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UNIQUENESS OF MEROMORPHIC FUNCTIONS WITH FINITE LOGARITHMIC ORDER REGARDING THEIR q-SHIFT DIFFERENCE AND DIFFERENTIAL POLYNOMIAL

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Abstract. In this paper, we investigate the uniqueness and value distribution of transcendental meromorphic functions with zero order by considering their q-shift difference and differential polynomial and obtain some results which improve and generalise the previous theorems given by Zheng and Xu [15].

1. Introduction and main results

We assume that the reader is accustomed with the Nevanlinna value distribution theory and knows the standard notations and definitions used in it such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, *etc.* (see Hayman [7], Yang [14], Yi and Yang [13]).
Let f and g be two transcendental meromorphic functions in the open complex plane. For $a \in \mathbb{C} \cup \{\infty\}$ and $k \in \mathbb{Z}^+ \cup \{\infty\}$ the set, $E(a,f) = \{z : f(z) - a = 0\}$, denotes all those a-points of f, where each a-point of f with multiplicity k is counted k times in the set and the set, $\overline{E}(a,f) = \{z : f(z) - a = 0\}$, denotes all those a-points of f, where the multiplicities are ignored. If $f(z) - a$ and $g(z) - a$ assumes the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicity) and we have $E(a,f) = E(a,g)$; Suppose, if $f(z) - a$ and $g(z) - a$ assumes the same zeros ignoring the multiplicities, then we say that $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicity) and we will have $\overline{E}(a,f) = \overline{E}(a,g)$.
A meromorphic function $a(z)$ is called a small function with respect to $f(z)$, if $T(r,a) = S(r,f)$, where $S(r,f)$ denotes any quantity which satisfies $S(r,f) = o(T(r,f))$ as $r \to +\infty$ possibly outside a set I with finite linear measure $\lim\limits_{r\to\infty} \int\limits_{(1,r]\cap I} \frac{dt}{t}$ $< \infty$. We also denote by $S_1(r,f)$ any quantity which satisfies $S_1(r,f) = o(T(r,f))$ for all r on a set F of logarithmic density 1.
We need the following standard definitions of Nevanlinna Theory.

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Definition 1.1. [3] *The logarithmic density of a set F is defined by,*

$$\lim_{r\to\infty}\frac{1}{log\, r}\int_{(1,r]\cap F}\frac{1}{t}dt.$$

Definition 1.2. [3] *The order $\rho(f)$ of a meromorphic function $f(z)$ is defined as,*

$$\rho(f)=\overline{\lim_{r\to\infty}}\frac{log\, T(r,f)}{log\, r}.$$

Definition 1.3. [3] *The logarithmic order of a meromorphic function $f(z)$ is defined by,*

$$\rho_{log}(f)=\overline{\lim_{r\to\infty}}\frac{log\, T(r,f)}{log\, log\, r}.$$

If $\rho_{log}(f)<\infty$, then $f(z)$ is said to be of finite logarithmic order. It is clear that, if a meromorphic function $f(z)$ has finite logarithmic order, then $f(z)$ has order zero. From the definition of logarithmic order, we can easily say that a constant function will have the logarithmic order zero and for a non-constant rational function it will be 1. A transcendental meromorphic function $f(z)$ will have the logarithmic order atleast 1. If $f(z)$ is a meromorphic function having finite positive logarithmic order $\rho_{log}(f)$, then $T(r,f)$ will have proximate logarithmic order $\rho_{log}(r)$. The logarithmic-type function of $T(r,f)$ is defined as $U(r,f)=(log\, r)^{\rho_{log}(r)}$. We will have $T(r,f)\leq U(r,f)$ for sufficiently larger r. The logarithmic exponent of convergence of a-points of $f(z)$ will be equal to the logarithmic order of $n(r,f=a)$, which is defined as,

$$\lambda_{log}(a)=\limsup_{r\to\infty}\frac{log\, n\left(r,\frac{1}{f-a}\right)}{log\, log\, r}.$$

It is known that for any meromorphic function $f(z)$ having finite positive order and for any $a\in\mathbb{C}$, the counting function $N(r,f=a)$ and $n(r,f=a)$, both have same order, the situation is different for functions having finite logarithmic order, that is logarithmic order of $N(r,f=a)$ is $\lambda_{log}(a)+1$, where $\lambda_{log}(a)$ is logarithmic order of $n(r,f=a)$, see [3].

