Journal of Fractional Calculus and Applications
Vol. 13(2) Jan 2022, pp. 45-65.
ISSN: 2090-5858.
http://math-frac.org/Journals/JFCA/

EXPLICIT FORMULAS FOR THE SOLUTIONS OF AUTONOMOUS LINEAR FRACTIONAL ORDER SYSTEMS

BICHITRA KUMAR LENKA

Abstract. We give explicit analytical formulas to the solutions of initial value problems of autonomous linear fractional order system where its state equations involve arbitrary fractional orders. We obtain the formulas by applying the Laplace transform to such fractional order system. The formulas are explicitly represented in terms of Mittag-Leffler functions and their higher integer order derivatives.

1. Introduction

Fractional calculus [1, 2] is a growing branch of mathematics that deals with fractional order derivative and/or integral of functions and studies the relationship between them, where fractional order could be integer numbers, rational numbers, irrational numbers etc. The concept of fractional calculus has been widely used to model physical problems, engineering systems and their applications that significantly leads to a set of linear or nonlinear fractional differential equations (fractional order systems) [1, 2, 3, 4, 5].

As a result of this, research importance has been focused on solving such types of fractional differential equations or systems. In general, most fractional differential equations or systems do not have exact solutions. However, in the literature, great efforts have been provided by researchers for solving linear or nonlinear fractional differential equations (fractional order systems) by introducing different methods. For example, effective methods such as Laplace transform [1, 6], Mellin transform [1], Sumudu transform [7], Operational method [8] etc., have been used to find the explicit solutions to scalar linear fractional differential equations. On the other hand, various methods (for instance, see [9, 10, 11, 12], and the references therein) have been applied to find the approximate analytical and numerical solutions of nonlinear fractional order systems.

2010 *Mathematics Subject Classification.* 26A33, 33E12, 34A05, 34A08, 34A30, 44A10.
Key words and phrases. Caputo fractional derivative; Linear fractional order system; Analytic solution; matrix Mittag-Leffer function.
Submitted Jan. 26, 2020. Revised Nov. 23, 2021.

In this paper, we consider the autonomous linear fractional order system

$$\begin{aligned} {}^{C}D^{\alpha_1}_{0,t}x_1(t) = &\quad a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) \\ {}^{C}D^{\alpha_2}_{0,t}x_2(t) = &\quad a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) \\ \vdots \quad & \\ {}^{C}D^{\alpha_n}_{0,t}x_n(t) = &\quad a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) \end{aligned} \tag{1.1}$$

subject to the initial conditions

$$x_i^{(k)}(0) = x_{ik}, \ \ k = 0, 1, 2, \cdots, r_i - 1, \ \ i = 1, 2, \cdots, n. \tag{1.2}$$

The system (1.1) can be rewritten in the matrix form

$${}^{C}D^{\widehat{\alpha}}_{0,t}x(t) = Ax(t) \tag{1.3}$$

where $x(t) = (x_1(t), \cdots, x_n(t))^T \in \mathbb{R}^n$, ${}^{C}D^{\widehat{\alpha}}_{0,t}x(t) = \left({}^{C}D^{\alpha_1}_{0,t}x_1(t), \cdots, {}^{C}D^{\alpha_n}_{0,t}x_n(t)\right)^T$, ${}^{C}D^{\alpha_i}_{0,t}$ is Caputo fractional derivative (see, section 2) of order α_i with $r_i - 1 < \alpha_i \leq r_i$, $r_i \in \mathbb{Z}^+$ for $i = 1, 2, \cdots, n$, and $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a constant matrix.

We are mainly interested in the structure and the closed-form solutions of autonomous linear fractional order system (1.1). In the literature, the linear fractional order system (1.1) have been studied by many researchers but most of the authors considered very specific type of problems depending on the restriction of range of fractional orders to certain intervals (see [14, 15, 16, 17, 18, 19, 20, 21]). In these references, we observe that the matrix Mittag-Leffler function plays a crucial role in the analytic representation of the solutions of commensurate fractional order system. It indicates that the computation of the matrix Mittag-Leffler function is an important problem from a theoretical as well as computational point of interest because, in order to find the components of the solution to such fractional order system, it is often necessary to compute the value of such matrix function. In this regard, there has been a great interest amongst the researchers for the evaluation of matrix Mittag-Leffler functions [20, 22, 23, 24].

In the above-mentioned works, we observe that the analytical formulas, as well as the structures for the components of the solution to the fractional order system (1.1), are not known whenever the state variables are associated with the same fractional order and different fractional orders.

Motivated by the above observation, in this work, we apply the Laplace transform to the fractional order system (1.1)-(1.2) and aim to find the explicit analytical formulas of its solutions. First, by using the concept of similarity transformation to the coefficient matrix of the system (1.1), we provide several formulas for the solution of the system (1.3) to the commensurate fractional order case. This is based on the factorization of the matrix Mittag-Leffler function. Then, we discuss the Laplace transform method for solving the linear fractional order system (1.1)-(1.2) to the incommensurate fractional order case and provide the explicit formulas to its solution.

2. Preliminary definitions and properties

In this section, we recall the definitions of fractional integral, fractional derivative, Laplace transform, Mittag-Leffler function and some of its important properties which will be used throughout the paper (for details, see [1, 2, 21, 18]). We denote $\mathbb{R}$ the set of all real numbers, $\mathbb{R}^+$ the set of all positive real numbers, $\mathbb{Z}^+$

the set of all positive integers, $\mathbb{C}$ the set of all complex numbers and $\arg(z)$ the argument of a complex number z.

Definition 2.1. *The Riemann-Liouville fractional integral with order α of function $x(t)$ is defined as*

$$^{RL}D_{0,t}^{-\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}x(\tau)d\tau,\ t>0, \tag{2.1}$$

where $0<\alpha\in\mathbb{R}^+$ and $\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt$ is the Gamma function with $\alpha>0$.

Definition 2.2. *The Caputo fractional derivative with order α of function $x(t)$ is defined as*

$$^{C}D_{0,t}^{\alpha}x(t) = \begin{cases} ^{RL}D_{0,t}^{-(n-\alpha)}\left(\frac{d^n}{dt^n}x(t)\right), & \alpha\in(n-1,n) \\ \frac{d^n}{dt^n}x(t), & \alpha=n \end{cases} \tag{2.2}$$

where $n\in\mathbb{Z}^+$ and $\Gamma(\cdot)$ is the Gamma function.

Definition 2.3. *The Laplace transform of a function $x(t)$ is defined as*

$$\mathcal{L}\{x(t)\} = \int_0^\infty e^{-st}x(t)dt \tag{2.3}$$

where t and s denote the variables in Laplace time and frequency domain respectively.

Property 2.1. *The Laplace transform of Caputo fractional derivative is given by*

$$\mathcal{L}\{^{C}D_{0,t}^{\alpha}x(t)\} = s^\alpha X(s) - \sum_{k=0}^{n-1}s^{\alpha-k-1}x^{(k)}(0), \tag{2.4}$$

where $n-1<\alpha\le n$ and $\mathcal{L}\{x(t)\}=X(s)$.

Definition 2.4. *The one parameter Mittag-Leffler function is defined as*

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+1)},\quad \alpha>0, z\in\mathbb{C}. \tag{2.5}$$

Definition 2.5. *The two parameter Mittag-Leffler function is defined as*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)},\quad \alpha>0, \beta>0, z\in\mathbb{C}. \tag{2.6}$$

It may be noted that when $\beta=1$, $E_{\alpha,1}(z)=E_\alpha(z)$. Further, if $\alpha=\beta=1$, then $E_{1,1}(z)=E_1(z)=e^z$.

Definition 2.6. *The two parameter matrix Mittag-Leffler function is defined as*

$$E_{\alpha,\beta}(A) = \sum_{k=0}^\infty \frac{A^k}{\Gamma(\alpha k+\beta)},\quad \alpha>0, \beta>0, A\in\mathbb{C}^{n\times n}. \tag{2.7}$$

Property 2.2. *The k^{th} derivative of the one parameter Mittag-Leffler function is given by*

$$E_\alpha^{(k)}(z) = \sum_{j=0}^\infty \frac{(j+k)!z^j}{j!\Gamma(\alpha j+\alpha k+1)},\quad k=0,1,2,\cdots \tag{2.8}$$

where $E_\alpha^{(k)}(z)=\frac{d^k}{dz^k}E_\alpha(z)$.

Property 2.3. *The k^{th} derivative of the two parameter Mittag-Leffler function is given by*

$$E^{(k)}_{\alpha,\beta}(z)=\sum_{j=0}^{\infty}\frac{(j+k)!z^j}{j!\Gamma(\alpha j+\alpha k+\beta)},\quad k=0,1,2,\cdots \tag{2.9}$$

where $E^{(k)}_{\alpha,\beta}(z)=\frac{d^k}{dz^k}E_{\alpha,\beta}(z)$.

Property 2.4. *The Laplace transforms of the Mittag-Leffler functions are given by*

$$\mathcal{L}\{t^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}(\pm\lambda t^{\alpha})\}=\frac{k!s^{\alpha-\beta}}{(s^{\alpha}\mp\lambda)^{k+1}},\quad \Re(s)>|\lambda|^{1/\alpha}. \tag{2.10}$$

where $E^{(k)}_{\alpha,\beta}(z)=\frac{d^k}{dz^k}E_{\alpha,\beta}(z)$.

Property 2.5. *The Laplace transform of the matrix Mittag-Leffler function is given by*

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(At^{\alpha})\}=s^{\alpha-\beta}\left(s^{\alpha}I-A\right)^{-1},\ \ \Re(s)>\|A\|^{1/\alpha}. \tag{2.11}$$

where $\Re(s)$ *represents the real part of the complex number* s.

