tau_{(\alpha,\beta)}\left[f_1\right]_{g_1}$.

Similarly, if we consider $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1} < \lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}$ with at least f_1 is of regular generalized relative growth (α, β) with respect to g_1 then one can easily verify that $\tau_{(\alpha,\beta)}\left[f_1 \pm f_2\right]_{g_1} = \tau_{(\alpha,\beta)}\left[f_2\right]_{g_1}$.

Case II. Let us consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_2 is of regular generalized relative growth (α,β) with respect to g_1. Also let $\varepsilon(>0)$ be arbitrary. As $T_{f_1\pm f_2}(r) \le T_{f_1}(r) + T_{f_2}(r) + O(1)$ for all large r, we obtain from (21) for all sufficiently large values of r that

$$T_{f_1\pm f_2}(r) \le$$
$$T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon\right)[\exp\beta(r)]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right](1+F). \quad (27)$$

where $F = \frac{T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}+\varepsilon\right)[\exp\beta(r)]^{\lambda_{(\alpha,\beta)}[f_2]_{g_i}}\right\}\right)\right]+O(1)}{T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon\right)[\exp\beta(r)]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right]}$, and in view of $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$, we can make the term F sufficiently small by taking r sufficiently large and therefore for similar reasoning of Case I we get from (27) that $\overline{\tau}_{(\alpha,\beta)}[f_1\pm f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}$ when $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ and at least f_2 is of regular generalized relative growth (α,β) with respect to g_1.

Likewise, if we consider $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1 then one can easily verify that $\overline{\tau}_{(\alpha,\beta)}[f_1\pm f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$.

Thus combining Case I and Case II, we obtain the first part of the theorem.

Case III. Let us consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore we can make the term $G = \frac{T_{g_2}\left[\alpha^{-1}\left(\log\left\{\left(\tau_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right]+O(1)}{T_{g_2}\left[\alpha^{-1}\left(\log\left\{\left(\tau_{(\alpha,\beta)}[f_1]_{g_2}-\varepsilon\right)[\exp\beta(r)]^{\lambda_{(\alpha,\beta)}[f_1]_{g_2}}\right\}\right)\right]}$ sufficiently small by taking r sufficiently large since $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. So $G < \varepsilon_1$. Since $T_{g_1\pm g_2}(r) \le T_{g_1}(r)+T_{g_2}(r)+O(1)$ for all large r, we get from (22) for all sufficiently large values of r that

$$T_{g_1\pm g_2}\left(\alpha^{-1}\left(\log\left\{\left(\tau_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right) \le$$
$$T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\tau_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right]+$$
$$T_{g_2}\left[\alpha^{-1}\left(\log\left\{\left(\tau_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right]+O(1)$$
$$i.e.,\ T_{g_1\pm g_2}\left(\alpha^{-1}\left(\log\left\{\left(\tau_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right)$$
$$\le (1+\varepsilon_1)\,T_{f_1}(r). \quad (28)$$

Therefore in view of Theorem 3 and using the similar technique of Case III of Theorem 17, we get from (28) that

$$\tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} \ge \tau_{(\alpha,\beta)}[f_1]_{g_1}. \quad (29)$$

Further, we may consider that $g = g_1 \pm g_2$. As $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, so $\tau_{(\alpha,\beta)}[f_1]_g = \tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} \ge \tau_{(\alpha,\beta)}[f_1]_{g_1}$. Further let $g_1 = (g \pm g_2)$. Therefore in view of Theorem 3 and $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ we obtain that $\lambda_{(\alpha,\beta)}[f_1]_g < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ holds. Hence in view of (29) $\tau_{(\alpha,\beta)}[f_1]_{g_1} \ge \tau_{(\alpha,\beta)}[f_1]_g = \tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2}$. Therefore $\tau_{(\alpha,\beta)}[f_1]_g = \tau_{(\alpha,\beta)}[f_1]_{g_1} \Rightarrow \tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_1}$.

Likewise, if we consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, then one can easily verify that $\tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_2}$.

Case IV. In this case further we consider $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Further we can make the term $H = \frac{T_{g_2}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon\right)[\exp\beta(r)]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right]+O(1)}{T_{g_2}\left[\alpha^{-1}\left(\log\left\{\left(\tau_{(\alpha,\beta)}[f_1]_{g_2}-\varepsilon\right)[\exp\beta(r)]^{\lambda_{(\alpha,\beta)}[f_1]_{g_2}}\right\}\right)\right]}$

sufficiently small by taking r sufficiently large, since $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore $H < \varepsilon_1$ for sufficiently large r. As $T_{g_1 \pm g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r) + O(1)$ for all large r, hence we obtain from (22) and (23), for a sequence of values of r tending to infinity that

$$T_{g_1 \pm g_2}\left(\alpha^{-1}\left(\log\left\{\left(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon\right)[\exp\beta(r)]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right) \leq$$
$$T_{g_1}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon\right)[\exp\beta(r)]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right] +$$
$$T_{g_2}\left[\alpha^{-1}\left(\log\left\{\left(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon\right)[\exp\beta(r)]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right] + O(1)$$
$$i.e.,\ T_{g_1 \pm g_2}\left(\alpha^{-1}\left(\log\left\{\left(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon\right)[\exp\beta(r)]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\right\}\right)\right)$$
$$\leq (1+\varepsilon_1)\, T_{f_1}(r), \quad (30)$$

and therefore using the similar technique for as executed in the proof of Case IV of Theorem 17, we get from (30) that $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}$ when $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$.

Similarly, if we consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, then one can easily verify that $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 6 and the above cases. □

In the next two theorems we reconsider the equalities in Theorem 1 to Theorem 4 under somewhat different conditions.

Theorem 19. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions.*
(A) *The following condition is assumed to be satisfied:*
(i) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$ holds and g_1 has the Property (A), then

$$\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}.$$

(B) *The following conditions are assumed to be satisfied:*
(i) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$ holds and $g_1 \pm g_2$ has the Property (A);
(ii) f_1 is of regular generalized relative growth (α,β) with respect to at least any one of g_1 or g_2, then

$$\rho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}.$$

Proof. Let f_1, f_2, g_1 and g_2 be any four entire functions satisfying the conditions of the theorem.