Definition 1.4. [8] *Let $n_{0j},n_{1j},...,n_{kj}$ be non-negative integers. The expression,*

$$M_j[f]=(f)^{n_{0j}}(f^{(1)})^{n_{1j}}...(f^{(k)})^{n_{kj}},$$

is called a differential monomial generated by f of degree $\gamma_{M_j}=\sum_{i=0}^{k}n_{ij}$ and weight $\Gamma_{M_j}=\sum_{i=0}^{k}(i+1)n_{ij}$. The sum,

$$H[f]=\sum_{i=1}^{l}b_jM_j[f],$$

is called a differential polynomial generated by f of degree $\gamma_p=max\{\ \gamma_{M_j}:1\leq j\leq l\ \}$ and weight $\Gamma_p=max\{\ \Gamma_{M_j}:1\leq j\leq l\ \}$, where $T(r,b_j)=S(r,f)$ for the co-efficients $b_j(j=1,2,...,l)$. The numbers, $\underline{\gamma}_p=min\{\gamma_{M_j}:1\leq j\leq l\ \}$ and k (the highest order of the derivarive of f in $H[f]$) are called, respectively, the lower degree and order of $H[f]$. We denote by $\sigma=max\{\Gamma_{M_j}-\gamma_{M_j}:1\leq j\leq l\ \}=max\{n_{1j}+2n_{2j}+...+kn_{kj}:1\leq j\leq l\ \}$.

$H[f]$ is said to be homogeneous if $\gamma_p = \underline{\gamma}_p$. Also $H[f]$ is called a quasi differential polynomial generated by f, if instead of assuming $T(r, b_j) = S(r, f)$, we just assume that $m(r, b_j) = S(r, f)$ for the co-efficients $b_j (j = 1, 2, ..., l)$.

In 1959, Hayman [6] discussed the Picard's value of meromorphic functions and their derivatives and he obtained the following well-known result.

Theorem A. [6] *Let $f(z)$ be a transcendental entire function. Then*
(a) for $n \geq 2$, $f'(z)f(z)^n$ assumes all finite values except possibly zero infinitely often;
(b) for $n \geq 3$ and $a \neq 0$, $f'(z) - af(z)^n$ assumes all finite values infinitely often.

Later in 1995, Chen and Fang [2] obtained the following result for transcendental meromorphic function.

Theorem B. [2] *Let $f(z)$ be a transcendental meromorphic function. If $n \geq 1$, is a positive integer, then $f'(z)f(z)^n - 1$ has infinitely many zeros.*

Around 2006, Halburd and Korhonen established the difference analogies of the Nevanlinna Theory (see [4], [5]). Since then the study of difference analogies became a subject of great interest for many mathematicians. In 2012, Xu and Zhang [11] studied the zeros of q-shift difference polynomials of meromorphic functions of finite logarithmic order and gave the following result.

Theorem C. [11] *If $f(z)$ is a transcendental meromorphic function of finite logarithmic order $\rho_{log}(f)$ with the logarithmic exponent of convergence of poles less than $\rho_{log}(f) - 1$ and q, c are non-zero complex constants, then for $n \geq 2$, $f(z)^n f(qz + c)$ assumes every value $b \in \mathbb{C}$ infinitely often.*

In 2014, Zheng and Xu [15] investigated the zeros of differential-q-shift-difference polynomials about $f(z)$, $f'(z)$, and $f(qz + c)$, where $f(z)$ is of finite positive logarithmic order and obtained the following results.

Theorem D. [15] *Let $f(z)$ be a transcendental meromorphic function of finite logarithmic order $\rho_{log}(f)$, with the logarithmic exponent of convergence of poles less than $\rho_{log}(f) - 1$. Set $F_1(z) = f(qz + c)^n f'(z)$. If $n \geq 3$, then $F_1(z) - a(z)$ has infinitely many zeros.*

Theorem E. [15] *Let $f(z)$ be a transcendental meromorphic function of finite logarithmic order $\rho_{log}(f)$, with the logarithmic exponent of convergence of poles less than $\rho_{log}(f) - 1$. Set $F_3(z) = f(z)^m f(qz+c)^n f'(z)$. If m, n satisfy $m \geq n+2$ or $n \geq m + 2$, then $F_3(z) - a(z)$ has infinitely many zeros.*

Let $P_n(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$, be a non-zero polynomial, where $a_0, a_1, ..., a_n (\neq 0)$ are complex constants and t_n be the number of the distinct zeros of $P_n(z)$. Then

Theorem F. [15] *Let $f(z)$ be a transcendental meromorphic function of finite logarithmic order $\rho_{log}(f)$, with the logarithmic exponent of convergence of poles less than $\rho_{log}(f) - 1$. Set $F_4(z) = f(z)^m P_n(f(qz + c)) \prod_{j=1}^{k} f^{(j)}(z)$. If $m \geq n + k + 1$, then $F_4(z) - a(z)$ has infinitely many zeros.*

Theorem G. [15] *Let $f(z)$ be a transcendental meromorphic function of finite logarithmic order $\rho_{log}(f)$, with the logarithmic exponent of convergence of poles less* *than $\rho_{log}(f)-1$. Set $F_5(z)=P_m(f(z))f(qz+c)^n\prod_{j=1}^{k}f^{(j)}(z)$. If $m\geq n+k+1$, then $F_5(z)-a(z)$ has infinitely many zeros.*

Zheng and Xu [15], further studied the uniqueness of differential q-shift difference polynomials of entire functions of order zero and gave the following results.