3. Main theoretical discussion

In this section, we discuss the theoretical approach for solving the system (1.1) corresponding to the initial conditions (1.2). We recall some of the interesting linear algebra results from the book [13]. Based on these results, we provide the structure as well as the formulas of solutions to the system (1.1). We present our discussion to the equal order case (commensurate order case) and different order case (incommensurate order case) separately.

3.1. **Commensurate fractional order case:** Note that in this case all the fractional orders α_i's involved in the system (1.1) or (1.3) satisfy $r-1<\alpha_1=\alpha_2=\cdots=\alpha_n=\alpha<r$, where $r_1=r_2=\cdots=r_n=r\in\mathbb{Z}^{+}$. Here, we write the initial conditions (1.2) in the vector form

$$x^{(k)}(0)=x_k, \tag{3.1}$$

where $k=0,1,2,\cdots,r-1$.

Applying the Laplace transform on the system (1.3) and using the initial conditions (3.1), we get

$$s^{\alpha}X(s)-\sum_{k=0}^{r-1}s^{\alpha-k-1}x^{(k)}(0)=AX(s) \tag{3.2}$$

Then, we have

$$X(s)=(s^{\alpha}I-A)^{-1}\cdot\sum_{k=0}^{r-1}s^{\alpha-k-1}x^{(k)}(0) \tag{3.3}$$

Taking the inverse Laplace transform on (3.3) and using the property 2.5, we obtain the exact solution

$$x(t)=\sum_{k=0}^{r-1}t^kE_{\alpha,k+1}(At^{\alpha})x^{(k)}(0). \tag{3.4}$$

It is quite interesting to see that the representation of solution (3.4) looks theoretically nice but in practice one needs to calculate the matrix Mittag-Leffler functions in order to carry out the components of the solution. To simplify the representation of solution (3.4), next we discuss several interesting results where the matrix Mittag-Leffler functions $E_{\alpha,k+1}(At^{\alpha})$ are evaluated when the matrix A is similar to a diagonal matrix or matrix in Jordan canonical form.

3.1.1. *Simplification of solution based on the computation of* $E_{\alpha,k+1}(At^{\alpha})$*:*

Lemma 3.1. [13] *If the eigenvalues* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *of an* $n \times n$ *real matrix* A *are real and distinct, then there exists an invertible matrix* P *such that* $P^{-1}AP = D$*, where* $D = diag\,(\lambda_1, \lambda_2, \cdots, \lambda_n)$*.*

Under the hypothesis of Lemma 3.1, we can write the solution (3.4) as

$$\begin{aligned} x(t) &= \sum_{k=0}^{r-1} t^k E_{\alpha,k+1}\left((PDP^{-1})t^{\alpha}\right) x^{(k)}(0) \\ &= \sum_{k=0}^{r-1} t^k P E_{\alpha,k+1}\left(Dt^{\alpha}\right) P^{-1} x^{(k)}(0), \end{aligned} \tag{3.5}$$

where

$$E_{\alpha,k+1}\left(Dt^{\alpha}\right) = \begin{pmatrix} E_{\alpha,k+1}(\lambda_1 t^{\alpha}) & & & \\ & E_{\alpha,k+1}(\lambda_2 t^{\alpha}) & & \\ & & \ddots & \\ & & & E_{\alpha,k+1}(\lambda_n t^{\alpha}) \end{pmatrix}. \tag{3.6}$$

Lemma 3.2. [13] *Let* A *be a real* $n \times n$ *matrix with real eigenvalues* λ_j *and corresponding multiplicities* n_j*,* $j = 1, 2, \cdots, w$*, then there exist an invertible matrix* P *such that* $P^{-1}AP = J = diag\,(J_1, \cdots, J_w)$*, where* $J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}_{n_i \times n_i}$ *for* $i = 1, 2, \cdots, w$*, and* $\sum_{i=1}^{w} n_i = n$*.*

Based on Lemma 3.2, the solution (3.4) becomes

$$\begin{aligned} x(t) &= \sum_{k=0}^{r-1} t^k E_{\alpha,k+1}\left((PJP^{-1})t^{\alpha}\right) x^{(k)}(0) \\ &= \sum_{k=0}^{r-1} t^k P E_{\alpha,k+1}\left(Jt^{\alpha}\right) P^{-1} x^{(k)}(0) \end{aligned} \tag{3.7}$$

where

$$E_{\alpha,k+1}\left(Jt^{\alpha}\right) = \begin{pmatrix} E_{\alpha,k+1}(J_1 t^{\alpha}) & & & \\ & E_{\alpha,k+1}(J_2 t^{\alpha}) & & \\ & & \ddots & \\ & & & E_{\alpha,k+1}(J_w t^{\alpha}) \end{pmatrix}. \tag{3.8}$$

To further simplify the structure of above matrix Mittag-Leffler function, we need to compute the matrix Mittag-Leffler functions involved in the diagonal Jordan blocks (3.8). Note that

$$\begin{aligned} E_{\alpha,k+1}(J_i t^\alpha) &= \sum_{m=0}^{\infty} \frac{(J_i t^\alpha)^m}{\Gamma(\alpha m + k + 1)} \\ &= \sum_{m=0}^{\infty} \frac{t^{\alpha m}}{\Gamma(\alpha m + k + 1)} J_i^m \end{aligned} \tag{3.9}$$

for $i = 1, 2, \cdots, w$. Here, we decompose the Jordan matrix J_i as a sum of identity matrix I_i and nilpotent matrix N_i of order n_i, and compute

$$\begin{aligned} J_i^m = (\lambda_i I_i + N_i)^m &= \sum_{u=0}^{m} \binom{m}{u} \lambda_i^{m-u} {I_i}^{m-u} N_i^u \\ &= \sum_{u=0}^{min\{m, n_i - 1\}} \binom{m}{u} \lambda_i^{m-u} {I_i}^{m-u} N_i^u \\ &= \begin{pmatrix} \lambda_i^m & \binom{m}{1}\lambda_i^{m-1} & \cdots & \binom{m}{n_i-1}\lambda_i^{m-n_i+1} \\ & \lambda_i^m & \ddots & \vdots \\ & & \ddots & \binom{m}{1}\lambda_i^{m-1} \\ & & & \lambda_i^m \end{pmatrix} \end{aligned} \tag{3.10}$$

for $i = 1, 2, \cdots, w$.

By substituting (3.10) into (3.9), we obtain

$$E_{\alpha,k+1}(J_i t^\alpha) = \begin{pmatrix} E_{\alpha,k+1}(\lambda_i t^\alpha) & \frac{t^\alpha}{1!} E^{(1)}_{\alpha,k+1}(\lambda_i t^\alpha) & \cdots & \frac{t^{\alpha(n_i-1)}}{(n_i-1)!} E^{(n_i-1)}_{\alpha,k+1}(\lambda_i t^\alpha) \\ & E_{\alpha,k+1}(\lambda_i t^\alpha) & \ddots & \vdots \\ & & \ddots & \frac{t^\alpha}{1!} E^{(1)}_{\alpha,k+1}(\lambda_i t^\alpha) \\ & & & E_{\alpha,k+1}(\lambda_i t^\alpha) \end{pmatrix} \tag{3.11}$$

for $i = 1, 2, \cdots, w$.

Thus, the representation of solution (3.4) is simplified to (3.7), where the diagonal Jordan block matrices involved in (3.8) has the representation (3.11).

Remark 3.1. *Note that if we allow complex eigenvalues of the real matrix A in Lemma 3.1 and Lemma 3.2, then one can observe that the structure of solution* (3.5) *or* (3.7) *appears to be complex valued. Since the representation of solution* (3.4) *is real valued, thus one expect to write the solution in terms of real valued functions.*

Remark 3.2. *Generally the following property:*

$$E_{\alpha,\beta}\left((a+b)t^\alpha\right) = E_{\alpha,\beta}\left(at^\alpha\right) E_{\alpha,\beta}\left(bt^\alpha\right)$$

does not hold for $\alpha > 0$ and $\beta > 0$ (for instance, see [25, 26, 27]*).*

Note that when $\alpha = \beta = 1$, we have $E_{\alpha,\beta}(a \pm ib) = e^{(a \pm ib)t}$ and the following properties are well known:

$$e^{(a \pm ib)t} = e^{at}\left(\cos(bt) \pm i \sin(bt)\right). \tag{3.12}$$

and

$$\frac{e^{(a+ib)t}+e^{(a-ib)t}}{2}=e^{at}\cos(bt), \tag{3.13}$$

$$\frac{e^{(a+ib)t}-e^{(a-ib)t}}{2i}=e^{at}\sin(bt). \tag{3.14}$$

In view of Remark 3.2, we observe the formulas like (3.12)-(3.14) does not hold for the Mittag-Leffler function $E_{\alpha,\beta}\left((a\pm ib)t^{\alpha}\right)$.

Next, we present an important result which possibly could overcome the complex valued representation of solution to the system (1.1) when some eigenvalues of the coefficient matrix A are complex numbers.

Lemma 3.3. *Let $D\in\mathbb{R}^{2\times 2}$ be a square matrix in the following form*

$$D=\begin{pmatrix} a & b\\ -b & a\end{pmatrix}, \tag{3.15}$$

then we have

$$E_{\alpha,\beta}\left(Dt^{\alpha}\right)=\begin{pmatrix} \sum\limits_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+\beta)}|\lambda|^{k}\cos\left(k\theta\right) & \sum\limits_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+\beta)}|\lambda|^{k}\sin\left(k\theta\right)\\ -\sum\limits_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+\beta)}|\lambda|^{k}\sin\left(k\theta\right) & \sum\limits_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+\beta)}|\lambda|^{k}\cos\left(k\theta\right)\end{pmatrix} \tag{3.16}$$

where $|\lambda|=\sqrt{a^2+b^2}$ and $\theta=\arg(a+ib)$.