Case I. Suppose that $\rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}$ $(0 < \rho_{(\alpha,\beta)}[f_1]_{g_1},\ \rho_{(\alpha,\beta)}[f_2]_{g_1} < \infty)$. Now in view of Theorem 2 it is easy to see that $\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}$. If possible let

$$\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}. \quad (31)$$

Let $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1}$. Then in view of the first part of Theorem 17 and (31) we obtain that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} = \sigma_{(\alpha,\beta)}[f_1 \pm f_2 \mp f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ which is a contradiction. Hence $\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}$. Similarly with the help of the first part of Theorem 17, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$. This proves the first part of the theorem.

Case II. Let us consider that $\rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}$ $(0 < \rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2} < \infty)$, f_1 is of regular generalized relative growth (α,β) with respect to at least any one of g_1 or g_2 and $(g_1 \pm g_2)$ and $g_1 \pm g_2$ satisfy the Property (A). Therefore in view of Theorem 4, it follows that $\rho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \geq \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}$ and if possible let

$$\rho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} > \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}. \tag{32}$$

Let us consider that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1]_{g_2}$. Then. in view of the proof of the second part of Theorem 17 and (32) we obtain that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} = \sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2 \mp g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_2}$ which is a contradiction. Hence $\rho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}$. Also in view of the proof of second part of Theorem 17 one can derive the same conclusion for the condition $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$ and therefore the second part of the theorem is established. □

Theorem 20. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions.*
(A) *The following conditions are assumed to be satisfied:*
(i) $(f_1 \pm f_2)$ is of regular generalized relative growth (α,β) with respect to at least any one of g_1 and g_2; and g_1, g_2 , $g_1 \pm g_2$ have the Property (A);
(ii) Either $\sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2}$;
(iii) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$;
(iv) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_2} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_2}$; then

$$\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2} = \rho_{(\alpha,\beta)}[f_2]_{g_2}.$$

(B) *The following conditions are assumed to be satisfied:*
(i) f_1 and f_2 are of regular generalized relative growth (α,β) with respect to at least any one of g_1 or g_2, and $g_1 \pm g_2$ has the Property (A);
(ii) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1 \pm g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1 \pm g_2}$;
(iii) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$;
(iv) Either $\sigma_{(\alpha,\beta)}[f_2]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_2}$; then

$$\rho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2} = \rho_{(\alpha,\beta)}[f_2]_{g_2}.$$

We omit the proof of Theorem 20 as it is a natural consequence of Theorem 19.

Theorem 21. *Let f_1, f_2 be any two meromorphic functions and g_1,g_2 be any two entire functions.*
(A) *The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 or f_2 is of regular generalized relative growth (α,β) with respect to g_1;
(ii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$ holds and g_1 has the Property (A), then

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}.$$

(B) *The following conditions are assumed to be satisfied:*
(i) f_1, g_1 and g_2 be any three entire functions such that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$ and $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$

exists;
(ii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1]_{g_2}$ *or* $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$ *holds and* $g_1 \pm g_2$ *has the Property (A), then*

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}.$$

Proof. Let f_1, f_2, g_1 and g_2 be any four entire functions satisfying the conditions of the theorem.

Case I. Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ $(0 < \lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1} < \infty)$ and at least f_1 or f_2 and $(f_1 \pm f_2)$ are of regular generalized relative growth (α,β) with respect to g_1. Now, in view of Theorem 1, it is easy to see that $\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. If possible let

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}. \tag{33}$$

Let $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1}$. Then in view of the proof of the first part of Theorem 18 and (33) we obtain that $\tau_{(\alpha,\beta)}[f_1]_{g_1} = \tau_{(\alpha,\beta)}[f_1 \pm f_2 \mp f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_2]_{g_1}$ which is a contradiction. Hence $\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. Similarly in view of the proof of the first part of Theorem 18 , one can establish the same conclusion under the hypothesis $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$. This proves the first part of the theorem.

Case II. Let us consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ $(0 < \lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2} < \infty$. Therefore in view of Theorem 3, it follows that $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \geq \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and if possible let

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} > \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}. \tag{34}$$

Suppose $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1]_{g_2}$. Then in view of the second part of Theorem 18 and (34), we obtain that $\tau_{(\alpha,\beta)}[f_1]_{g_1} = \tau_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2 \mp g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_2}$ which is a contradiction. Hence $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Analogously with the help of the second part of Theorem 18, the same conclusion can also be derived under the condition $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{g_2}^{(p,q)}(f_1)$ and therefore the second part of the theorem is established. □

Theorem 22. *Let* f_1, f_2 *be any two meromorphic functions and* g_1, g_2 *be any two entire functions.*
(A) *The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 *or* f_2 *is of regular generalized relative growth* (α,β) *with respect to* g_1 *and* g_2*. Also* g_1, g_2, $g_1 \pm g_2$ *have satisfy the Property (A);*
(ii) Either $\tau_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2}$ *or* $\overline{\tau}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2}$*;*
(iii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1}$ *or* $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$ *;*
(iv) Either $\tau_{(\alpha,\beta)}[f_1]_{g_2} \neq \tau_{(\alpha,\beta)}[f_2]_{g_2}$ *or* $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_2}$ *; then*

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2} = \lambda_{(\alpha,\beta)}[f_2]_{g_2}.$$

(B) *The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 *or* f_2 *are of regular generalized relative growth* (α,β) *with respect to* $g_1 \pm g_2$*, and* $g_1 \pm g_2$ *has satisfy the Property (A);*
(ii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1 \pm g_2}$ *or* $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1 \pm g_2}$ *holds;*

(iii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1]_{g_2}$ *or* $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$ *holds;*
(iv) Either $\tau_{(\alpha,\beta)}[f_2]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_2}$ *or* $\overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_2}$ *holds, then*

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2} = \lambda_{(\alpha,\beta)}[f_2]_{g_2}.$$

We omit the proof of Theorem 22 as it is a natural consequence of Theorem 21.