Theorem H. [15] *Let $f(z)$ and $g(z)$ be transcendental entire functions of order zero and $n\geq 5$. If $f(qz+c)^n f'(z)$ and $g(qz+c)^n g'(z)$ share a non-zero polynomial $p(z)$ CM, then $f(qz+c)^n f'(z)=g(qz+c)^n g'(z)$.*

Theorem I. [15] *Let $f(z)$ and $g(z)$ be transcendental entire functions of order zero and $m\geq n+2t_n+5$. If $f(z)^m P_n(f(qz+c))f'(z)$ and $g(z)^m P_n(g(qz+c))g'(z)$ share a non-zero polynomial $p(z)$ CM, then $f(z)^m P_n(f(qz+c))f'(z)=g(z)^m P_n(g(qz+c))g'(z)$.*

Theorem J. [15] *Let $f(z)$ and $g(z)$ be transcendental entire functions of order zero and $n\geq m+2t_m+5$. If $P_m(f(z))f(qz+c)^n f'(z)$ and $P_m(g(z))g(qz+c)^n g'(z)$ share a non-zero polynomial $p(z)$ CM, then $P_m(f(z))f(qz+c)^n f'(z)=P_m(g(z))g(qz+c)^n g'(z)$.*

The motivation to this paper is [10], where Thin, states that the inequality $\overline{N}(r,P(f))\leq mT(r,f)+S(r,f)$ (where $P(z)$ is a polynomial with m distinct zeros and $f(z)$ is a transcendental meromorphic function), is very weak and indeed we have the equality $\overline{N}(r,P(f))=\overline{N}(r,f)$. Thus, we can easily get, $\overline{N}(r,H[f])=\overline{N}(r,f)$, where $H[f]$ is a differential polynomial generated by a transcendental meromorphic function f.

In this paper, we extend the above theorems H-J from entire functions to meromorphic functions and also extend the differential monomials $f'(z)$ and $g'(z)$ in theorems D-J to differential polynomials $H[f]$ and $H[g]$ respectively and we obtain the following generalised results.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function of finite logarithmic order $\rho_{log}(f)$, with the logarithmic exponent of convergence of poles less than $\rho_{log}(f)-1$ and $a(z)$ be a small function with respect to $f(z)$. Set $\mathcal{F}_1(z)=f^n(qz+c)H[f]$. If $n\geq\gamma_p+\sigma+1$, then $\mathcal{F}_1(z)-a(z)$ has infinitely many zeros.*

Remark 1.1. *In Theorem 1.1, if $H[f]=f'(z)$, then we get $(\gamma_p=\gamma_{M_1}=1)$, $(\Gamma_p=\Gamma_{M_1}=2)$ and $(\sigma=\Gamma_{M_1}-\gamma_{M_1}=1)$, thus $n\geq 3$ and hence Theorem 1.1 reduces to Theorem D.*

Theorem 1.2. *Let $f(z)$ be a transcendental meromorphic function of finite logarithmic order $\rho_{log}(f)$, with the logarithmic exponent of convergence of poles less than $\rho_{log}(f)-1$ and $a(z)$ be a small function with respect to $f(z)$. Set $\mathcal{F}_2(z)=f(z)^m f^n(qz+c)H[f]$. If m, n satisfy $m\geq n+\gamma_p+\sigma$ (or) $n\geq m+\gamma_p+\sigma$, then $\mathcal{F}_2(z)-a(z)$ has infinitely many zeros.*

Remark 1.2. *In Theorem 1.2, if $H[f]=f'(z)$, then we get, $(\gamma_p=\gamma_{M_1}=1)$, $(\Gamma_p=\Gamma_{M_1}=2)$ and $(\sigma=\Gamma_{M_1}-\gamma_{M_1}=1)$, thus $m\geq n+2$ or $n\geq m+2$ and hence Theorem 1.2 reduces to Theorem E.*

Theorem 1.3. *Let $f(z)$ be a transcendental meromorphic function of finite logarithmic order $\rho_{log}(f)$, with the logarithmic exponent of convergence of poles less* *than $\rho_{log}(f)-1$, $a(z)$ be a small function with respect to $f(z)$ and $P_n(z)=a_nz^n+a_{n-1}z^{n-1}+...+a_1z+a_0$, where $a_0,a_1,...,a_n(\neq 0)$ are complex constants, be a polynomial of degree n and t_m distinct zeros. Set $\mathcal{F}_3(z)=f(z)^mP_n(f(qz+c))H[f]$. If $m\geq n+\gamma_p+1$, then $\mathcal{F}_3(z)-a(z)$ has infinitely many zeros.*

Remark 1.3. *In Theorem 1.3, if $H[f]=\prod_{j=1}^{k}f^{(j)}(z)$, then we get $(\gamma_p=\gamma_{M_1}=k)$, $(\Gamma_p=\Gamma_{M_1}=\frac{k(k+1)}{2}+k)$, and $(\sigma=\Gamma_{M_1}-\gamma_{M_1}=\frac{k(k+1)}{2})$, thus $n\geq m+k+1$ and hence Theorem 1.3 reduces to Theorem F.*