Proof. Let us decompose the matrix $D=aI+bJ$, where $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$. By definition of Mittag-Leffler function, we have

$$E_{\alpha,\beta}\left(Dt^{\alpha}\right)=\sum_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+\beta)}\left(aI+bJ\right)^{k}. \tag{3.17}$$

Since $J^2=-I$, $J^3=-J$, $J^{4+k}=J^k$ for $k=1,2,\cdots$, and $IJ=JI$. Here, we compute

$$\begin{aligned}(aI+bJ)^{k}&=\sum_{j=0}^{k}\begin{pmatrix}k\\ j\end{pmatrix}a^{k-j}b^{j}J^{4+j}\\ &=\sum_{j=0,\ k\geq 2j}^{k}\begin{pmatrix}k\\ 2j\end{pmatrix}a^{k-2j}b^{2j}J^{4+2j}\\ &+\sum_{j=0,\ k\geq 2j+1}^{k}\begin{pmatrix}k\\ 2j+1\end{pmatrix}a^{k-2j-1}b^{2j+1}J^{4+2j+1}.\end{aligned} \tag{3.18}$$

Then, by substituting (3.18) into (3.17), we get

$$\begin{aligned}E_{\alpha,\beta}\left(Dt^{\alpha}\right)&=\sum_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+\beta)}\sum_{j=0,\ k\geq 2j}^{k}(-1)^{j}\begin{pmatrix}k\\ 2j\end{pmatrix}a^{k-2j}b^{2j}I\\ &+\sum_{k=0}^{\infty}\frac{t^{\alpha k}}{\Gamma(\alpha k+\beta)}\sum_{j=0,\ k\geq 2j+1}^{k}(-1)^{j}\begin{pmatrix}k\\ 2j+1\end{pmatrix}a^{k-2j-1}b^{2j+1}J\end{aligned} \tag{3.19}$$

Note that $a \pm ib$ are the eigenvalues of the matrix D. It is not difficult to observe that

$$(a+ib)^k = \sum_{j=0}^{k} \binom{k}{j} a^{k-j} (ib)^j = \Re(a+ib)^k + i\Im(a+ib)^k \tag{3.20}$$

where

$$\Re(a+ib)^k = \sum_{j=0,\ k\geq 2j}^{k} (-1)^j \binom{k}{2j} a^{k-2j} b^{2j}, \tag{3.21}$$

$$\Im(a+ib)^k = \sum_{j=0,\ k\geq 2j+1}^{k} (-1)^j \binom{k}{2j+1} a^{k-2j-1} b^{2j+1}. \tag{3.22}$$

Let us write $\lambda = a + ib$. Then, we have

$$\lambda^k = |\lambda|^k \left(cos(k\theta) + isin(k\theta)\right) \tag{3.23}$$

where $|\lambda| = \sqrt{a^2+b^2}$ and $\theta = \arg(a+ib)$. Comparing the real and imaginary parts of (3.20) and (3.23), and then using in (3.19) we obtain (3.16). $\square$

Corollary 3.1. *Let $b \in \mathbb{R}$ and $D = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$, then we have*

$$E_{\alpha,\beta}\left(Dt^{\alpha}\right) = \begin{pmatrix} \sum\limits_{j=0}^{\infty} (-1)^j \frac{(bt^{\alpha})^{2j}}{\Gamma(2j\alpha+\beta)} & \sum\limits_{j=0}^{\infty} (-1)^j \frac{(bt^{\alpha})^{2j+1}}{\Gamma((2j+1)\alpha+\beta)} \\ -\sum\limits_{j=0}^{\infty} (-1)^j \frac{(bt^{\alpha})^{2j+1}}{\Gamma((2j+1)\alpha+\beta)} & \sum\limits_{j=0}^{\infty} (-1)^j \frac{(bt^{\alpha})^{2j}}{\Gamma(2j\alpha+\beta)} \end{pmatrix}. \tag{3.24}$$

Lemma 3.4. *Let $D \in \mathbb{R}^{2\times 2}$ be a square matrix in the following form*

$$D = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \tag{3.25}$$

then we have

$$E^{(j)}_{\alpha,\beta}\left(Dt^{\alpha}\right) = \begin{pmatrix} \sum\limits_{k=0}^{\infty} \frac{(k+j)!t^{\alpha k}}{k!\Gamma(\alpha k+\alpha j+\beta)} |\lambda|^k \cos(k\theta) & \sum\limits_{k=0}^{\infty} \frac{(k+j)!t^{\alpha k}}{k!\Gamma(\alpha k+\alpha j+\beta)} |\lambda|^k \sin(k\theta) \\ -\sum\limits_{k=0}^{\infty} \frac{(k+j)!t^{\alpha k}}{k!\Gamma(\alpha k+\alpha j+\beta)} |\lambda|^k \sin(k\theta) & \sum\limits_{k=0}^{\infty} \frac{(k+j)!t^{\alpha k}}{k!\Gamma(\alpha k+\alpha j+\beta)} |\lambda|^k \cos(k\theta) \end{pmatrix} \tag{3.26}$$

where $|\lambda| = \sqrt{a^2+b^2}$ and $\theta = \arg(a+ib)$ and $j = 0, 1, 2, \cdots$.

Remark 3.3. *In Lemma 3.3 and Lemma 3.4, if we take $B = D^T$, then we have $E_{\alpha,\beta}\left(Bt^{\alpha}\right) = E_{\alpha,\beta}\left(Dt^{\alpha}\right)^T$ and $E^{(j)}_{\alpha,\beta}\left(Bt^{\alpha}\right) = E^{(j)}_{\alpha,\beta}\left(Dt^{\alpha}\right)^T$.*

Remark 3.4. *It is evident from Lemma 3.3 and Lemma 3.4 that*

$$E^{(j)}_{\alpha,\beta}\left(Dt^{\alpha}\right) = \begin{pmatrix} \Re E^{(j)}_{\alpha,\beta}\left((a+ib)t^{\alpha}\right) & \Im E^{(j)}_{\alpha,\beta}\left((a+ib)t^{\alpha}\right) \\ -\Im E^{(j)}_{\alpha,\beta}\left((a+ib)t^{\alpha}\right) & \Re E^{(j)}_{\alpha,\beta}\left((a+ib)t^{\alpha}\right) \end{pmatrix} \tag{3.27}$$

where

$$\Re E^{(j)}_{\alpha,\beta}\left((a+ib)t^{\alpha}\right) = \frac{1}{2}\left[E^{(j)}_{\alpha,\beta}\left((a+ib)t^{\alpha}\right) + E^{(j)}_{\alpha,\beta}\left((a-ib)t^{\alpha}\right)\right]$$

and

$$\Im E^{(j)}_{\alpha,\beta}\left((a+ib)t^{\alpha}\right) = \frac{1}{2i}\left[E^{(j)}_{\alpha,\beta}\left((a+ib)t^{\alpha}\right) - E^{(j)}_{\alpha,\beta}\left((a-ib)t^{\alpha}\right)\right]$$

are real valued functions, and denotes the real and imaginary parts of the complex valued Mittag-Leffler function $E^{(j)}_{\alpha,\beta}\left((a+ib)t^{\alpha}\right)$ *for* $j=0,1,2,\cdots$.

Lemma 3.5. [13] *If the* $2n\times 2n$ *real matrix* A *has* $2n$ *distinct complex eigenvalues* $\lambda_j=a_j+ib_j$ *and* $\bar{\lambda}=a_j-ib_j$, $j=1,2,\cdots,n$, *then there exists an invertible matrix* P *such that* $P^{-1}AP=diag\begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix}$, *a real* $2n\times 2n$ *matrix with* 2×2 *blocks along the diagonal.*

Let us assume the hypothesis of Lemma 3.5 for the system (1.1). Then, by using Lemma 3.3 in the solution (3.4), we get

$$x(t)=\sum_{k=0}^{r-1} t^k P E_{\alpha,k+1}\left(Dt^{\alpha}\right) P^{-1} x^{(k)}(0) \tag{3.28}$$

where $D=diag\left(D_1,D_2,\cdots,D_n\right)$, $D_j=\begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix}$,

$$E_{\alpha,k+1}\left(Dt^{\alpha}\right)=diag\left(E_{\alpha,k+1}(D_1t^{\alpha}),E_{\alpha,k+1}(D_2t^{\alpha}),\cdots,E_{\alpha,k+1}(D_nt^{\alpha})\right),$$

$$\begin{aligned} &E_{\alpha,k+1}\left(D_jt^{\alpha}\right)\\ &=\begin{pmatrix} \sum\limits_{m=0}^{\infty}\frac{t^{\alpha m}}{\Gamma(\alpha m+k+1)}|\lambda_j|^m\cos\left(m\theta_j\right) & \sum\limits_{m=0}^{\infty}\frac{t^{\alpha m}}{\Gamma(\alpha m+k+1)}|\lambda_j|^m\sin\left(m\theta_j\right) \\ -\sum\limits_{m=0}^{\infty}\frac{t^{\alpha m}}{\Gamma(\alpha m+k+1)}|\lambda_j|^m\sin\left(m\theta_j\right) & \sum\limits_{m=0}^{\infty}\frac{t^{\alpha m}}{\Gamma(\alpha m+k+1)}|\lambda_j|^m\cos\left(m\theta_j\right) \end{pmatrix}, \end{aligned}$$

and $|\lambda_j|=\sqrt{a_j^2+b_j^2}$ and $\theta_j=\arg(a_j+ib_j)$, $j=1,2,\cdots,n$.