Theorem 23. *Let* f_1, f_2 *be any two meromorphic functions and* g_1, g_2 *be any two entire functions. Also let* $\rho_{(\alpha,\beta)}[f_1]_{g_1}$, $\rho_{(\alpha,\beta)}[f_2]_{g_1}$, $\rho_{(\alpha,\beta)}[f_1]_{g_2}$ *and*$\rho_{(\alpha,\beta)}[f_2]_{g_2}$ *be all non zero and finite.*
(A) *Assume the functions* f_1, f_2 *and* g_1 *satisfy the following conditions:*
(i) Any one of $\rho_{(\alpha,\beta)}[f_i]_{g_1} > \rho_{(\alpha,\beta)}[f_j]_{g_1}$ *hold for* $i,\ j = 1,2$ *and* $i \neq j$*;*
(ii) g_1 *satisfies the Property (A), then*

$$\sigma_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_i]_{g_1} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1,2.$$

Similarly,

$$\sigma_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \sigma_{(\alpha,\beta)}[f_i]_{g_1} \text{ and } \overline{\sigma}_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1,2$$

holds provided (i) $\frac{f_1}{f_2}$ *is meromorphic, (ii)* $\rho_{(\alpha,\beta)}[f_i]_{g_1} > \rho_{(\alpha,\beta)}[f_j]_{g_1} \mid i,\ 1,2;\ j = 1,2;\ i \neq j$ *and (iii)* g_1 *satisfy the Property (A).*
(B) *Assume the functions* g_1, g_2 *and* f_1 *satisfy the following conditions:*
(i) Any one of $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j}$ *hold and at least* f_1 *is of regular generalized relative growth* (α,β) *with respect to any one of* g_j *for* $i,\ j = 1,2$ *and* $i \neq j$*, and* g_i *satisfies the Property (A);*
(ii) $g_1 \cdot g_2$ *satisfies the Property (A), then*

$$\sigma_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_i} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2.$$

Similarly,

$$\sigma_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \sigma_{(\alpha,\beta)}[f_1]_{g_i} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2$$

holds provided (i) $\frac{g_1}{g_2}$ *is entire and satisfy the Property (A), (ii) At least* f_1 *is of regular generalized relative growth* (α,β) *with respect to* g_2*, (iii)* $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j} \mid i = 1,2;\ j = 1,2;\ i \neq j$ *and (iv)* g_1 *satisfy the Property (A).*
(C) *Assume the functions* f_1, f_2*,* g_1 *and* g_2 *satisfy the following conditions:*
(i) $g_1 \cdot g_2$ *satisfies the Property (A);*
(ii) Any one of $\rho_{(\alpha,\beta)}[f_1]_{g_i} < \rho_{(\alpha,\beta)}[f_1]_{g_j}$ *hold and at least* f_1 *is of regular generalized relative growth* (α,β) *with respect to* g_j *for* $i = 1,\ 2,\ j = 1,2$ *and* $i \neq j$*;*
(iii) Any one of $\rho_{(\alpha,\beta)}[f_2]_{g_i} < \rho_{(\alpha,\beta)}[f_2]_{g_j}$ *hold and at least* f_2 *is of regular generalized relative growth* (α,β) *with respect to* g_j *for* $i = 1,\ 2,\ j = 1,2$ *and* $i \neq j$*;*
(iv) $\rho_{(\alpha,\beta)}[f_i]_{g_1} > \rho_{(\alpha,\beta)}[f_j]_{g_1}$ *and* $\rho_{(\alpha,\beta)}[f_i]_{g_2} > \rho_{(\alpha,\beta)}[f_j]_{g_2}$ *holds simultaneously for* $i = 1,2;\ j = 1,2$ *and* $i \neq j$*;*
(v) $\rho_{(\alpha,\beta)}[f_l]_{g_m} =$
$\max\left[\min\left\{\rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2}\right\}, \min\left\{\rho_{(\alpha,\beta)}[f_2]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_2}\right\}\right] \mid l,m = 1,2;$
then

$$\sigma_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \sigma_{(\alpha,\beta)}[f_l]_{g_m} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_l]_{g_m} \mid l,m = 1,2.$$

Similarly,

$$\sigma_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{\frac{g_1}{g_2}} = \sigma_{(\alpha,\beta)}\left[f_l\right]_{g_m} \text{ and } \overline{\sigma}_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{\frac{g_1}{g_2}} = \overline{\sigma}_{(\alpha,\beta)}\left[f_l\right]_{g_m} \mid l, m = 1, 2.$$

holds provided $\frac{f_1}{f_2}$ *is meromorphic function and* $\frac{g_1}{g_2}$ *is entire function which satisfy the following conditions:*
(i) $\frac{g_1}{g_2}$ *satisfies the Property (A);*
(ii) At least f_1 *is of regular generalized relative growth* (α, β) *with respect to* g_2 *and* $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1} \neq \rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}$*;*
(iii) At least f_2 *is of regular generalized relative growth* (α, β) *with respect to* g_2 *and* $\rho_{(\alpha,\beta)}\left[f_2\right]_{g_1} \neq \rho_{(\alpha,\beta)}\left[f_2\right]_{g_2}$ *;*
(iv) $\rho_{(\alpha,\beta)}\left[f_i\right]_{g_1} < \rho_{(\alpha,\beta)}\left[f_j\right]_{g_1}$ *and* $\rho_{(\alpha,\beta)}\left[f_i\right]_{g_2} < \rho_{(\alpha,\beta)}\left[f_j\right]_{g_2}$ *holds simultaneously for* $i = 1, 2$*;* $j = 1, 2$ *and* $i \neq j$*;*
(v) $\rho_{(\alpha,\beta)}\left[f_l\right]_{g_m} =$
$\max\left[\min\left\{\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}, \rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}\right\}, \min\left\{\rho_{(\alpha,\beta)}\left[f_2\right]_{g_1}, \rho_{(\alpha,\beta)}\left[f_2\right]_{g_2}\right\}\right] \mid l, m = 1, 2.$

Proof. Let us suppose that $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}$, $\rho_{(\alpha,\beta)}\left[f_2\right]_{g_1}$, $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}$ and$\rho_{(\alpha,\beta)}\left[f_2\right]_{g_2}$ are all non zero and finite.