Theorem 1.4. *Let $f(z)$ be a transcendental meromorphic function of finite logarithmic order $\rho_{log}(f)$, with the logarithmic exponent of convergence of poles less than $\rho_{log}(f)-1$, $a(z)$ be a small function with respect to $f(z)$ and $P_n(z)=a_nz^n+a_{n-1}z^{n-1}+...+a_1z+a_0$, where $a_0,a_1,...,a_n(\neq 0)$ are complex constants, be a polynomial of degree n and t_m distinct zeros. Set $\mathcal{F}_4(z)=P_m(f(z))f^n(qz+c)H[f]$. If $m\geq n+\gamma_p+1$, then $\mathcal{F}_4(z)-a(z)$ has infinitely many zeros.*

Remark 1.4. *In Theorem 1.4, if $H[f]=\prod_{j=1}^{k}f^{(j)}(z)$, then we get $(\gamma_p=\gamma_{M_1}=k)$, $(\Gamma_p=\Gamma_{M_1}=\frac{k(k+1)}{2}+k)$, and $(\sigma=\Gamma_{M_1}-\gamma_{M_1}=\frac{k(k+1)}{2})$, thus $n\geq m+k+1$ and hence Theorem 1.4 reduces to Theorem G.*

Theorem 1.5. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of order zero and n be a positive integer. If $f^n(qz+c)H[f]$ and $g^n(qz+c)H[g]$ share a non-zero polynomial $q(z)$, ∞ CM and $n\geq 3\gamma_p+3\sigma+5$, then $f^n(qz+c)H[f]=g^n(qz+c)H[g]$.*

Corollary 1.5. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of order zero and n be a positive integer. If $f^n(qz+c)H[f]$ and $g^n(qz+c)H[g]$ share a non-zero polynomial $q(z)$ CM and $n\geq 3\gamma_p+3\sigma+3$, then $f^n(qz+c)H[f]=g^n(qz+c)H[g]$.*

Theorem 1.6. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of order zero, n be a positive integer, $P_m(z)$ be a polynomial of degree m and t_m distinct zeros. If $f^n(z)P_m(f(qz+c))H[f]$ and $g^n(z)P_m(g(qz+c))H[g]$ share a non-zero polynomial $q(z)$, ∞ CM and $n\geq 2t_m+m+3\gamma_p+3\sigma+5$, then $f^n(z)P_m(f(qz+c))H[f]=g^n(z)P_m(g(qz+c))H[g]$.*

Corollary 1.6. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of order zero, n be a positive integer, $P_m(z)$ be a polynomial of degree m and t_m distinct zeros. If $f^n(z)P_m(f(qz+c))H[f]$ and $g^n(z)P_m(g(qz+c))H[g]$ share a non-zero polynomial $q(z)$ CM and $n\geq 2t_m+m+3\gamma_p+3$, then $f^n(z)P_m(f(qz+c))H[f]=g^n(z)P_m(g(qz+c))H[g]$.*

Theorem 1.7. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of order zero, n be a positive integer, $P_m(z)$ be a polynomial of degree m and t_m distinct zeros. If $f^n(qz+c)P_m(f(z))H[f]$ and $g^n(qz+c)P_m(g(z))H[g]$ share a non-zero polynomial $q(z)$, ∞ CM and $n\geq 2t_m+m+3\gamma_p+3\sigma+5$, then $f^n(qz+c)P_m(f(z))H[f]=g^n(qz+c)P_m(g(z))H[g]$.*

Corollary 1.7. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of order zero, n be a positive integer, $P_m(z)$ be a polynomial of degree m and t_m distinct zeros. If $f^n(qz+c)P_m(f(z))H[f]$ and $g^n(qz+c)P_m(g(z))H[g]$ share a non-zero polynomial $q(z)$ CM and $n\geq 2t_m+m+3\gamma_p+3$, then $f^n(qz+c)P_m(f(z))H[f]=g^n(qz+c)P_m(g(z))H[g]$.*

2. **Lemmas**

This section provides all the necessary lemmas used in the sequel.

Lemma 2.1. [13] *Let f be a non-constant meromorphic function and let $a_1, a_2, ..., a_n$ be finite complex numbers, $a_n \neq 0$. Then*

$$T(r, a_n f^n + ... + a_2 f^2 + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2. [11] *Let $f(z)$ be a transcendental meromorphic function of finite logarithmic order and q, c be two non-zero complex constants. Then*

$$T(r, f(qz + c)) = T(r, f) + S_1(r, f),$$
$$N(r, f(qz + c)) = N(r, f) + S_1(r, f),$$
$$N\left(r, \frac{1}{f(qz + c)}\right) = N\left(r, \frac{1}{f}\right) + S_1(r, f).$$

Lemma 2.3. [9] *Let $f(z)$ be a non-constant zero order meromorphic function and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$m\left(r, \frac{f(qz + c)}{f(z)}\right) = S_1(r, f).$$

