

**NONTRIVIAL SOLUTIONS FOR A NONLINEAR ν TH ORDER
ATICI-ELOE FRACTIONAL DIFFERENCE EQUATION
SATISFYING LEFT FOCAL BOUNDARY CONDITIONS**

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Dedicated to my friend and collaborator, Professor Rodica Luca.

ABSTRACT. For $1 < \nu \leq 2$ a real number and $T \geq 3$ a natural number, nontrivial solutions are established via a Krasnosel'skii-Zabreiko fixed point theorem for a nonlinear ν th order Atici-Eloe fractional difference equation, $\Delta^\nu u(t) + f(u(t + \nu - 1)) = 0$, $t \in \{0, 1, \dots, T\}$, and satisfying the left focal boundary conditions $\Delta u(\nu - 2) = u(\nu + T) = 0$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $\lim_{|r| \rightarrow \infty} \frac{f(r)}{r}$ exists.

1. INTRODUCTION

In this paper, for $1 < \nu \leq 2$ a real number and $T \geq 3$ a natural number, we are concerned with the existence of nontrivial solutions of the nonlinear ν th order Atici-Eloe fractional difference equation,

$$\Delta^\nu u(t) + f(u(t + \nu - 1)) = 0, \quad t \in \{0, 1, \dots, T\}, \quad (1)$$

satisfying the left focal boundary conditions

$$\Delta u(\nu - 2) = 0, \quad u(\nu + T) = 0, \quad (2)$$

where Δ is the forward difference operator with step-size 1, Δ^ν is the Atici-Eloe fractional difference and $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

Interest in fractional difference equations devoted to both their theoretical development and their applications is currently quite high, with much of this interest spawned by the definitions, in the context of discrete domains, of fractional sums and fractional differences in the pioneering papers by Atici and Eloe [1, 2]. Further development and extensions were done in the seminal papers by Goodrich [8, 9, 10]. Atici and Eloe dealt first with a theory for initial value problems for fractional difference equations, followed in their second paper by applications of their definitions in obtaining positive solutions for Dirichlet boundary value problems for

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fractional difference equations. Their second work centered on a Guo-Krasnosel'skii fixed point argument which required the construction of a Green's function for their fractional problem. Other subsequent research by Goodrich that focused on questions expanding the Atıcı and Eloe work can be found in [11, 12, 13, 14, 15]. A couple of recent papers by Henderson [17] and Henderson and Neugebauer [20] were devoted to the existence of local solutions for boundary value problems for Atıcı-Eloe fractional difference equations, and other recent works devoted to Atıcı-Eloe difference equations can be found in, to cite a few, [6, 19, 29].

Discrete fractional calculus and fractional difference equations frequently appear in modeling natural processes such as found in the paper by Atıcı and S. Şengül [3] and in the papers by Magin [23] and Metzler *et al.* [24]. Especially prominent is the current use of boundary value problems for discrete fractional difference equations in their applications to discrete control processes; see, for example, the monographs devoted to discrete fractional control [4, 5, 25, 26, 27].

In this paper, we apply a Krasnosel'skii and Zabreiko fixed point theorem [22] in establishing the existence of nontrivial solutions of (1), (2). Effective use has been made of the Krasnosel'skii and Zabreiko fixed point theorem in showing existence of solutions of boundary value problems (for ordinary differential equations, for difference equations and for dynamic equations) in the context of when the nonlinearity is almost linear at infinity; see, for example [7, 16, 18, 21, 28].

2. SOME PRELIMINARIES AND THE GREEN'S FUNCTION

We begin this section with the Atıcı-Eloe definitions of fractional sum and fractional difference in the context of a discrete domain.

Definition 2.1. *Let $n \in \mathbb{N}$ and $n - 1 < \kappa \leq n$ be a real number, and let $a \in \mathbb{R}$. For $t \in \{a + \kappa, a + \kappa + 1, \dots\}$, the κ th order **Atıcı-Eloe fractional sum**, $\Delta^{-\kappa}u$, of the function u is defined by*

$$\Delta^{-\kappa}u(t) := \frac{1}{\Gamma(\kappa)} \sum_{s=a}^{t-\kappa} (t-s-1)^{(\kappa-1)} u(s),$$

where $t^{(\kappa)} := \frac{\Gamma(t+1)}{\Gamma(t+1-\kappa)}$ is the falling function.