Case I. Suppose that $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1} > \rho_{(\alpha,\beta)}\left[f_2\right]_{g_1}$. Also let g_1 satisfy the Property (A). Since $T_{f_1 \cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$ for all large r, therefore applying the same procedure as adopted in Case I of Theorem 17 we get that

$$\sigma_{(\alpha,\beta)}\left[f_1 \cdot f_2\right]_{g_1} \leq \sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1}. \tag{35}$$

Further without loss of any generality, let $f = f_1 \cdot f_2$ and $\rho_{(\alpha,\beta)}\left[f_2\right]_{g_1} < \rho_{(\alpha,\beta)}\left[f_1\right]_{g_1} = \rho_{(\alpha,\beta)}\left[f\right]_{g_1}$. Then in view of (35), we obtain that $\sigma_{(\alpha,\beta)}\left[f\right]_{g_1} = \sigma_{(\alpha,\beta)}\left[f_1 \cdot f_2\right]_{g_1} \leq \sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1}$. Also $f_1 = \frac{f}{f_2}$ and $T_{f_2}(r) = T_{\frac{1}{f_2}}(r) + O(1)$. Therefore $T_{f_1}(r) \leq T_f(r) + T_{f_2}(r) + O(1)$ and in this case also we obtain from (35) that $\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1} \leq \sigma_{(\alpha,\beta)}\left[f\right]_{g_1} = \sigma_{(\alpha,\beta)}\left[f_1 \cdot f_2\right]_{g_1}$. Hence$\sigma_{(\alpha,\beta)}\left[f\right]_{g_1} = \sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1}$ $\Rightarrow \sigma_{(\alpha,\beta)}\left[f_1 \cdot f_2\right]_{g_1} = \sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1}$.

Similarly, if we consider $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1} < \rho_{(\alpha,\beta)}\left[f_2\right]_{g_1}$, then one can verify that $\sigma_{(\alpha,\beta)}\left[f_1 \cdot f_2\right]_{g_1} = \sigma_{(\alpha,\beta)}\left[f_2\right]_{g_1}$.

Next we may suppose that $f = \frac{f_1}{f_2}$ with f_1, f_2 and f are all meromorphic functions.

Sub Case I$_{\mathbf{A}}$. Let $\rho_{(\alpha,\beta)}\left[f_2\right]_{g_1} < \rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}$. Therefore in view of Theorem 9, $\rho_{(\alpha,\beta)}\left[f_2\right]_{g_1} < \rho_{(\alpha,\beta)}\left[f_1\right]_{g_1} = \rho_{(\alpha,\beta)}\left[f\right]_{g_1}$. We have $f_1 = f \cdot f_2$. So, $\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1}$ $= \sigma_{(\alpha,\beta)}\left[f\right]_{g_1} = \sigma_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1}$.

Sub Case I$_{\mathbf{B}}$. Let $\rho_{(\alpha,\beta)}\left[f_2\right]_{g_1} > \rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}$. Therefore in view of Theorem 9, $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1} < \rho_{(\alpha,\beta)}\left[f_2\right]_{g_1} = \rho_{(\alpha,\beta)}\left[f\right]_{g_1}$. Since $T_f(r) = T_{\frac{1}{f}}(r) + O(1) = T_{\frac{f_2}{f_1}}(r) + O(1)$, So $\sigma_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \sigma_{(\alpha,\beta)}\left[f_2\right]_{g_1}$.

Case II. Let $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1} > \rho_{(\alpha,\beta)}\left[f_2\right]_{g_1}$. Also let g_1 satisfy the Property (A). As $T_{f_1 \cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$ for all large r, therefore applying the same procedure as explored in Case II of Theorem 17, one can easily verify that $\overline{\sigma}_{(\alpha,\beta)}\left[f_1 \cdot f_2\right]_{g_1} =$

$\overline{\sigma}_{(\alpha,\beta)}\left[f_1\right]_{g_1}$ and $\overline{\sigma}_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1}=\overline{\sigma}_{(\alpha,\beta)}\left[f_i\right]_{g_1}\mid i=1,2$ under the conditions specified in the theorem.

Similarly, if we consider $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}<\rho_{(\alpha,\beta)}\left[f_2\right]_{g_1}$, then one can verify that $\overline{\sigma}_{(\alpha,\beta)}\left[f_1\cdot f_2\right]_{g_1}=\overline{\sigma}_{g_1}^{(p,q)}\left(f_2\right)$ and $\overline{\sigma}_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1}=\overline{\sigma}_{g_1}^{(p,q)}\left(f_2\right)$.

Therefore the first part of theorem follows from Case I and Case II.

Case III. Let $g_1\cdot g_2$ satisfy the Property (A) and $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}<\rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_2. Since $T_{g_1\cdot g_2}(r)\leq T_{g_1}(r)+T_{g_2}(r)$ for all large r, therefore applying the same procedure as adopted in Case III of Theorem 17 we get that

$$\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1\cdot g_2}\geq\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1}. \tag{36}$$

Further without loss of any generality, let $g=g_1\cdot g_2$ and $\rho_{(\alpha,\beta)}\left[f_1\right]_g=\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}<\rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}$. Then in view of (36), we obtain that $\sigma_{(\alpha,\beta)}\left[f_1\right]_g=\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1\cdot g_2}\geq\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1}$. Also $g_1=\frac{g}{g_2}$ and $T_{g_2}(r)=T_{\frac{1}{g_2}}(r)+O(1)$. Therefore $T_{g_1}(r)\leq T_g(r)+T_{g_2}(r)+O(1)$ and in this case we obtain from (36) that $\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1}\geq\sigma_{(\alpha,\beta)}\left[f_1\right]_g=\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1\cdot g_2}$. Hence $\sigma_{(\alpha,\beta)}\left[f_1\right]_g=\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1}\Rightarrow\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1\cdot g_2}=\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1}$.