Lemma 3.6. [13] *If the real matrix* A *has distinct real eigenvalues* λ_j, $j=1,2,\cdots,m$, *and distinct complex eigenvalues* $\lambda_j=a_j+ib_j$, $\bar{\lambda_j}=a_j-ib_j$, $j=m+1,m+2,\cdots,n$, *then there exists an invertible matrix* P *such that*

$$P^{-1}AP=diag\left(\lambda_1,\cdots,\lambda_m,B_{m+1},\cdots,B_n\right),$$

where the 2×2 *blocks matrix* $B_j=\begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix}$ *for* $j=m+1,\cdots,n$.

In view of Lemma 3.6 and Lemma 3.3, the representation of solution (3.4) reduces to

$$x(t)=\sum_{k=0}^{r-1} t^k P E_{\alpha,k+1}\left(Dt^{\alpha}\right) P^{-1} x^{(k)}(0) \tag{3.29}$$

where

$$\begin{aligned} &E_{\alpha,k+1}\left(Dt^{\alpha}\right)\\ &=\begin{pmatrix} E_{\alpha,k+1}\left(\lambda_1t^{\alpha}\right) & & & & & \\ & \ddots & & & & \\ & & E_{\alpha,k+1}\left(\lambda_mt^{\alpha}\right) & & & \\ & & & E_{\alpha,k+1}\left(B_{m+1}t^{\alpha}\right) & & \\ & & & & \ddots & \\ & & & & & E_{\alpha,k+1}\left(B_nt^{\alpha}\right) \end{pmatrix}, \end{aligned} \tag{3.30}$$

$$E_{\alpha,k+1}\left(B_j t^{\alpha}\right)=\begin{pmatrix} \sum\limits_{l=0}^{\infty}\frac{t^{\alpha l}}{\Gamma(\alpha l+k+1)}|\lambda_j|^l\cos\left(l\theta_j\right) & \sum\limits_{l=0}^{\infty}\frac{t^{\alpha l}}{\Gamma(\alpha l+k+1)}|\lambda_j|^l\sin\left(l\theta_j\right) \\ -\sum\limits_{l=0}^{\infty}\frac{t^{\alpha l}}{\Gamma(\alpha l+k+1)}|\lambda_j|^l\sin\left(l\theta_j\right) & \sum\limits_{l=0}^{\infty}\frac{t^{\alpha l}}{\Gamma(\alpha l+k+1)}|\lambda_j|^l\cos\left(l\theta_j\right) \end{pmatrix}, \tag{3.31}$$

and $|\lambda_j|=\sqrt{a_j^2+b_j^2}$ and $\theta_j=\arg(a_j+ib_j)$, $j=m+1,m+2,\cdots,n$.

Lemma 3.7. [13] *Let A be a real matrix with real eigenvalues λ_j, $j=1,2,\cdots,m$ and complex eigenvalues $\lambda_j=a_j+ib_j$ $\bar{\lambda_j}=a_j-ib_j$, $j=m+1,m+2,\cdots,n$, then there exists an invertible matrix P such that $P^{-1}AP=diag\left(B_1,B_2,\cdots,B_r\right)$, where the elementary Jordan blocks $B=B_j$, $j=1,2,\cdots,r$ are either of the form*

$$B=\begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \tag{3.32}$$

for λ one of the real eigenvalues of A or of the form

$$B=\begin{pmatrix} D & I & & \\ & D & \ddots & \\ & & \ddots & I \\ & & & D \end{pmatrix} \tag{3.33}$$

with $D=\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for $\lambda=a+ib$ one of the eigenvalues of A.

Here, based on the Lemma 3.7 and Lemma 3.4, the solution (3.4) reduces to

$$x(t)=\sum_{k=0}^{r-1}t^kPE_{\alpha,k+1}\left(Qt^{\alpha}\right)P^{-1}x^{(k)}(0) \tag{3.34}$$

where $Q=diag\left(B_1,B_2,\cdots,B_r\right)$ and

$$E_{\alpha,k+1}\left(Qt^{\alpha}\right)=\left(E_{\alpha,k+1}\left(B_1t^{\alpha}\right),E_{\alpha,k+1}\left(B_2t^{\alpha}\right),\cdots,E_{\alpha,k+1}\left(B_rt^{\alpha}\right)\right). \tag{3.35}$$

In (3.35), if $B_j=B$ is a $p\times p$ matrix of the form (3.32) and λ is a real eigenvalue of A then $B=\lambda I+N$ and

$$E_{\alpha,k+1}(Bt^{\alpha})=\begin{pmatrix} E_{\alpha,k+1}(\lambda t^{\alpha}) & \frac{t^{\alpha}}{1!}E^{(1)}_{\alpha,k+1}(\lambda t^{\alpha}) & \cdots & \frac{t^{\alpha(p-1)}}{(p-1)!}E^{(p-1)}_{\alpha,k+1}(\lambda t^{\alpha}) \\ & E_{\alpha,k+1}(\lambda t^{\alpha}) & \ddots & \vdots \\ & & \ddots & \frac{t^{\alpha}}{1!}E^{(1)}_{\alpha,k+1}(\lambda t^{\alpha}) \\ & & & E_{\alpha,k+1}(\lambda t^{\alpha}) \end{pmatrix} \tag{3.36}$$

where I is $p\times p$ identity matrix and N is the $p\times p$ nilpotent matrix of order p.

Similarly, in (3.35) if $B_j = B$ is a $2p\times 2p$ matrix of the form (3.33) and $\lambda = a+ib$ is a complex eigenvalue of A, then we have

$$E_{\alpha,k+1}(Bt^{\alpha}) = \begin{pmatrix} E_{\alpha,k+1}(Dt^{\alpha}) & \frac{t^{\alpha}}{1!}E^{(1)}_{\alpha,k+1}(Dt^{\alpha}) & \cdots & \frac{t^{\alpha(p-1)}}{(p-1)!}E^{(p-1)}_{\alpha,k+1}(Dt^{\alpha}) \\ & E_{\alpha,k+1}(Dt^{\alpha}) & \ddots & \vdots \\ & & \ddots & \frac{t^{\alpha}}{1!}E^{(1)}_{\alpha,k+1}(Dt^{\alpha}) \\ & & & E_{\alpha,k+1}(Dt^{\alpha}) \end{pmatrix} \tag{3.37}$$

where the matrix Mittag-Leffler functions involved in (3.37) can have the forms given in Lemma 3.3 and Lemma 3.4.

In the next, we give an alternative way to obtain the analytic solution of the linear fractional system (1.1) corresponding to specified initial conditions (1.2) using the Laplace transform technique. In this approach, actually one does not need the formula for the solution (3.4) and the above mentioned Lemmas to solve any specific problems which are of the form (1.1) or (1.3).

3.1.2. *Computation of $\mathcal{L}^{-1}\left(s^{\alpha}I - A\right)^{-1}$ and explicit formula for solution:* Observe that by applying the Laplace transform to the system (1.3), we arrive at the expression (3.3). Then, we proceed to compute the matrix $\left(s^{\alpha}I - A\right)^{-1}$ in (3.3) as follows:

We know that

$$\left(s^{\alpha}I - A\right)^{-1} = \frac{adj\,(s^{\alpha}I - A)}{\det\,(s^{\alpha}I - A)}. \tag{3.38}$$

Let us denote $P(s^{\alpha}) = \det\,(s^{\alpha}I - A)$. Then, under the transformation $\lambda = s^{\alpha}$, we have

$$\left(\lambda I - A\right)^{-1} = \frac{adj\,(\lambda I - A)}{\det\,(\lambda I - A)}. \tag{3.39}$$

Now one can observe that $P(\lambda) = \det\,(\lambda I - A)$ is a polynomial in λ of degree n and each entry of $adj\,(\lambda I - A)$ is a polynomial in λ of degree at most $n-1$. Suppose

$$P(\lambda) = \det\,(\lambda I - A) = \prod_{i=1}^{m}\left(\lambda - \lambda_i\right)^{p_i}, \tag{3.40}$$

where λ_i's are the distinct eigenvalues of the matrix A, with corresponding multiplicities $p_1, p_2, \cdots, p_m$. Let us decompose the matrix $\left(\lambda I - A\right)^{-1}$ in terms of partial fractions

$$\left(\lambda I - A\right)^{-1} = \sum_{i=1}^{m}\sum_{j=1}^{p_i}\frac{1}{(\lambda - \lambda_i)^j}K_{ij}, \tag{3.41}$$

where each K_{ij} is a matrix of the partial fraction expansion coefficients. Note that the coefficients K_{ij} can be evaluated by the following formula

$$K_{ij} = \frac{1}{(p_i - j)!}\frac{d^{p_i - j}}{d\lambda^{p_i - j}}\left\{(\lambda - \lambda_i)^{p_i}\frac{adj\,(\lambda I - A)}{\det\,(\lambda I - A)}\right\}\Big|_{\lambda = \lambda_i}, \tag{3.42}$$

Then, in view of (3.41), we write

$$\left(s^{\alpha}I - A\right)^{-1} = \sum_{i=1}^{m}\sum_{j=1}^{p_i}\frac{1}{(s^{\alpha} - \lambda_i)^j}K_{ij}, \tag{3.43}$$

By substituting (3.43) in (3.3), we get

$$X(s)=\sum_{k=0}^{r-1}\sum_{i=1}^{m}\sum_{j=1}^{p_i}\frac{s^{\alpha-k-1}}{(s^\alpha-\lambda_i)^j}K_{ij}x^{(k)}(0). \tag{3.44}$$

Taking the inverse transform on (3.44) and using the Property 2.4, we can easily obtain the solution

$$x(t)=\sum_{k=0}^{r-1}\sum_{i=1}^{m}\sum_{j=1}^{p_i}\frac{t^{\alpha(j-1)+k}}{(j-1)!}E_{\alpha,k+1}^{(j-1)}\left(\lambda_i t^\alpha\right)K_{ij}x^{(k)}(0). \tag{3.45}$$

3.2. **Incommensurate fractional order case:** Applying the Laplace transform on the system (1.1) and using the initial conditions (1.2), we obtain the following system

$$\left.\begin{aligned}
s^{\alpha_1}X_1(s)-\sum_{k=0}^{r_1-1}s^{\alpha_1-k-1}x_1^{(k)}(0)&=\sum_{k=1}^{n}a_{1k}X_k(s)\\
s^{\alpha_2}X_2(s)-\sum_{k=0}^{r_2-1}s^{\alpha_2-k-1}x_2^{(k)}(0)&=\sum_{k=1}^{n}a_{2k}X_k(s)\\
&\vdots\\
s^{\alpha_n}X_n(s)-\sum_{k=0}^{r_n-1}s^{\alpha_n-k-1}x_n^{(k)}(0)&=\sum_{k=1}^{n}a_{nk}X_k(s)
\end{aligned}\right\} \tag{3.46}$$

where $X_i(s)=\mathcal{L}\{x_i(t)\}$ for $i=1,2,\cdots,n$.