The κ th order **Atıcı-Eloe fractional difference**, $\Delta^\kappa u$, of the function u is defined by

$$\Delta^\kappa u(t) := \Delta^{n-(n-\kappa)} u(t) := \Delta^n (\Delta^{-(n-\kappa)} u(t)),$$

where Δ is the forward difference defined by $\Delta u(t) = u(t+1) - u(t)$, and $\Delta^i u(t) = \Delta(\Delta^{i-1} u(t))$, $i = 2, 3, \dots$

Remark 2.2. *We note that, for u defined on $\{a, a + 1, \dots\}$ $\Delta^{-\kappa}u$ is defined on $\{a + \kappa, a + \kappa + 1, \dots\}$.*

In [10], for $1 < \nu \leq 2$, Goodrich constructed the Green's function, $G(t, s)$, for

$$-\Delta^\nu y(t) = 0, \quad t \in \{0, 1, \dots, T\}, \tag{3}$$

satisfying the left focal boundary conditions (2). In particular, by direct computation, Goodrich obtained

$$G(t, s) = \frac{c(\nu, T)}{\Gamma(\nu)} \begin{cases} \Gamma(\nu)(\nu + T - s - 1)^{(\nu-1)}, \\ \quad t = \nu - 2, \nu - 1, \\ [(2 - \nu)t^{(\nu-1)} + (\nu - 1)^{(\nu-2)}] (\nu + T - s - 1)^{(\nu-1)}, \\ \quad 0 \leq t - \nu \leq s \leq T, \\ [(2 - \nu)t^{(\nu-1)} + (\nu - 1)^{(\nu-2)}] (\nu + T - s - 1)^{(\nu-1)} \\ \quad - \frac{1}{c(\nu, T)} (t - s - 1)^{(\nu-2)}, \\ \quad 0 \leq s < t - \nu \leq T, \end{cases} \quad (4)$$

with

$$c(\nu, T) = \frac{1}{(\nu + T)^{(\nu-2)} [(\nu - 1) + (2 - \nu)(T + 2)]}.$$

Goodrich also obtained the following properties of $G(t, s)$ which will be of importance to us:

(a) For each $s \in \{0, \dots, T\}$,

$$\Delta_t G(\nu - 2, s) = 0 \text{ and } G(\nu + T, s) = 0.$$

(b) $G(t, s) > 0$, for $(t, s) \in \{\nu - 2, \dots, \nu + T - 1\} \times \{0, \dots, T\}$.

(c) $\max_{t \in \{\nu-2, \dots, \nu+T\}} G(t, s) = G(s + \nu - 1, s)$, for $s \in \{0, \dots, T\}$.

As stated in the Introduction, we will apply a Krasnosel'skii-Zabreiko fixed point theorem [22] in establishing the existence of nontrivial solutions of (1), (2). We now state that fixed point theorem.

Theorem 2.3. *Let X be a Banach space and $F : X \rightarrow X$ be a completely continuous operator. If there exists a bounded linear operator $A : X \rightarrow X$ such that 1 is not an eigenvalue and*

$$\lim_{\|u\| \rightarrow \infty} \frac{\|F(u) - A(u)\|}{\|u\|} = 0,$$

then F has a fixed point in X .

We will apply Theorem 2.3 to a nonlinear summation operator whose kernel is the Green's function, $G(t, s)$.

For our setting, let a Banach space $(X, \|\cdot\|)$ be defined by

$$X := \{h : \{\nu - 2, \dots, T + \nu\} \rightarrow \mathbb{R} \mid \Delta h(\nu - 2) = h(T + \nu) = 0\}, \quad (5)$$

with norm

$$\|h\| := \max_{x \in \{\nu-2, \dots, T+\nu\}} |h(x)|. \quad (6)$$

It is standard that $u \in X$ is a fixed point of the completely continuous operator $F : X \rightarrow X$ defined by

$$(Fu)(t) := \sum_{s=0}^T G(t, s) f(u(s + \nu - 1)), \quad t \in \{\nu - 2, \dots, T + \nu\}, \quad (7)$$

if and only if u is a solution of (1), (2) (for detailed proofs, see the papers [10] and [20]).

3. EXISTENCE RESULTS

In this section, we apply Theorem 2.3 to the operator F defined in (7) and to an associated linear operator to obtain solutions of (1), (2).

Theorem 3.1. *Assume $f : \mathbb{R} \rightarrow \mathbb{R}$ and $\lim_{|r| \rightarrow \infty} \frac{f(r)}{r} = m$. If*

$$|m| < b := \frac{1}{\sum_{s=0}^T G(s + \nu - 1, s)},$$

then the boundary value problem (1), (2) has a solution u , and moreover, $u \neq 0$, when $f(0) \neq 0$.

Proof. Let the Banach space $(X, \|\cdot\|)$ and the completely continuous operator $F : X \rightarrow X$ be as defined in Section 2 in (5), (6) and (7), respectively.

Associated with (1), (2), we consider a linear ν th order equation,

$$\Delta^\nu u(t) + mu(t + \nu - 1) = 0, \quad t \in \{0, \dots, T\}, \tag{8}$$

satisfying the boundary conditions (2), and we define a completely continuous linear operator $A : X \rightarrow X$ by

$$(Au)(t) := m \sum_{s=0}^T G(t, s)u(s + \nu - 1), \quad t \in \{\nu - 2, \dots, T + \nu\}. \tag{9}$$

Of course, solutions of (8), (2) are fixed points of A , and conversely.

Our first claim is that 1 is not an eigenvalue of A . There are two cases to consider for this claim: (a) $m = 0$, and (b) $m \neq 0$.

For (a), if $m = 0$, since the boundary value problem (3), (2) has only the trivial solution, it is immediate that 1 is not an eigenvalue of A .