Similarly, if we consider $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}>\rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1, then one can verify that $\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1\cdot g_2}=\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_2}$.

Next we may suppose that $g=\frac{g_1}{g_2}$ with g_1, g_2, g are all entire functions satisfying the conditions specified in the theorem.

Sub Case III$_{\mathbf{A}}$. Let $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}<\rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}$. Therefore in view of Theorem 12, $\rho_{(\alpha,\beta)}\left[f_1\right]_g=\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}<\rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}$. We have $g_1=g\cdot g_2$. So $\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_1}=\sigma_{(\alpha,\beta)}\left[f_1\right]_g=\sigma_{(\alpha,\beta)}\left[f_1\right]_{\frac{g_1}{g_2}}$.

Sub Case III$_{\mathbf{B}}$. Let $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}>\rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}$. Therefore in view of Theorem 12, $\rho_{(\alpha,\beta)}\left[f_1\right]_g=\rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}<\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}$. Since $T_g(r)=T_{\frac{1}{g}}(r)+O(1)=T_{\frac{g_2}{g_1}}(r)+O(1)$, So $\sigma_{(\alpha,\beta)}\left[f_1\right]_{\frac{g_1}{g_2}}=\sigma_{(\alpha,\beta)}\left[f_1\right]_{g_2}$.

Case IV. Suppose $g_1\cdot g_2$ satisfy the Property (A). Also let $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}<\rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_2. As $T_{g_1\cdot g_2}(r)\leq T_{g_1}(r)+T_{g_2}(r)$ for all large r, the same procedure as explored in Case IV of Theorem 17, one can easily verify that $\overline{\sigma}_{(\alpha,\beta)}\left[f_1\right]_{g_1\cdot g_2}=\overline{\sigma}_{(\alpha,\beta)}\left[f_1\right]_{g_1}$ and $\overline{\sigma}_{(\alpha,\beta)}\left[f_1\right]_{\frac{g_1}{g_2}}=\overline{\sigma}_{(\alpha,\beta)}\left[f_1\right]_{g_i}\mid i=1,2$ under the conditions specified in the theorem.

Likewise, if we consider $\rho_{(\alpha,\beta)}\left[f_1\right]_{g_1}>\rho_{(\alpha,\beta)}\left[f_1\right]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1, then one can verify that $\overline{\sigma}_{(\alpha,\beta)}\left[f_1\right]_{g_1\cdot g_2}=\overline{\sigma}_{(\alpha,\beta)}\left[f_1\right]_{g_2}$ and $\overline{\sigma}_{(\alpha,\beta)}\left[f_1\right]_{\frac{g_1}{g_2}}=\overline{\sigma}_{(\alpha,\beta)}\left[f_1\right]_{g_2}$. Therefore the second part of theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 13 and Theorem 15 and the above cases. □

Theorem 24. *Let f_1,f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions. Also let $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_1}$, $\lambda_{(\alpha,\beta)}\left[f_2\right]_{g_1}$, $\lambda_{(\alpha,\beta)}\left[f_1\right]_{g_2}$ and $\lambda_{(\alpha,\beta)}\left[f_2\right]_{g_2}$ be all non zero and finite.*

(A) *Assume the functions f_1,f_2 and g_1 satisfy the following conditions:*

(i) Any one of $\lambda_{(\alpha,\beta)}[f_i]_{g_1} > \lambda_{(\alpha,\beta)}[f_j]_{g_1}$ *hold and at least any one of* f_j *is of regular generalized relative growth* (α,β) *with respect to* g_1 *for* $i,\ j = 1,2$ *and* $i \neq j$*;*
(ii) g_1 *satisfies the Property (A), then*

$$\tau_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_i]_{g_1} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1,2.$$

Similarly,

$$\tau_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \tau_{(\alpha,\beta)}[f_i]_{g_1} \text{ and } \overline{\tau}_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1,2$$

holds provided $\frac{f_1}{f_2}$ *is meromorphic, at least* f_2 *is of regular generalized relative growth* (α,β) *with respect to* g_1 *where* g_1 *satisfy the Property (A) and* $\lambda_{(\alpha,\beta)}[f_i]_{g_1} > \lambda_{(\alpha,\beta)}[f_j]_{g_1} \mid i = 1,2;\ j = 1,2;\ i \neq j.$
(B) *Assume the functions* g_1, g_2 *and* f_1 *satisfy the following conditions:*
(i) Any one of $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j}$ *hold for* $i,\ j = 1,2,\ i \neq j$*; and* g_i *satisfy the Property (A)*
(ii) $g_1 \cdot g_2$ *satisfy the Property (A), then*

$$\tau_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_i} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2.$$

Similarly,

$$\tau_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \tau_{(\alpha,\beta)}[f_1]_{g_i} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2$$

holds provided $\frac{g_1}{g_2}$ *is entire and satisfy the Property (A),* g_1 *satisfy the Property (A) and* $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j} \mid i = 1,2;\ j = 1,2;\ i \neq j.$
(C) *Assume the functions* $f_1, f_2,\ g_1$ *and* g_2 *satisfy the following conditions:*
(i) $g_1 \cdot g_2,\ g_1$ *and* g_2 *are satisfy the Property (A);*
(ii) Any one of $\lambda_{(\alpha,\beta)}[f_i]_{g_1} > \lambda_{(\alpha,\beta)}[f_j]_{g_1}$ *hold and at least any one of* f_j *is of regular generalized relative growth* (α,β) *with respect to* g_1 *for* $i = 1,\ 2,\ j = 1,2$ *and* $i \neq j$*;*
(iii) Any one of $\lambda_{(\alpha,\beta)}[f_i]_{g_2} > \lambda_{(\alpha,\beta)}[f_j]_{g_2}$ *hold and at least any one of* f_j *is of regular generalized relative growth* (α,β) *with respect to* g_2 *for* $i = 1,\ 2,\ j = 1,2$ *and* $i \neq j$*;*
(iv) $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j}$ *and* $\lambda_{(\alpha,\beta)}[f_2]_{g_i} < \lambda_{(\alpha,\beta)}[f_2]_{g_j}$ *holds simultaneously for* $i = 1,2;\ j = 1,2$ *and* $i \neq j$*;*
(v) $\lambda_{(\alpha,\beta)}[f_l]_{g_m} =$
$\min\left[\max\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\right\}, \max\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\right\}\right] \mid l,m = 1,2;$
then