The system (3.46) can be rewritten in the matrix form

$$\Delta(s)\cdot X(s)=b(s) \tag{3.47}$$

where

$$\Delta(s)=\begin{bmatrix} s^{\alpha_1}-a_{11} & -a_{12} & \cdots & -a_{1n}\\ -a_{21} & s^{\alpha_2}-a_{22} & \cdots & -a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ -a_{n1} & -a_{n2} & \cdots & s^{\alpha_n}-a_{nn}\end{bmatrix} \tag{3.48}$$

$$X(s)=\begin{bmatrix} X_1(s)\\ X_2(s)\\ \vdots\\ X_n(s)\end{bmatrix}, b(s)=\begin{bmatrix} b_1(s)\\ b_2(s)\\ \vdots\\ b_n(s)\end{bmatrix}, \tag{3.49}$$

and $b_i(s)=\sum_{k=0}^{r_i-1}s^{\alpha_i-k-1}x_i^{(k)}(0)$ for $i=1,2,\cdots,n$.

Then, it follows from (3.47) that

$$X_i(s)=\frac{\det(\Delta_i(s))}{\det(\Delta(s))} \tag{3.50}$$

for $i=1,2,\cdots,n$, where $\Delta_i(s)$ is the matrix formed by replacing the i^{th} column of $\Delta(s)$ by the column vector $b(s)$.

Here, we assume fractional orders $\alpha_i=\frac{u_i}{v_i}$, where $\gcd(u_i,v_i)=1$ for $i=1,2,\cdots,n$. Let us denote $M=l.c.m(v_1,v_2,\cdots,v_n)$, and set $\gamma=\frac{1}{M}$.

Observe that $\det(\Delta(s^{1/\gamma}))$ is a polynomial in s of degree $N = M(\sum_{i=1}^{n} \alpha_i)$. Let us write this polynomial as

$$\det(\Delta(s^{1/\gamma})) = (s-\lambda_1)^{p_1}(s-\lambda_2)^{p_2}\cdots(s-\lambda_m)^{p_m}, \tag{3.51}$$

where $\sum_{i=1}^{m} p_i = N$. Then, by using (3.51) in (3.50), we get

$$X_i(s) = \frac{\det(\Delta_i(s))}{(s^\gamma-\lambda_1)^{p_1}(s^\gamma-\lambda_2)^{p_2}\cdots(s^\gamma-\lambda_m)^{p_m}}, \tag{3.52}$$

for $i = 1, 2, \cdots, n$.

Let us expand $\det(\Delta_i(s))$ and write in the form

$$\begin{aligned}\det(\Delta_i(s)) &= \sum_{k=0}^{r_1-1} s^{\alpha_1-k-1}x_1^{(k)}(0)\cdot q_1^i(s) + \cdots + \sum_{k=0}^{r_n-1} s^{\alpha_n-k-1}x_n^{(k)}(0)\cdot q_n^i(s) \\ &= \sum_{l=1}^{n}\sum_{k=0}^{r_l-1} s^{\alpha_l-k-1}x_l^{(k)}(0)\cdot q_l^i(s) \qquad (3.53)\end{aligned}$$

for $i = 1, 2, \cdots, n$, where $q_l^i(s^{1/\gamma}))$ is a polynomial in s of degree $d_l^i < N$.

Then, it follows from (3.52) and (3.53) that

$$X_i(s) = \frac{\sum_{l=1}^{n}\sum_{k=0}^{r_l-1} s^{\alpha_l-k-1}x_l^{(k)}(0)\cdot q_l^i(s)}{(s^\gamma-\lambda_1)^{p_1}(s^\gamma-\lambda_2)^{p_2}\cdots(s^\gamma-\lambda_m)^{p_m}} \tag{3.54}$$

for $i = 1, 2, \cdots, n$.

Now by using the following partial fraction decomposition of the j^{th} term in (3.54),

$$\begin{aligned}\frac{q_j^i(s)}{(s^\gamma-\lambda_1)^{p_1}(s^\gamma-\lambda_2)^{p_2}\cdots(s^\gamma-\lambda_m)^{p_m}} &= \sum_{v=1}^{p_1}\frac{c_{j1}^{vi}}{(s^\gamma-\lambda_1)^v} + \sum_{v=1}^{p_2}\frac{c_{j2}^{vi}}{(s^\gamma-\lambda_2)^v} \\ &\quad + \cdots + \sum_{v=1}^{p_m}\frac{c_{jm}^{vi}}{(s^\gamma-\lambda_m)^v} \\ &= \sum_{u=1}^{m}\sum_{v=1}^{p_u}\frac{c_{ju}^{vi}}{(s^\gamma-\lambda_u)^v} \qquad (3.55)\end{aligned}$$

where

$$c_{ju}^{vi} = \frac{1}{(p_u-v)!}\,\frac{d^{p_u-v}}{ds^{p_u-v}}\left[(s-\lambda_u)^{p_u}\left\{\frac{q_j^i(s^{1/\gamma})}{\prod_{l=1}^{m}(s-\lambda_l)^{p_l}}\right\}\right]\Bigg|_{s=\lambda_u} \tag{3.56}$$

for $v = 1, 2, \cdots, p_u$, $u = 1, 2, \cdots, m$ and $i = 1, 2, \cdots, n$,

we get

$$X_i(s) \;=\; \sum_{l=1}^{n}\sum_{k=0}^{r_l-1}\sum_{u=1}^{m}\sum_{v=1}^{p_u}\frac{c_{lu}^{vi}}{(s^\gamma-\lambda_u)^v}\cdot s^{\alpha_l-k-1}x_l^{(k)}(0). \tag{3.57}$$

Taking the inverse Laplace transform on (3.57) and using Property (2.4), we obtain the solution

$$x_i(t) = \sum_{l=1}^{n}\sum_{k=0}^{r_l-1}\sum_{u=1}^{m}\sum_{v=1}^{p_u}\frac{c_{lu}^{vi}}{(v-1)!}t^{\gamma v-\alpha_l+k}E_{\gamma,\gamma-\alpha_l+k+1}^{(v-1)}(\lambda_u t^\gamma)\cdot x_l^{(k)}(0), \tag{3.58}$$

for $i = 1, 2, \cdots, n$.

4. Illustrative examples

Example 4.1. *Consider the commensurate fractional order system*

$$\begin{aligned} {}^{C}D_{0,t}^{\alpha}\, x_1(t) &= \quad x_1(t) + 4x_2(t) \\ {}^{C}D_{0,t}^{\alpha}\, x_2(t) &= \ -x_1(t) - 3x_2(t) \end{aligned} \tag{4.1}$$

with initial conditions

$$x_i^{(k)}(0) = x_{ik}, \quad k = 0, 1, 2, \cdots, r-1, \quad i = 1, 2. \tag{4.2}$$

where fractional order $r - 1 < \alpha \leq r$ and $r \in \mathbb{Z}^+$.

Here, by applying the Laplace transform to the system (4.1)*, we obtain the solution in vector form (for instance, see subsection 3.1)*

$$x(t) = \sum_{k=0}^{r-1} t^k E_{\alpha,k+1}(At^{\alpha}) x^{(k)}(0). \tag{4.3}$$

where $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and the coefficient matrix $A = \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix}$. Since $\lambda = -1$ is the only eigenvalue of the matrix A with multiplicity 2*, we find the generalized eigenvectors $v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Therefore, we have $P = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$ and $P^{-1}AP = J$ with $J = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. Then, in view of Lemma 3.2, it follows from* (4.3) *that*

$$x(t) = \sum_{k=0}^{r-1} t^k P E_{\alpha,k+1}(Jt^{\alpha}) P^{-1} x^{(k)}(0). \tag{4.4}$$

Simplification of (4.4) *gives the following solution components*

$$\begin{aligned} x_1(t) = &\sum_{k=0}^{r-1} t^k \Big\{ E_{\alpha,k+1}(-t^{\alpha}) + 2t^{\alpha} E_{\alpha,k+1}^{(1)}(-t^{\alpha}) \Big\} x_1^{(k)}(0) \\ &+ \Big\{ 4t^{\alpha} E_{\alpha,k+1}^{(1)}(-t^{\alpha}) \Big\} x_2^{(k)}(0), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} x_2(t) = &\sum_{k=0}^{r-1} t^k \Big\{ -t^{\alpha} E_{\alpha,k+1}^{(1)}(-t^{\alpha}) \Big\} x_1^{(k)}(0) \\ &+ \Big\{ E_{\alpha,k+1}(-t^{\alpha}) - 2t^{\alpha} E_{\alpha,k+1}^{(1)}(-t^{\alpha}) \Big\} x_2^{(k)}(0). \end{aligned} \tag{4.6}$$