Lemma 2.4. [13] *Let $f(z)$ be a non-constant meromorphic function in the complex plane and k be a positive integer. Then*

$$T(r, f^{(k)}) \leq T(r, f) + k\overline{N}(r, f) + S(r, f),$$
$$N(r, f^{(k)}) \leq N(r, f) + k\overline{N}(r, f).$$

Lemma 2.5. [1] *Let f be a non constant meromorphic function and $H[f]$ be a differential polynomial in f. Then*

$$m\left(r, \frac{H[f]}{f^{\gamma_p}}\right) \leq (\gamma_p - \underline{\gamma}_p) m\left(r, \frac{1}{f}\right) + S(r, f),$$
$$m\left(r, \frac{H[f]}{f^{\underline{\gamma}_p}}\right) \leq (\gamma_p - \underline{\gamma}_p) m\left(r, f\right) + S(r, f),$$
$$N\left(r, \frac{H[f]}{f^{\gamma_p}}\right) \leq (\gamma_p - \underline{\gamma}_p) N\left(r, \frac{1}{f}\right) + \sigma\left[\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right)\right] + S(r, f),$$
$$N(r, H[f]) \leq \gamma_p N(r, f) + \sigma\overline{N}(r, f) + S(r, f),$$
$$T(r, H[f]) \leq \gamma_p T(r, f) + \sigma\overline{N}(r, f) + S(r, f),$$

where σ=max $\{n_{1j} + 2n_{2j} + 3n_{3j} + ... + kn_{kj}\,; 1 \leq j \leq l\}$.

Lemma 2.6. [3] *If $f(z)$ is a transcendental meromorphic function of finite logarithmic order $\rho_{log}(f)$, then for any two distinct small functions $a(z)$ and $b(z)$ with respect to $f(z)$, we have*

$$T(r, f) \leq N\left(r, \frac{1}{f - a}\right) + N\left(r, \frac{1}{f - b}\right) o(U(r, f)),$$

where $U(r, f) = (log\, r)^{\rho_{log}(f)}$ is a logarithmic-type function of $T(r, f)$. Further, if $T(r, f)$ has a finite lower logarithmic order

$$\mu = \lim_{r\to\infty} \frac{log\, T(r, f)}{log\, log\, r},$$

with $\rho_{log}(f) - \mu < 1$*, then*

$$T(r,f) \le N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) o(T(r,f)).$$

Remark 2.1. *Here the complex values* a *and* b *can be easily changed into* $a(z)$ *and* $b(z)$*, where* $a(z)$ *and* $b(z)$ *are two distinct small functions with respect to* $f(z)$*.*

Lemma 2.7. *Let* $f(z)$ *be a transcendental meromorphic function of order zero. Set* $\mathcal{F}_1 = f^n(qz+c)H[f]$*. Then, we have*

$$(n-\gamma_p-\sigma)T(r,f) + S_1(r,f) \le T(r,\mathcal{F}_1) \le (n+\gamma_p+\sigma)T(r,f) + S_1(r,f). \quad (1)$$

Proof. If $f(z)$ is a meromorphic function of order zero, then from Lemmas 2.1, 2.2 and 2.5, we have

$$T(r,\mathcal{F}_1) \le nT(r,f(qz+c)) + T(r,H[f]) \le (n+\gamma_p+\sigma)T(r,f) + S_1(r,f).$$

Once again from Lemmas 2.1, 2.2 and 2.5, we have

$$\begin{aligned}(n+1)T(r,f) &= T(r,f^{n+1}(qz+c)) + S_1(r,f)\\ &\le T(r,\mathcal{F}_1) + T\left(r, \frac{f(qz+c)}{H[f]}\right) + S_1(r,f)\\ &\le T(r,\mathcal{F}_1) + T(r,f) + \gamma_p T(r,f) + \sigma\overline{N}(r,f) + S_1(r,f)\\ &\le T(r,\mathcal{F}_1) + (\gamma_p+\sigma+1)T(r,f) + S_1(r,f).\end{aligned}$$

Thus, we get (1). This completes the proof of Lemma 2.7. □

Lemma 2.8. *Let* $f(z)$ *be a transcendental meromorphic function of order zero. Set* $\mathcal{F}_2 = f^m(z)f^n(qz+c)H[f]$*. Then, we have*

$$T(r,\mathcal{F}_2) \le (m+n+\gamma_p+\sigma)T(r,f) + S_1(r,f) \quad (2)$$

and

$$(|m-n|-\gamma_p-\sigma)T(r,f) + S_1(r,f) \le T(r,\mathcal{F}_2). \quad (3)$$

Proof. If $f(z)$ is a meromorphic function of order zero, then from Lemmas 2.1, 2.2 and 2.5, we have

$$T(r,\mathcal{F}_2) \le mT(r,f)+nT(r,f(qz+c))+T(r,H[f]) \le (m+n+\gamma_p+\sigma)T(r,f)+S_1(r,f).$$