$$\tau_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \tau_{(\alpha,\beta)}[f_l]_{g_m} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \overline{\tau}_{(\alpha,\beta)}[f_l]_{g_m} \mid l,m = 1,2.$$

Similarly,

$$\tau_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{\frac{g_1}{g_2}} = \tau_{(\alpha,\beta)}[f_l]_{g_m} \text{ and } \overline{\tau}_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{\frac{g_1}{g_2}} = \overline{\tau}_{(\alpha,\beta)}[f_l]_{g_m} \mid l,m = 1,2.$$

holds provided $\frac{f_1}{f_2}$ *is meromorphic and* $\frac{g_1}{g_2}$ *is entire functions which satisfy the following conditions:*
(i) $\frac{g_1}{g_2}$*,* g_1 *and* g_2 *satisfy the Property (A);*
(ii) At least f_2 *is of regular generalized relative growth* (α,β) *with respect to* g_1 *and* $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \neq \lambda_{(\alpha,\beta)}[f_2]_{g_1}$*;*

(iii) At least f_2 *is of regular generalized relative growth* (α,β) *with respect to* g_2 *and* $\lambda_{(\alpha,\beta)}[f_1]_{g_2} \neq \lambda_{(\alpha,\beta)}[f_2]_{g_2}$ *;*
(iv) $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j}$ *and* $\lambda_{(\alpha,\beta)}[f_2]_{g_i} < \lambda_{(\alpha,\beta)}[f_2]_{g_j}$ *holds simultaneously for* $i = 1,2$; $j = 1,2$ *and* $i \neq j$*;*
(v) $\lambda_{(\alpha,\beta)}[f_l]_{g_m} =$
$\min\left[\max\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\right\}, \max\left\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\right\}\right] \mid l,m = 1,2.$

Proof. Let us consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$, $\lambda_{(\alpha,\beta)}[f_2]_{g_1}$, $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $\lambda_{(\alpha,\beta)}[f_2]_{g_2}$ are all non zero and finite.
Case I. Suppose $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_2 is of regular generalized relative growth (α,β) with respect to g_1 and g_1 satisfy the Property (A). Since $T_{f_1\cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$ for all large r, therefore applying the same procedure as adopted in Case I of Theorem 18 we get that

$$\tau_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \leq \tau_{(\alpha,\beta)}[f_1]_{g_1}. \tag{37}$$

Further without loss of any generality, let $f = f_1 \cdot f_2$ and $\lambda_{(\alpha,\beta)}[f_2]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f]_{g_1}$. Then in view of (37), we obtain that $\tau_{(\alpha,\beta)}[f]_{g_1} = \tau_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \leq \tau_{(\alpha,\beta)}[f_1]_{g_1}$. Also $f_1 = \frac{f}{f_2}$ and $T_{f_2}(r) = T_{\frac{1}{f_2}}(r) + O(1)$. Therefore $T_{f_1}(r) \leq T_f(r) + T_{f_2}(r) + O(1)$ and in this case we obtain from the above arguments that $\tau_{(\alpha,\beta)}[f_1]_{g_1} \leq \tau_{(\alpha,\beta)}[f]_{g_1} = \tau_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1}$. Hence $\tau_{(\alpha,\beta)}[f]_{g_1} = \tau_{(\alpha,\beta)}[f_1]_{g_1}$ $\Rightarrow \tau_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_1]_{g_1}$.

Similarly, if we consider $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_1 is of regular generalized relative growth with respect to g_1, then one can easily verify that $\tau_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_2]_{g_1}$.

Next we may suppose that $f = \frac{f_1}{f_2}$ with f_1, f_2 and f are all meromorphic functions satisfying the conditions specified in the theorem.

Sub Case I$_{\mathbf{A}}$. Let $\lambda_{(\alpha,\beta)}[f_2]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_1}$. Therefore in view of Theorem 8, $\lambda_{(\alpha,\beta)}[f_2]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f]_{g_1}$. We have $f_1 = f \cdot f_2$. So $\tau_{(\alpha,\beta)}[f_1]_{g_1} = \tau_{(\alpha,\beta)}[f]_{g_1} = \tau_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1}$.

Sub Case I$_{\mathbf{B}}$. Let $\lambda_{(\alpha,\beta)}[f_2]_{g_1} > \lambda_{(\alpha,\beta)}[f_1]_{g_1}$. Therefore in view of Theorem 8, $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f]_{g_1}$. Since $T_f(r) = T_{\frac{1}{f}}(r) + O(1) = T_{\frac{f_2}{f_1}}(r) + O(1)$, So $\tau_{(\alpha,\beta)}\left[\frac{f_1}{f_2}\right]_{g_1} = \tau_{(\alpha,\beta)}[f_2]_{g_1}$.

Case II. Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_2 is of regular generalized relative growth (α,β) with respect to g_1 where g_1 satisfy the Property (A). As $T_{f_1\cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$ for all large r, so applying the same procedure as adopted in Case II of Theorem 18 we can easily verify that $\overline{\tau}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}$ and $\tau_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}} = \tau_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2$ under the conditions specified in the theorem.

Similarly, if we consider $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1, then one can easily verify that $\overline{\tau}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$.