Example 4.2. *Consider the fractional order system*

$$\begin{aligned} {}^{C}D_{0,t}^{\alpha}\, x_1(t) &= \quad x_1(t) + 4x_2(t) \\ {}^{C}D_{0,t}^{\alpha}\, x_2(t) &= \ -x_1(t) - 3x_2(t) \end{aligned} \tag{4.7}$$

Here, we consider the following cases:

(a) *Let $\alpha = 1$: In this case, we assume the initial conditions for the system* (4.7) *of the form*

$$x_1^{(0)}(0) = x_{10}, \quad x_2^{(0)}(0) = x_{20}. \tag{4.8}$$

Application of Laplace transform to the system (4.7) *gives*

$$X(s) = (sI - A)^{-1} \cdot x^{(0)}(0) \tag{4.9}$$

where $X(s) = \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix}$, $A = \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix}$ *and* $x^{(0)}(0) = \begin{pmatrix} x_1^{(0)}(0) \\ x_2^{(0)}(0) \end{pmatrix}$.
Using the formula (3.38) *in* (4.9), *we obtain*

$$X(s) = \frac{1}{(s+1)^2} \begin{pmatrix} s+3 & 4 \\ -1 & s-1 \end{pmatrix} x^{(0)}(0). \tag{4.10}$$

Let us decompose the following matrix in terms of partial fraction expansion

$$\frac{1}{(s+1)^2} \begin{pmatrix} s+3 & 4 \\ -1 & s-1 \end{pmatrix} = \frac{1}{s+1} K_{11} + \frac{1}{(s+1)^2} K_{12}, \tag{4.11}$$

where

$$K_{11} = \frac{d}{ds} \left\{ \begin{pmatrix} s+3 & 4 \\ -1 & s-1 \end{pmatrix} \right\} \Bigg|_{s=-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.12}$$

$$K_{12} = \left\{ \begin{pmatrix} s+3 & 4 \\ -1 & s-1 \end{pmatrix} \right\} \Bigg|_{s=-1} = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}. \tag{4.13}$$

By substituting (4.11) *in* (4.10) *and then taking the inverse Laplace transform, we get*

$$\begin{aligned} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} &= e^{-t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^{(0)}(0) \\ x_2^{(0)}(0) \end{pmatrix} + te^{-t} \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1^{(0)}(0) \\ x_2^{(0)}(0) \end{pmatrix} \\ &= \begin{pmatrix} e^{-t}(1+2t)x_{10} + 4te^{-t}x_{20} \\ -te^{-t}x_{10} + e^{-t}(1-2t)x_{20} \end{pmatrix}. \end{aligned} \tag{4.14}$$

(b) *Suppose* $\alpha \in (1,2)$*: Here, we take initial conditions for the system* (4.7) *in the following form*

$$x_1^{(0)}(0) = x_{10}, \;\; x_1^{(1)}(0) = x_{11}, \;\; x_2^{(0)}(0) = x_{20}, \;\; x_2^{(1)}(0) = x_{21}. \tag{4.15}$$

By applying the Laplace transform to the system (4.7) *and using the initial conditions* (4.15), *we obtain*

$$X(s) = (s^\alpha I - A)^{-1} \left[s^{\alpha-1} x^{(0)}(0) + s^{\alpha-2} x^{(1)}(0) \right] \tag{4.16}$$

where $X(s) = \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix}$, $A = \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix}$, $x^{(0)}(0) = \begin{pmatrix} x_1^{(0)}(0) \\ x_2^{(0)}(0) \end{pmatrix}$ *and* $x^{(1)}(0) = \begin{pmatrix} x_1^{(1)}(0) \\ x_2^{(1)}(0) \end{pmatrix}$.

Then, by using the formula (3.38) *in* (4.16), *we get*

$$X(s) = \frac{1}{(s^\alpha+1)^2} \begin{pmatrix} s^\alpha+3 & 4 \\ -1 & s^\alpha-1 \end{pmatrix} \left[s^{\alpha-1} x^{(0)}(0) + s^{\alpha-2} x^{(1)}(0) \right]. \tag{4.17}$$

Using the partial fraction decomposition, we write

$$\frac{1}{(s^\alpha+1)^2} \begin{pmatrix} s^\alpha+3 & 4 \\ -1 & s^\alpha-1 \end{pmatrix} = \frac{1}{s^\alpha+1} K_{11} + \frac{1}{(s^\alpha+1)^2} K_{12}, \tag{4.18}$$

where

$$K_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_{12} = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}.$$

By substituting (4.18) *in* (4.17)*, we get*

$$\begin{aligned} X(s) = & \left[\frac{s^{\alpha-1}}{s^\alpha+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{s^{\alpha-1}}{(s^\alpha+1)^2} \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \right] x^{(0)}(0) \\ & + \left[\frac{s^{\alpha-2}}{s^\alpha+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{s^{\alpha-2}}{(s^\alpha+1)^2} \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \right] x^{(1)}(0). \end{aligned} \tag{4.19}$$

Taking the inverse Laplace transform on both sides of (4.19)*, we obtain*

$$\begin{aligned} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = & \left[E_{\alpha,1}(-t^\alpha) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t^\alpha E^{(1)}_{\alpha,1}(-t^\alpha) \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \right] \begin{pmatrix} x_1^{(0)}(0) \\ x_2^{(0)}(0) \end{pmatrix} \\ & + \left[t E_{\alpha,2}(-t^\alpha) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t^{\alpha+1} E^{(1)}_{\alpha,2}(-t^\alpha) \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \right] \begin{pmatrix} x_1^{(1)}(0) \\ x_2^{(1)}(0) \end{pmatrix}. \end{aligned} \tag{4.20}$$

Then, the solution components are

$$\begin{aligned} x_1(t) = & \left\{ E_{\alpha,1}(-t^\alpha) + 2t^\alpha E^{(1)}_{\alpha,1}(-t^\alpha) \right\} x_1^{(0)}(0) \\ & + \left\{ t E_{\alpha,2}(-t^\alpha) + 2t^{\alpha+1} E^{(1)}_{\alpha,2}(-t^\alpha) \right\} x_1^{(1)}(0) \\ & + \left\{ 4t^\alpha E^{(1)}_{\alpha,1}(-t^\alpha) \right\} x_2^{(0)}(0) + \left\{ 4t^{\alpha+1} E^{(1)}_{\alpha,2}(-t^\alpha) \right\} x_2^{(1)}(0), \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} x_2(t) = & \left\{ -t^\alpha E^{(1)}_{\alpha,1}(-t^\alpha) \right\} x_1^{(0)}(0) + \left\{ -t^{\alpha+1} E^{(1)}_{\alpha,2}(-t^\alpha) \right\} x_1^{(1)}(0) \\ & + \left\{ E_{\alpha,1}(-t^\alpha) - 2t^\alpha E^{(1)}_{\alpha,1}(-t^\alpha) \right\} x_2^{(0)}(0) \\ & + \left\{ t E_{\alpha,2}(-t^\alpha) - 2t^{\alpha+1} E^{(1)}_{\alpha,2}(-t^\alpha) \right\} x_2^{(1)}(0). \end{aligned} \tag{4.22}$$

(c) *Suppose* $\alpha \in (0,1)$*:*

For the system (4.7)*, we consider the following initial conditions*

$$x_1^{(0)}(0) = x_{10}, \quad x_2^{(0)}(0) = x_{20}. \tag{4.23}$$

Similar to the case (b), here one can easily obtain the solution components

$$x_1(t) = \left\{ E_{\alpha,1}(-t^\alpha) + 2t^\alpha E^{(1)}_{\alpha,1}(-t^\alpha) \right\} x_1^{(0)}(0) + \left\{ 4t^\alpha E^{(1)}_{\alpha,1}(-t^\alpha) \right\} x_2^{(0)}(0), \tag{4.24}$$

and

$$x_2(t) = \left\{ -t^\alpha E^{(1)}_{\alpha,1}(-t^\alpha) \right\} x_1^{(0)}(0) + \left\{ E_{\alpha,1}(-t^\alpha) - 2t^\alpha E^{(1)}_{\alpha,1}(-t^\alpha) \right\} x_2^{(0)}(0). \tag{4.25}$$