Thus, we have (2). Once again from Lemmas 2.1, 2.2 and 2.5, we have

$$\begin{aligned}(n+m+1)T(r,f) &= T(r,f^{n+m+1}) = T\left(r, \frac{f^{n+1}(z)\mathcal{F}_2}{f^n(qz+c)H[f]}\right)\\ &\le T(r,\mathcal{F}_2) + T(r,f^{n+1}(z)) + T(r,f^n(qz+c))\\ &\quad + T(r,H[f]) + S_1(r,f)\\ &\le T(r,\mathcal{F}_2) + (2n+\gamma_p+\sigma+1)T(r,f) + S_1(r,f).\end{aligned}$$

Thus, we have (3), where we assume $m \ge n$ without the loss of generality.
This completes the proof of Lemma 2.8. □

Lemma 2.9. *Let* $f(z)$ *be a transcendental meromorphic function of order zero. Set* $\mathcal{F}_3 = f^m(z)P_n(f(qz+c))H[f]$*. Then, we have*

$$(m-n-\gamma_p)T(r,f) \le T(r,\mathcal{F}_3) + \sigma\overline{N}(r,f) + S_1(r,f) \quad (4)$$

and

$$T(r,\mathcal{F}_3) \le (m+n+\gamma_p+\sigma)T(r,f) + S_1(r,f). \quad (5)$$

Proof. If $f(z)$ is a transcendental meromorphic function of order zero, then from Lemmas 2.1, 2.2 and 2.5, we have

$$T(r,\mathcal{F}_3) \le mT(r,f)+nT(r,f(qz+c))+T(r,H[f]) \le (m+n+\gamma_p+\sigma)T(r,f)+S_1(r,f).$$

Thus we have (5). Once again from Lemmas 2.1, 2.2 and 2.5, we have

$$\begin{aligned}(m+k)T(r,f) = T(r,f^{m+k}) &\le T\left(r,\frac{f^k(z)\mathcal{F}_3}{P_n(f(qz+c))H[f]}\right)\\ &\le T(r,\mathcal{F}_3)+T(r,P_n(f(qz+c)))+T(r,f^k(z))+T(r,H[f])\\ &\le T(r,\mathcal{F}_3)+(n+k+\gamma_p)T(r,f)+\sigma\overline{N}(r,f)+S_1(r,f).\end{aligned}$$

This completes the proof of Lemma 2.9. □

Remark 2.2. *In Lemma 2.9, if* $H[f]=\prod_{j=1}^{k} f^{(j)}(z)$ *then, we get* $(\gamma_p=\gamma_{M_1}=k)$, $(\Gamma_p=\Gamma_{M_1}=\frac{k(k+1)}{2}+k)$ *and* $(\sigma=\Gamma_{M_1}-\gamma_{M_1}=\frac{k(k+1)}{2})$ *then Lemma 2.9. reduces to Lemma 2.8 in* [15].

Lemma 2.10. *Let* $f(z)$ *be a transcendental meromorphic function of order zero. Set* $\mathcal{F}_4=P_m(f(z))f^n(qz+c)H[f]$. *Then, we have*

$$(n-m-\gamma_p)T(r,f) \le T(r,\mathcal{F}_4)+\sigma\overline{N}(r,f)+S_1(r,f), \tag{6}$$

$$T(r,\mathcal{F}_4) \le (m+n+\gamma_p+\sigma)T(r,f)+S_1(r,f). \tag{7}$$

Proof. Lemma 2.10 can be proved in a similar fashion to Lemma 2.9. □

Remark 2.3. *In Lemma 2.10 if* $H[f]=\prod_{j=1}^{k} f^{(j)}(z)$ *then, we get* $(\gamma_p=\gamma_{M_1}=k)$, $(\Gamma_p=\Gamma_{M_1}=\frac{k(k+1)}{2}+k)$ *and* $(\sigma=\Gamma_{M_1}-\gamma_{M_1}=\frac{k(k+1)}{2})$ *then Lemma 2.10 reduces to Lemma 2.9 in* [15].

3. Proof of Theorems

3.1. Proof of Theorem 1.1.

Proof. From Lemma 2.7, we can conclude that $T(r,\mathcal{F}_1)=O(T(r,f))$ holds for all r on a set of logarithmic density 1. Since $f(z)$ is transcendental and $n \ge \gamma_p+\sigma+1$, from Lemma 2.7, $\mathcal{F}_1$ is transcendental. Since the logarithmic exponent of convergence of poles of $f(z)$ less than $\rho_{log}(f)-1$, we have

$$\limsup_{r\to\infty}\frac{log\,N(r,f)}{log\,log\,r} < \rho_{log}(f).$$

Assume that $\mathcal{F}_1(z)-a(z)$ has only finitely many zeros. Then, from Lemmas 2.2, 2.5, 2.6 and 2.7, we have

$$\begin{aligned}(n-\gamma_p-\sigma)T(r,f) &\le T(r,\mathcal{F}_1)+S_1(r,f)\\ &\le N(r,\mathcal{F}_1)+N\left(r,\frac{1}{\mathcal{F}_1-a}\right)+o(U(r,f))+S_1(r,f)\\ &\le N(r,f^n(qz+c))+N(r,H[f])+N\left(r,\frac{1}{\mathcal{F}_1-a}\right)\\ &\quad+o(U(r,f))+S_1(r,f).\end{aligned}$$