Therefore the first part of theorem follows Case I and Case II.
Case III. Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $g_1 \cdot g_2$ satisfy the Property (A).Since

$T_{g_1\cdot g_2}(r)\leq T_{g_1}(r)+T_{g_2}(r)$ for all large r, therefore applying the same procedure as adopted in Case III of Theorem 18 we get that

$$\tau_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2}\leq \tau_{(\alpha,\beta)}[f_1]_{g_1}. \quad (38)$$

Further without loss of any generality, let $g=g_1\cdot g_2$ and $\lambda_{(\alpha,\beta)}[f_1]_g=\lambda_{(\alpha,\beta)}[f_1]_{g_1}<\lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Then in view of (38), we obtain that $\tau_{(\alpha,\beta)}[f_1]_g=\tau_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2}\geq \tau_{(\alpha,\beta)}[f_1]_{g_1}$. Also $g_1=\frac{g}{g_2}$ and $T_{g_2}(r)=T_{\frac{1}{g_2}}(r)+O(1)$. Therefore $T_{g_1}(r)\leq T_g(r)+T_{g_2}(r)+O(1)$ and in this case we obtain from above arguments that $\tau_{(\alpha,\beta)}[f_1]_{g_1}\geq \tau_{(\alpha,\beta)}[f_1]_g=\tau_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2}$. Hence $\tau_{(\alpha,\beta)}[f_1]_g=\tau_{(\alpha,\beta)}[f_1]_{g_1}$ $\Rightarrow \tau_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2}=\tau_{(\alpha,\beta)}[f_1]_{g_1}$.

If $\lambda_{(\alpha,\beta)}[f_1]_{g_1}>\lambda_{(\alpha,\beta)}[f_1]_{g_2}$, then one can easily verify that $\tau_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2}$ $=\tau_{(\alpha,\beta)}[f_1]_{g_2}$.

Next we may suppose that $g=\frac{g_1}{g_2}$ with g_1, g_2, g are all entire functions satisfying the conditions specified in the theorem.

Sub Case $\mathbf{III_A}$. Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1}<\lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore in view of Theorem 10, $\lambda_{(\alpha,\beta)}[f_1]_g=\lambda_{(\alpha,\beta)}[f_1]_{g_1}<\lambda_{(\alpha,\beta)}[f_1]_{g_2}$. We have $g_1=g\cdot g_2$. So $\tau_{(\alpha,\beta)}[f_1]_{g_1}=\tau_{(\alpha,\beta)}[f_1]_g=\tau_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}}$.

Sub Case $\mathbf{III_B}$. Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1}>\lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore in view of Theorem 10, $\lambda_{(\alpha,\beta)}[f_1]_g=\lambda_{(\alpha,\beta)}[f_1]_{g_2}<\lambda_{(\alpha,\beta)}[f_1]_{g_1}$. Since $T_g(r)=T_{\frac{1}{g}}(r)+O(1)=T_{\frac{g_2}{g_1}}(r)+O(1)$, So $\tau_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}}=\tau_{(\alpha,\beta)}[f_1]_{g_2}$.

Case IV. Suppose $\lambda_{(\alpha,\beta)}[f_1]_{g_1}<\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $g_1\cdot g_2$ satisfy the Property (A). Since $T_{g_1\cdot g_2}(r)\leq T_{g_1}(r)+T_{g_2}(r)$ for all large r, then adopting the same procedure as of Case IV of Theorem 18, we obtain that $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2}=\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}$ and $\overline{\tau}_{(\alpha,\beta)}[f_1]_{\frac{g_1}{g_2}}=\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_i}\mid i=1,2$.

Similarly if we consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}>\lambda_{(\alpha,\beta)}[f_1]_{g_2}$, then one can easily verify that $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2}=\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$.

Therefore the second part of the theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 14 , Theorem 16 and the above cases. □

Theorem 25. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions.*
(A) *The following condition is assumed to be satisfied:*
(i) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1}\neq\sigma_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}\neq\overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$ holds;
(ii) g_1 satisfies the Property (A), then

$$\rho_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1}=\rho_{(\alpha,\beta)}[f_1]_{g_1}=\rho_{(\alpha,\beta)}[f_2]_{g_1}.$$

(B) *The following conditions are assumed to be satisfied:*
(i) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1}\neq\sigma_{(\alpha,\beta)}[f_1]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}\neq\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$ holds;
(ii) f_1 is of regular generalized relative growth (α,β) with respect to at least any one of g_1 or g_2. Also $g_1\cdot g_2$ satisfy the Property (A). Then we have

$$\rho_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2}=\rho_{(\alpha,\beta)}[f_1]_{g_1}=\rho_{(\alpha,\beta)}[f_1]_{g_2}.$$

Proof. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions satisfying the conditions of the theorem.

Case I. Suppose that $\rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}$ $(0 < \rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_2]_{g_1} < \infty)$ and g_1 satisfy the Property (A). Now in view of Theorem 9, it is easy to see that $\rho_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \leq \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}$. If possible let

$$\rho_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} < \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}. \tag{39}$$

Let $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1}$. Now in view of the first part of Theorem 23 and (39) we obtain that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} = \sigma_{(\alpha,\beta)}\left[\frac{f_1 \cdot f_2}{f_2}\right]_{g_1} = \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ which is a contradiction. Hence $\rho_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1}$. Similarly with the help of the first part of Theorem 23, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$. This prove the first part of the theorem.
Case II. Let us consider that $\rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}$ $(0 < \rho_{(\alpha,\beta)}[f_1]_{g_1}, \rho_{(\alpha,\beta)}[f_1]_{g_2} < \infty)$, f_1 is of regular generalized relative growth (α,β) with respect to at least any one of g_1 or g_2. Also $g_1 \cdot g_2$ satisfy the Property (A). Therefore in view of Theorem 11, it follows that $\rho_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \geq \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}$ and if possible let

$$\rho_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} > \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}. \tag{40}$$