For (b), if $m \neq 0$ and (8), (2) has a nontrivial solution, u , then $\|u\| > 0$. And so, we have

$$\begin{aligned} \|u\| &= \|Au\| \\ &= \max_{t \in \{\nu-2, \dots, T+\nu\}} \left| m \sum_{s=0}^T G(t, s)u(s + \nu - 1) \right| \\ &= |m| \max_{t \in \{\nu-2, \dots, T+\nu\}} \left| \sum_{s=0}^T G(t, s)u(s + \nu - 1) \right| \\ &\leq |m| \|u\| \sum_{s=0}^T G(s + \nu - 1, s) \\ &< b \|u\| \frac{1}{b} \\ &= \|u\|, \end{aligned}$$

which is a contradiction. So, again, 1 is not an eigenvalue of A .

Next, we exhibit that

$$\lim_{\|u\| \rightarrow \infty} \frac{\|F(u) - A(u)\|}{\|u\|} = 0.$$

To that end, let $\epsilon > 0$ be given. Since $\lim_{|r| \rightarrow \infty} \frac{f(r)}{r} = m$, there exists an $M_1 > 0$ such that, for $|r| > M_1$,

$$|f(r) - mr| < \epsilon|r|. \quad (10)$$

Set

$$M = \sup_{|r| \leq M_1} |f(r)|,$$

and let $L > M_1$ be such that

$$\frac{M + |m|M_1}{L} < \epsilon.$$

If we choose $u \in X$ with $\|u\| > L$, then for $s \in \{0, \dots, T\}$,

(i) if $|u(s + \nu - 1)| \leq M_1$, we have

$$\begin{aligned} |f(u(s + \nu - 1)) - mu(s + \nu - 1)| &\leq |f(u(s + \nu - 1))| + |m||u(s + \nu - 1)| \\ &\leq M + |m|M_1 \\ &< \epsilon L \\ &< \epsilon\|u\|, \end{aligned}$$

and

(ii) if $|u(s + \nu - 1)| > M_1$, we have from (10) that

$$|f(u(s + \nu - 1)) - mu(s + \nu - 1)| < \epsilon|u(s + \nu - 1)| \leq \epsilon\|u\|.$$

Thus, from (i) and (ii), for all $s \in \{0, \dots, T\}$,

$$|f(u(s + \nu - 1)) - mu(s + \nu - 1)| \leq \epsilon\|u\|. \quad (11)$$

It follows from (11) that, for $u \in X$ with $\|u\| > L$,

$$\begin{aligned} \|F(u) - A(u)\| &= \max_{t \in \{\nu-2, \dots, T+\nu\}} \left| \sum_{s=0}^T G(t, s) [f(u(s + \nu - 1)) - mu(s + \nu - 1)] \right| \\ &\leq \max_{t \in \{\nu-2, \dots, T+\nu\}} \sum_{s=0}^T G(t, s) |f(u(s + \nu - 1)) - mu(s + \nu - 1)| \\ &\leq \epsilon\|u\| \sum_{s=0}^T G(s + \nu - 1, s) \\ &= \frac{\epsilon}{b}\|u\|. \end{aligned}$$

As a consequence,

$$\lim_{\|u\| \rightarrow \infty} \frac{\|F(u) - A(u)\|}{\|u\|} = 0.$$

By Theorem 2.3, F has a fixed point $u \in X$, and in turn, u is a desired solution of (1), (2). Moreover, if in addition, $f(0) \neq 0$, it is immediate that $u \neq 0$.

The proof is complete. \square

As a corollary, we will show that when $f \geq 0$, then (1), (2) has positive solutions.

Corollary 3.2. *Assume $f : [0, \infty) \rightarrow [0, \infty)$ is continuous and $\lim_{r \rightarrow \infty} \frac{f(r)}{r} = 0$. Then the boundary value problem (1), (2) has a nonnegative solution u , and moreover, u is a positive solution, when $f(0) \neq 0$.*

Proof. Let $\bar{f} : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$\bar{f}(r) := \begin{cases} f(r), & r \geq 0, \\ f(-r), & r < 0. \end{cases}$$

Then, \bar{f} is continuous on \mathbb{R} and $\lim_{|r| \rightarrow \infty} \frac{\bar{f}(r)}{r} = 0$. It follows from Theorem 3.1 that the fractional equation

$$\Delta^\nu u(t) + \bar{f}(u(t + \nu - 1)) = 0, \quad t \in \{0, \dots, T\}, \quad (12)$$

satisfying the boundary conditions (2) has a solution u . In particular, u satisfies

$$u(t) = \sum_{s=0}^T G(t, s) \bar{f}(u(s + \nu - 1)), \quad t \in \{\nu - 2, \dots, T + \nu\},$$

and hence $u(t) \geq 0$, $t \in \{\nu - 2, \dots, T + \nu\}$. In view of that, $\bar{f}(u(s + \nu - 1)) = f(u(s + \nu - 1))$, $s \in \{0, \dots, T\}$, and so u satisfies (1), (2). That is, u is a nonnegative solution of (1), (2). And as before, if $f(0) \neq 0$, then $u \neq 0$. \square

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