Example 4.3. *Consider the fractional order system*

$$\begin{pmatrix} {}^C D^{\alpha_1}_{0,t}\, x_1(t) \\ {}^C D^{\alpha_2}_{0,t}\, x_2(t) \\ {}^C D^{\alpha_3}_{0,t}\, x_3(t) \end{pmatrix} = \begin{pmatrix} -1 & 4 & -1 \\ 0 & -4 & 1 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \tag{4.26}$$

with initial conditions

$$x_1^{(0)}(0) = x_{10}, \quad x_1^{(1)}(0) = x_{11}, \quad x_2^{(0)}(0) = x_{20}, \quad x_3^{(0)}(0) = x_{30}, \tag{4.27}$$

where fractional orders $\alpha_1 = \frac{u_1}{v_1} = \frac{3}{2}$, $\alpha_2 = \frac{u_2}{v_2} = 1$ *and* $\alpha_3 = \frac{u_3}{v_3} = \frac{1}{2}$. *Here, by applying the Laplace transform on both the sides of system* (4.26) *and using the initial conditions* (4.27), *we get (for details see subsection 3.2)*

$$X_1(s) = \frac{(s^{\alpha_1-1}x_1^{(0)}(0) + s^{\alpha_1-2}x_1^{(1)}(0)) \cdot q_1^1(s)}{\det(\Delta(s))} + \frac{(s^{\alpha_2-1}x_2^{(0)}(0)) \cdot q_2^1(s)}{\det(\Delta(s))} + \frac{(s^{\alpha_3-1}x_3^{(0)}(0)) \cdot q_3^1(s)}{\det(\Delta(s))}, \tag{4.28}$$

$$X_2(s) = \frac{(s^{\alpha_1-1}x_1^{(0)}(0) + s^{\alpha_1-2}x_1^{(1)}(0)) \cdot q_1^2(s)}{\det(\Delta(s))} + \frac{(s^{\alpha_2-1}x_2^{(0)}(0)) \cdot q_2^2(s)}{\det(\Delta(s))} + \frac{(s^{\alpha_3-1}x_3^{(0)}(0)) \cdot q_3^2(s)}{\det(\Delta(s))}, \tag{4.29}$$

and

$$X_3(s) = \frac{(s^{\alpha_1-1}x_1^{(0)}(0) + s^{\alpha_1-2}x_1^{(1)}(0)) \cdot q_1^3(s)}{\det(\Delta(s))} + \frac{(s^{\alpha_2-1}x_2^{(0)}(0)) \cdot q_2^3(s)}{\det(\Delta(s))} + \frac{(s^{\alpha_3-1}x_3^{(0)}(0)) \cdot q_3^3(s)}{\det(\Delta(s))}, \tag{4.30}$$

where

$$q_1^1(s) = (s^{\alpha_2} + 4)(s^{\alpha_3} + 3), \;\; q_2^1(s) = 4(s^{\alpha_3} + 3), \;\; q_3^1(s) = -s^{\alpha_2},$$
$$q_1^2(s) = 0, \;\; q_2^2(s) = (s^{\alpha_1} + 1)(s^{\alpha_3} + 3), \;\; q_3^2(s) = s^{\alpha_1} + 1,$$
$$q_1^3(s) = 0, \;\; q_2^3(s) = 0, \;\; q_3^3(s) = (s^{\alpha_1} + 1)(s^{\alpha_2} + 4).$$

Note that the characteristic matrix for the system (4.26) *is*

$$\Delta(s) = \begin{pmatrix} s^{\alpha_1} + 1 & -4 & 1 \\ 0 & s^{\alpha_2} + 4 & -1 \\ 0 & 0 & s^{\alpha_3} + 3 \end{pmatrix}. \tag{4.31}$$

Let $\gamma = \frac{1}{M} = \frac{1}{2}$. *Then, we have*

$$\det\left(\Delta(s^{\frac{1}{\gamma}})\right) = \det \begin{pmatrix} s^{M\alpha_1} + 1 & -4 & 1 \\ 0 & s^{M\alpha_2} + 4 & -1 \\ 0 & 0 & s^{M\alpha_3} + 3 \end{pmatrix} = (s^3 + 1)(s^2 + 4)(s + 3), \tag{4.32}$$

and

$$\det\left(\Delta(s)\right) = \prod_{i=1}^{6}(s^{1/2} - \lambda_i), \tag{4.33}$$

where

$$\lambda_1 = -1, \;\; \lambda_2 = 1/2 - \sqrt{3}/2i, \;\; \lambda_3 = 1/2 + \sqrt{3}/2i, \lambda_4 = 2i, \;\; \lambda_5 = -2i, \;\; \lambda_6 = -3. \tag{4.34}$$

By using the formula (3.56), *we can write*

$$\frac{q_1^1(s)}{\det(\Delta(s))} = \frac{1/3}{s^{1/2} + 1} + \frac{(-1 - \sqrt{3}i)/6}{s^{1/2} - (1/2 + \sqrt{3}/2i)} + \frac{(-1 + \sqrt{3}i)/6}{s^{1/2} - (1/2 - \sqrt{3}/2i)}, \tag{4.35}$$

$$\frac{q_2^1(s)}{\det(\Delta(s))}=\frac{4/15}{s^{1/2}+1}+\frac{(-10-6\sqrt{3}i)/39}{s^{1/2}-(1/2+\sqrt{3}/2i)}+\frac{(-10+6\sqrt{3}i)/39}{s^{1/2}-(1/2-\sqrt{3}/2i)}+\frac{(8-i)/65}{s^{1/2}-2i}+\frac{(8+i)/65}{s^{1/2}+2i}, \tag{4.36}$$

$$\frac{q_3^1(s)}{\det(\Delta(s))}=\frac{-1/30}{s^{1/2}+1}+\frac{(-23+7\sqrt{3}i)/1014}{s^{1/2}-(1/2+\sqrt{3}/2i)}+\frac{(-23-7\sqrt{3}i)/1014}{s^{1/2}-(1/2-\sqrt{3}/2i)}+\frac{(22-19i)/845}{s^{1/2}-2i}+\frac{(22+19i)/845}{s^{1/2}+2i}+\frac{9/338}{s^{1/2}+3}, \tag{4.37}$$

$$\frac{q_2^2(s)}{\det(\Delta(s))}=\frac{-i/4}{s^{1/2}-2i}+\frac{i/4}{s^{1/2}+2i}, \tag{4.38}$$

$$\frac{q_3^2(s)}{\det(\Delta(s))}=\frac{(-2+3i)/52}{s^{1/2}-2i}+\frac{(-2-3i)/52}{s^{1/2}+2i}+\frac{1/13}{s^{1/2}+3}, \tag{4.39}$$

$$\frac{q_3^3(s)}{\det(\Delta(s))}=\frac{1}{s^{1/2}+3}. \tag{4.40}$$

Substituting these expressions in (4.28), (4.29) and (4.30), and then by taking the inverse Laplace transform, we obtain the following solution components

$$\begin{aligned}
x_1(t)=&\Big\{-\frac{1}{3}t^{-1/2}E_{1/2,1/2}\left(-t^{1/2}\right)+\frac{(1-\sqrt{3}i)}{6}t^{-1/2}E_{1/2,1/2}\left(((1+\sqrt{3}i)/2)t^{1/2}\right)\\
&+\frac{(1+\sqrt{3}i)}{6}t^{-1/2}E_{1/2,1/2}\left(((1-\sqrt{3}i)/2)t^{1/2}\right)\Big\}x_1^{(0)}(0)\\
&+\Big\{\frac{1}{3}E_{1/2,1}\left(-t^{1/2}\right)+\frac{(-1-\sqrt{3}i)}{6}E_{1/2,1}\left(((1+\sqrt{3}i)/2)t^{1/2}\right)\\
&+\frac{(-1+\sqrt{3}i)}{6}E_{1/2,1}\left(((1-\sqrt{3}i)/2)t^{1/2}\right)\Big\}x_1^{(1)}(0)\\
&+t^{-1/2}\Big\{\frac{4}{15}E_{1/2,1/2}\left(-t^{1/2}\right)+\frac{(-10-6\sqrt{3}i)}{39}E_{1/2,1/2}\left(((1+\sqrt{3}i)/2)t^{1/2}\right)\\
&+\frac{(-10+6\sqrt{3}i)}{39}E_{1/2,1/2}\left(((1-\sqrt{3}i)/2)t^{1/2}\right)+\frac{(8-i)}{65}E_{1/2,1/2}\left((2i)t^{1/2}\right)\\
&+\frac{(8+i)}{65}E_{1/2,1/2}\left((-2i)t^{1/2}\right)\Big\}x_2^{(0)}(0)\\
&+\Big\{\frac{-1}{30}E_{1/2,1}\left(-t^{1/2}\right)+\frac{(-23+7\sqrt{3}i)}{1014}E_{1/2,1}\left(((1+\sqrt{3}i)/2)t^{1/2}\right)\\
&+\frac{(-23-7\sqrt{3}i)}{1014}E_{1/2,1}\left(((1-\sqrt{3}i)/2)t^{1/2}\right)+\frac{(22-19i)}{845}E_{1/2,1}\left((2i)t^{1/2}\right)\\
&+\frac{(22+19i)}{845}E_{1/2,1}\left((-2i)t^{1/2}\right)+\frac{9}{338}E_{1/2,1}\left(-3t^{1/2}\right)\Big\}x_3^{(0)}(0),
\end{aligned} \tag{4.41}$$

$$\begin{aligned}
x_2(t)=&\,t^{-1/2}\left\{-\frac{i}{4}E_{1/2,1/2}\left((2i)t^{1/2}\right)+\frac{i}{4}E_{1/2,1/2}\left((-2i)t^{1/2}\right)\right\}x_2^{(0)}(0)\\
&+\Big\{\frac{(-2+3i)}{52}E_{1/2,1}\left((2i)t^{1/2}\right)+\frac{(-2-3i)}{52}E_{1/2,1}\left((-2i)t^{1/2}\right)\\
&+\frac{1}{13}E_{1/2,1}\left(-3t^{1/2}\right)\Big\}x_3^{(0)}(0),
\end{aligned} \tag{4.42}$$

and

$$x_3(t) = E_{1/2,1}\left(-3t^{1/2}\right)x_3^{(0)}(0). \tag{4.43}$$

Note that the solution components $x_1(t)$ and $x_2(t)$ in (4.41) *and* (4.42) *are represented in terms of complex valued functions. Here, by using the Remark 3.4 and Lemma 3.4, we represent the solution components in terms of real valued functions*