Further suppose that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore in view of the proof of the second part of Theorem 23 and (40), we obtain that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} = \sigma_{(\alpha,\beta)}[f_1]_{\frac{g_1 \cdot g_2}{g_2}} = \sigma_{(\alpha,\beta)}[f_1]_{g_2}$ which is a contradiction. Hence $\rho_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2}$. Likewise in view of the proof of second part of Theorem 23, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$. This proves the second part of the theorem. □

Theorem 26. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions.*
(A) *The following conditions are assumed to be satisfied:*
(i) $(f_1 \cdot f_2)$ is of regular generalized relative growth (α,β) with respect to at least any one g_1 or g_2;
(ii) $(g_1 \cdot g_2)$, g_1 and g_2 all satisfy the Property (A);
(iii) Either $\sigma_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_2}$;
(iv) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$;
(v) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_2} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_2}$; then

$$\rho_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2} = \rho_{(\alpha,\beta)}[f_2]_{g_2}.$$

(B) *The following conditions are assumed to be satisfied:*
(i) $(g_1 \cdot g_2)$ satisfies the Property (A);
(ii) f_1 and f_2 are of regular generalized relative growth (α,β) with respect to at least any one g_1 or g_2;
(iii) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1 \cdot g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1 \cdot g_2}$;
(iv) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$;
(v) Either $\sigma_{(\alpha,\beta)}[f_2]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_2}$; then

$$\rho_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \rho_{(\alpha,\beta)}[f_1]_{g_1} = \rho_{(\alpha,\beta)}[f_2]_{g_1} = \rho_{(\alpha,\beta)}[f_1]_{g_2} = \rho_{(\alpha,\beta)}[f_2]_{g_2}.$$

We omit the proof of Theorem 26 as it is a natural consequence of Theorem 25.

Theorem 27. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions.*
(A) *The following conditions are assumed to be satisfied:*
(i) *At least any one of f_1 or f_2 is of regular generalized relative growth (α,β) with respect to g_1;*
(ii) *If either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$ holds.*
(iii) *g_1 satisfies the Property (A), then*

$$\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}.$$

(B) *The following conditions are assumed to be satisfied:*
(i) *f_1 is any meromorphic function and g_1, g_2 are any two entire functions such that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$ and $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ exist and $g_1 \cdot g_2$ satisfy the Property (A);*
(ii) *If either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$ holds, then*

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}.$$

Proof. Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions satisfy the conditions of the theorem.

Case I. Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ $(0 < \lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1} < \infty)$, g_1 satisfies the Property (A) and at least f_1 or f_2 be of regular generalized relative growth (α,β) with respect to g_1. Now in view of Theorem 7 it is easy to see that $\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \leq \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. If possible let

$$\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}. \tag{41}$$

Also let $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1}$. Then in view of the proof of first part of Theorem 24 and (41), we obtain that $\tau_{(\alpha,\beta)}[f_1]_{g_1} = \tau_{(\alpha,\beta)}\left[\frac{f_1 \cdot f_2}{f_2}\right]_{g_1} = \tau_{(\alpha,\beta)}[f_2]_{g_1}$ which is a contradiction. Hence $\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. Analogously, in view of the proof of first part of Theorem 24 and using the same technique as above, one can easily derive the same conclusion under the hypothesis $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$. Hence the first part of the theorem is established.
Case II. Let us consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ $(0 < \lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2} < \infty$ and $g_1 \cdot g_2$ satisfy the Property (A). Therefore in view of Theorem 10, it follows that $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \geq \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and if possible let

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} > \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}. \tag{42}$$

Further let $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1]_{g_2}$. Then in view of second part of Theorem 24 and (42), we obtain that $\tau_{(\alpha,\beta)}[f_1]_{g_1} = \tau_{(\alpha,\beta)}[f_1]_{\frac{g_1 \cdot g_2}{g_2}} = \tau_{(\alpha,\beta)}[f_1]_{g_2}$ which is a contradiction. Hence $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Similarly by second part of Theorem 24, we get the same conclusion when $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$ and therefore the second part of the theorem follows. □

Theorem 28. *Let f_1, f_2 be any two meromorphic functions and g_1, g_2 be any two entire functions.*
(A) *The following conditions are assumed to be satisfied:*
(i) *$g_1 \cdot g_2$, g_1 and g_2 satisfy the Property (A);*
(ii) *At least any one of f_1 or f_2 is of regular generalized relative growth (α,β) with respect to g_1 and g_2;*
(iii)*Either $\tau_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_2}$;*

(iv) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1}$ *or* $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$ *;*
(v) Either $\tau_{(\alpha,\beta)}[f_1]_{g_2} \neq \tau_{(\alpha,\beta)}[f_2]_{g_2}$ *or* $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_2}$ *; then*

$$\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2} = \lambda_{(\alpha,\beta)}[f_2]_{g_2}.$$

(B) *The following conditions are assumed to be satisfied:*
(i) $g_1 \cdot g_2$ *satisfies the Property (A);*
(ii) At least any one of f_1 *or* f_2 *is of regular generalized relative growth* (α,β) *with respect to* $g_1 \cdot g_2$*;*
(iii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1 \cdot g_2}$ *or* $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1 \cdot g_2}$ *holds;*
(iv) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1]_{g_2}$ *or* $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$ *holds;*
(v) If either $\tau_{(\alpha,\beta)}[f_2]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_2}$ *or* $\overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_2}$ *holds, then*

$$\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2} = \lambda_{(\alpha,\beta)}[f_2]_{g_2}.$$

We omit the proof of Theorem 28 as it is a natural consequence of Theorem 27.

Remark 1. *If we take* $\frac{f_1}{f_2}$ *instead of* $f_1 \cdot f_2$ *and* $\frac{g_1}{g_2}$ *instead of* $g_1 \cdot g_2$ *where* $\frac{f_1}{f_2}$ *is meromorphic and* $\frac{g_1}{g_2}$ *is entire function, and the other conditions of Theorem 25, Theorem 26, Theorem 27 and Theorem 28 remain the same, then conclusion of Theorem 25, Theorem 26, Theorem 27 and Theorem 28 remains valid.*

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