Since $\mathcal{F}_1(z) - a(z)$ has finitely many zeros, hence $N\left(r, \frac{1}{\mathcal{F}_1 - a}\right) = S_1(r, \mathcal{F}_1) = S_1(r, f)$, and hence, the above inequality reduces to,

$$(n - \gamma_p - \sigma)T(r, f) \leq (n + \gamma_p + \sigma)N(r, f) + o(U(r, f)) + S_1(r, f).$$

Since, $n \geq \gamma_p + \sigma + 1$, from the above inequality, we get

$$\limsup_{r\to\infty} \frac{log\, T(r, f)}{log\, log\, r} \leq \limsup_{r\to\infty} \frac{log\, N(r, f)}{log\, log\, r} \leq \rho_{log}(f),$$

which contradicts the fact that $T(r, f)$ has finite logarithmic order $\rho_{log}(f)$. Thus, $\mathcal{F}_1(z) - a(z)$ has infinitely many zeros. i.e., $f^n(qz + c)H[f] - a(z)$ has infinitely many zeros. □

3.2. **Proofs of Theorems 1.2 - 1.4.** Theorems 1.2, 1.3, and 1.4 can be proved easily by using a similar argument of that of Theorem 1.1, by applying the Lemmas 2.8, 2.9 and 2.10 respectively.

3.3. **Proofs of Theorems 1.5 - 1.7.** Here, we only give the proof of Theorem 1.6, because the method of proof of Theorem 1.5, 1.6 and 1.7 are very similar. Their corollaries can also be proved in a similar fashion by taking $N(r, f) = S(r, f)$.

3.4. **Proof of Theorem 1.6.**

Proof. Let us consider,

$$F(z) = f^n(z)P_m(f(qz + c))H[f] \quad \text{and} \quad G(z) = g^n(z)P_m(g(qz + c))H[g]. \tag{8}$$

Now,

$$\begin{aligned} T(r, F(z)) &= T(r, f^n(z)P_m(f(qz + c))H[f]) \\ &\leq T(r, f^n(z)) + T(r, P_m(f(qz + c))) + T(r, H[f]) + S_1(r, f). \end{aligned} \tag{9}$$

From Lemmas 2.1, 2.2 and 2.5, we get

$$T(r, F(z)) \leq nT(r, f) + mT(r, f) + \gamma_p T(r, f) + \sigma\overline{N}(r, f) + S_1(r, f). \tag{10}$$

Therefore,

$$T(r, F(z)) \leq (n + m + \gamma_p + \sigma)T(r, f) + S_1(r, f). \tag{11}$$

Once again from Lemmas 2.1, 2.2, 2.5 and the First fundamental theorem,

$$\begin{aligned} (n + k)T(r, f) &= T(r, f^{n+k}) \\ &= T\left(r, \frac{f^k \cdot F(z)}{P_m(f(qz + c)) \cdot H[f]}\right) \\ &\leq T(r, F(z)) + T(r, P_m(f(qz + c))) + T(r, f^k) + T(r, H[f]) \\ &\quad + S_1(r, f) \\ &\leq T(r, F(z)) + mT(r, f) + kT(r, f) + \gamma_p T(r, f) + \sigma\overline{N}(r, f) \\ &\quad + S_1(r, f) \end{aligned}$$

$$(n + k)T(r, f) \leq T(r, F(z)) + (m + k + \gamma_p + \sigma)T(r, f) + S_1(r, f). \tag{12}$$

Therefore,

$$(n - m - \gamma_p - \sigma)T(r, f) + S_1(r, f) \leq T(r, F(z)). \tag{13}$$

From (11) and (13), we get

$$(n - m - \gamma_p - \sigma)T(r, f) + S_1(r, f) \leq T(r, F(z)) \leq (n + m + \gamma_p + \sigma)T(r, f) + S_1(r, f). \tag{14}$$

From (14), we have $S_1(r,F) = S_1(r,f)$. Similarly, we have $S_1(r,G) = S_1(r,g)$ and

$$(n-m-\gamma_p-\sigma)T(r,g)+S_1(r,g) \leq T(r,G(z)) \leq (n+m+\gamma_p+\sigma)T(r,g)+S_1(r,g). \tag{15}$$