$$\begin{aligned}
x_1(t) = &\Big\{-\frac{1}{3}t^{-1/2}E_{1/2,1/2}\left(-t^{1/2}\right) + \frac{1}{3}t^{-1/2}\sum_{k=0}^{\infty}\frac{t^{k/2}}{\Gamma(\frac{k+1}{2})}\cos(\frac{k\pi}{3}) \\
&+ \frac{1}{\sqrt{3}}t^{-1/2}\sum_{k=0}^{\infty}\frac{t^{k/2}}{\Gamma(\frac{k+1}{2})}\sin(\frac{k\pi}{3})\Big\}x_1^{(0)}(0) \\
&+ \Big\{\frac{1}{3}E_{1/2,1}\left(-t^{1/2}\right) - \frac{1}{3}\sum_{k=0}^{\infty}\frac{t^{k/2}}{\Gamma(\frac{k}{2}+1)}\cos(\frac{k\pi}{3}) \\
&+ \frac{1}{\sqrt{3}}\sum_{k=0}^{\infty}\frac{t^{k/2}}{\Gamma(\frac{k}{2}+1)}\sin(\frac{k\pi}{3})\Big\}x_1^{(1)}(0) \\
&+ t^{-1/2}\Big\{\frac{4}{15}E_{1/2,1/2}\left(-t^{1/2}\right) - \frac{20}{39}\sum_{k=0}^{\infty}\frac{t^{k/2}}{\Gamma(\frac{k+1}{2})}\cos(\frac{k\pi}{3}) \\
&+ \frac{12\sqrt{3}}{39}\sum_{k=0}^{\infty}\frac{t^{k/2}}{\Gamma(\frac{k+1}{2})}\sin(\frac{k\pi}{3}) + \frac{16}{65}\sum_{k=0}^{\infty}\frac{2^k t^{k/2}}{\Gamma(\frac{k+1}{2})}\cos(\frac{k\pi}{2}) \\
&+ \frac{2}{65}\sum_{k=0}^{\infty}\frac{2^k t^{k/2}}{\Gamma(\frac{k+1}{2})}\sin(\frac{k\pi}{2})\Big\}x_2^{(0)}(0) \\
&+ \Big\{-\frac{1}{30}E_{1/2,1}\left(-t^{1/2}\right) - \frac{46}{1014}\sum_{k=0}^{\infty}\frac{t^{k/2}}{\Gamma(\frac{k}{2}+1)}\cos(\frac{k\pi}{3}) \\
&- \frac{14\sqrt{3}}{1014}\sum_{k=0}^{\infty}\frac{t^{k/2}}{\Gamma(\frac{k}{2}+1)}\sin(\frac{k\pi}{3}) + \frac{44}{845}\sum_{k=0}^{\infty}\frac{2^k t^{k/2}}{\Gamma(\frac{k}{2}+1)}\cos(\frac{k\pi}{2}) \\
&+ \frac{38}{845}\sum_{k=0}^{\infty}\frac{2^k t^{k/2}}{\Gamma(\frac{k}{2}+1)}\sin(\frac{k\pi}{2}) + \frac{9}{338}E_{1/2,1}\left(-3t^{1/2}\right)\Big\}x_3^{(0)}(0),
\end{aligned} \tag{4.44}$$

$$\begin{aligned}
x_2(t) =& t^{-1/2}\left\{\frac{1}{2}\sum_{k=0}^{\infty}\frac{2^k t^{k/2}}{\Gamma(\frac{k+1}{2})}\sin(\frac{k\pi}{2})\right\}x_2^{(0)}(0) + \Big\{-\frac{2}{26}\sum_{k=0}^{\infty}\frac{2^k t^{k/2}}{\Gamma(\frac{k}{2}+1)}\cos(\frac{k\pi}{2}) \\
&- \frac{3}{26}\sum_{k=0}^{\infty}\frac{2^k t^{k/2}}{\Gamma(\frac{k}{2}+1)}\sin(\frac{k\pi}{2}) + \frac{1}{13}E_{1/2,1}\left(-3t^{1/2}\right)\Big\}x_3^{(0)}(0),
\end{aligned} \tag{4.45}$$

and

$$x_3(t) = E_{1/2,1}\left(-3t^{1/2}\right)x_3^{(0)}(0). \tag{4.46}$$

5. Conclusions

In this paper, we have studied the initial value problem of arbitrary order autonomous linear fractional order system. By applying the Laplace transform to such a system, we have discussed the analytic solutions of such a system to the cases when all the fractional orders are equal (commensurate case) and are different (incommensurate case). For the commensurate fractional order case, the analytic solution to such a system is expressed in terms of matrix Mittag-Leffler functions. Whenever the coefficient matrix of such a system is similar to a diagonal matrix or a Jordan canonical form, then the matrix Mittag-Leffler functions are computed and based on this, we have presented several interesting results. Further, we have presented an alternative way for solving such a system, which is based on the partial fraction decomposition of its characteristic matrix, and interestingly, in this case, one does not really need to compute the matrix Mittag-Leffler function. For the incommensurate fractional order case, we have presented the explicit analytical formulas for the solution components to such a system whenever all the fractional orders are rational numbers. Finally, we have demonstrated the theoretical approach by presenting illustrative examples.

Acknowledgement

The author wishes to thank the Reviewer for the careful and insightful review, which has helped to improve the manuscript. The author would like to thank the Editor Prof. A. M. A. El-Sayed for allowing a revision.

References

[1] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
[2] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, Amsterdam: Elsevier, 2006.
[3] C. A. Monje, Y. Q.Chen, B. M. Vinagre, D. Xue, V. Feliu, Fractional-order Systems and Controls: Fundamentals and Applications, Springer, 2010.
[4] I. Petráš, Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation, Springer, 2011.
[5] F. Mainardi, Fractional Relaxation-Oscillation and Fractional Diffusion-Wave Phenomena, Chaos, Solitons & Fractals, Vol. 7, 1461–1477, 1996.
[6] S. Kazem, Exact Solution of Some Linear Fractional Differential Equations by Laplace Transform, International Journal of Nonlinear Science, Vol. 16, 3–11, 2013.
[7] Q. D. Katatbeh, F. B. M. Belgacem, Applications of the Sumudu transform to fractional differential equations, Nonlinear Studies, Vol. 18, 99–112, 2011.
[8] Y. Luchko, R. Gorenflo, An operational method for solving fractional differential equations with the Caputo derivatives, Acta Mathematica Vietnamica, Vol. 24, 207–233, 1999.
[9] K. Diethelm, N. J. Ford, A. D. Freed, A Predictor-Corrector Approach for the Numerical Solution of Fractional Differential Equations, Nonlinear Dynamics, Vol. 29, 3–22, 2002.
[10] V. D. Gejji, Y. Sukale, S. Bhalekar, A new predictor-corrector method for fractional differential equations, Applied Mathematics and Computation, Vol. 244, 158–182, 2014.
[11] V. S. Ertürk, S. Momani, Solving systems of fractional differential equations using differential transform method, Journal of Computational and Applied Mathematics, Vol. 215, 142–151, 2008.
[12] Z. M. Odibat, S. Momani, Application of Variational Iteration Method to Nonlinear Differential Equations of Fractional Order, International Journal of Nonlinear Sciences and Numerical Simulation, Vol. 7, 27–34, 2006.
[13] L. Perko, Differential Equations and Dynamical Systems, Springer, 2001.
[14] V. D. Gejji, H. Jafari, Adomian decomposition: a tool for solving a system of fractional differential equations, Journal of Mathematical Analysis and Applications, Vol. 301, 508–518, 2005.
[15] B. Bonilla, M. Rivero, J. J. Trujillo, On systems of linear fractional differential equations with constant coefficients, Applied Mathematics and Computation, Vol. 187, 68–78, 2007.
[16] Z. M. Odibat, Analytic study on linear systems of fractional differential equations, Computers and Mathematics with Applications, Vol. 59, 1171–1183, 2010.

[17] A. Charef, D. Boucherma, Analytical solution of the linear fractional system of commensurate order, Computers and Mathematics with Applications, Vol. 62, 4415–4428, 2011.
[18] K. Li, J. Peng, Laplace transform and fractional differential equations, Applied Mathematics Letters, Vol. 24, 2019–2023, 2011.
[19] J. Duan, A generalization of the Mittag-Leffler function and solution of system of fractional differential equations, Advances in Difference Equations, 2018, Article number:239, 2018. https://doi:10.1186/s13662-018-1693-9.
[20] J. Duan, L. Chen, Solution of Fractional Differential Equation Systems and Computation of Matrix Mittag-Leffler Functions, Symmetry, 10, 503, 2018. https://doi.org/10.3390/sym10100503.
[21] A. Chikrii, I. Matychyn, Riemann-Liouville, Caputo, and Sequential fractional derivatives in differential games, Annals of the International Society of Dynamic Games: Advances in Dynamic Games, Birkhäuser, Boston, 61–81, 2011.
[22] R. Garrappa, M. Popolizio, Computing the Matrix Mittag-Leffler Function with Applications to Fractional Calculus, Journal of Scientific Computing, Vol. 77, 129–153, 2018.
[23] M. Popolizio, On the Matrix Mittag-Leffler Function: Theoretical Properties and Numerical Computation, Mathematics, 7, 1140, 2019. https://doi.org/10.3390/math7121140.
[24] I. Matychyn, V. Onyshchenko, Matrix Mittag-Leffler function in fractional systems and its computation, Bulletin Of The Polish Academy Of Sciences Technical Sciences, Vol. 66, 495–500, 2018.
[25] J. Peng, K. Li, A note on property of the Mittag-Leffler function, Journal of Mathematical Analysis and Applications, Vol. 370, 635–638, 2010.
[26] S. K. Elagan, On the invalidity of semigroup property for the Mittag-Leffler function with two parameters, Journal of the Egyptian Mathematical Society, Vol. 24, 200–203, 2016.
[27] A. Sadeghi, J. R. Cardoso, Some notes on properties of the matrix Mittag-Leffler function, Applied Mathematics and Computation, Vol. 338, 733–738, 2018.

Bichitra Kumar Lenka
Nonlinear Dynamics Lab,
Indian Institute of Science Education and Research Kolkata,
Mohanpur, Nadia-741246, West Bengal, India
E-mail address: bichitra.lenka99@gmail.com