Since $f(z)$ and $g(z)$ are transcendental meromorphic functions of zero order and share $q(z)$, ∞ CM, we have

$$\frac{F(z)/q(z)-1}{G(z)/q(z)-1} = \beta, \tag{16}$$

where β is a non-zero constant.
Case 1. If $\beta = 1$, then we have $F(z) = G(z)$, which implies

$$f^n(z)P_m(f(qz+c))H[f] = g^n(z)P_m(g(qz+c))H[g].$$

Case 2. If $\beta \neq 1$, then we have

$$F(z) - q(z) = \beta G(z) - \beta q(z) \tag{17}$$

$$F(z) - (1-\beta)q(z) = \beta G(z) \tag{18}$$

Since $P_m(z)$ has t_m distinct zeros, hence by using Second fundamental theorem, we have

$$\begin{aligned}
T(r,F(z)) &\leq \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-q(z)(1-\beta)}\right) + S_1(r,F)\\
&\leq \overline{N}(r,f) + \overline{N}(r,f(qz+c)) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{P_m(f(qz+c))}\right)\\
&\quad + \overline{N}\left(r,\frac{1}{H[f]}\right) + \overline{N}\left(r,\frac{1}{G}\right) + S_1(r,f)\\
&\leq \overline{N}(r,f) + \overline{N}(r,f(qz+c)) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{P_m(f(qz+c))}\right)\\
&\quad + \overline{N}\left(r,\frac{1}{H[f]}\right) + \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{P_m(g(qz+c))}\right) + \overline{N}\left(r,\frac{1}{H[g]}\right)\\
&\quad + S_1(r,f) + S_1(r,g)\\
&\leq \overline{N}(r,f) + \overline{N}(r,f(qz+c)) + \overline{N}\left(r,\frac{1}{f}\right) + \sum_{i=1}^{t_m}\overline{N}\left(r,\frac{1}{f(qz+c)-a_i}\right)\\
&\quad + T(r,H[f]) + \overline{N}\left(r,\frac{1}{g}\right) + \sum_{i=1}^{t_m}\overline{N}\left(r,\frac{1}{g(qz+c)-a_i}\right) + T(r,H[g])\\
&\quad + S_1(r,f) + S_1(r,g).
\end{aligned}$$

Hence,

$$T(r,F(z)) \leq (t_m+\gamma_p+\sigma+3)T(r,f)+(t_m+\gamma_p+\sigma+1)T(r,g)+S_1(r,f)+S_1(r,g), \tag{19}$$

where $a_1, a_2, ..., a_{t_m}$ are the distinct zeros of $P_m(z)$.
Similarly, we have

$$T(r,G(z)) \leq (t_m+\gamma_p+\sigma+3)T(r,g)+(t_m+\gamma_p+\sigma+1)T(r,f)+S_1(r,f)+S_1(r,g). \tag{20}$$

From (14), (15), (19) and (20), we get

$$(n-m-\gamma_p-\sigma)\{T(r,f)+T(r,g)\} \leq (2t_m+2\gamma_p+2\sigma+4)\{T(r,f)+T(r,g)\} + S_1(r,f)+S_1(r.g), \tag{21}$$

which contradicts with $n \geq 2t_m+m+3\gamma_p+3\sigma+5$.
Hence $\beta=1$, which implies, $f^n(z)P_m(f(qz+c))H[f] = g^n(z)P_m(g(qz+c))H[g]$.
This completes the proof of Theorem 1.6. □

References

[1] S. S. Bhoosnurmath and A. J. Patil, On the growth and value distribution of meromorphic functions and their differential polynomials, J. Indian Math. Soc. (N.S.) **74** (2007), no. 3-4, 167–184 (2008).

[2] H. H. Chen and M. L. Fang, The value distribution of $f^n f'$, Sci. China Ser. A **38** (1995), no. 7, 789–798.

[3] P. T.-Y. Chern, On meromorphic functions with finite logarithmic order, Trans. Amer. Math. Soc. **358** (2006), no. 2, 473–489.

[4] R. G. Halburd and R. J. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, J. Math. Anal. Appl. **314** (2006), no. 2, 477–487.

[5] R. G. Halburd and R. J. Korhonen, Nevanlinna theory for the difference operator, Ann. Acad. Sci. Fenn. Math. **31** (2006), no. 2, 463–478.

[6] W. K. Hayman, Picard values of meromorphic functions and their derivatives, Ann. of Math. (2) **70** (1959), 9–42.

[7] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[8] I. Lahiri, Value distribution of certain differential polynomials, Int. J. Math. Math. Sci. **28** (2001), no. 2, 83–91.

[9] K. Liu and X.-G. Qi, Meromorphic solutions of q-shift difference equations, Ann. Polon. Math. **101** (2011), no. 3, 215–225.

[10] N. V. Thin, Uniqueness of meromorphic functions and Q-difference polynomials sharing small functions, Bull. Iranian Math. Soc. **43** (2017), no. 3, 629–647.

[11] J. Xu and X. Zhang, The zeros of q-shift difference polynomials of meromorphic functions, Adv. Difference Equ. **2012**, 2012:200, 10 pp.

[12] C. C. Yang, On deficiencies of differential polynomials. II, Math. Z. **125** (1972), 107–112.

[13] C.-C. Yang and H.-X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, 557, Kluwer Academic Publishers Group, Dordrecht, 2003.

[14] L. Yang, *Value distribution theory*, translated and revised from the 1982 Chinese original, Springer-Verlag, Berlin, 1993.

[15] X.-M. Zheng and H.-Y. Xu, On value distribution and uniqueness of meromorphic function with finite logarithmic order concerning its derivative and q-shift difference, J. Inequal. Appl. **2014**, 2014:295, 11 pp.

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