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COEFFICIENT BOUNDS AND DISTORTION THEOREMS FOR CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS

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Abstract. This aim of this paper is to establish the coefficient estimates, distortion theorems, growth theorems and inclusion relations for certain new subclasses of univalent functions in the open unit disc $E = \{z : |z| < 1\}$. The results established here, will generalize earlier known results.

1. Introduction

Let $\mathcal{A}$ denotes the class of analytic functions in the unit disc $E = \{z : |z| < 1\}$, which are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and normalized by $f(0) = f'(0) - 1 = 0$. $\mathcal{S}$ is the class of functions in $\mathcal{A}$, which are univalent in E.

By $\mathcal{U}$, we denote the class of Schwarzian functions $w(z) = \sum_{n=1}^{\infty} c_n z^n$, which are analytic in the unit disc E such that $w(0) = 0, |w(z)| < 1$.

For two functions f and g analytic in E, f is said to be subordinate to g (symbolically $f \prec g$) if a Schwarzian function $w(z) \in \mathcal{U}$ can be found for which $f(z) = g(w(z))$. Under the assumption that g is univalent, we have $f \prec g$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$.

The well known classes of starlike and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{K}$ respectively and defined as

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in E \right\}$$

and

$$\mathcal{K} = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in E \right\}.$$

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The classes $\mathcal{S}^*$ and $\mathcal{K}$ are related by the Alexander relation [1] as $f \in \mathcal{K}$ if and only if $zf' \in \mathcal{S}^*$.

Kaplan [3] introduced the class $\mathcal{C}$ of functions $f \in \mathcal{A}$ such that $Re\left(\frac{f'(z)}{h'(z)}\right) > 0$, $h \in \mathcal{K}$ or equivalently $Re\left(\frac{zf'(z)}{g(z)}\right) > 0$ where $g = zh'$ is starlike. The functions in the class $\mathcal{C}$ are known as close-to-convex functions. Subsequently, Noor [6] introduced the class $\mathcal{C}^*$ of quasi-convex functions defined as

$$\mathcal{C}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{h'(z)}\right) > 0, h \in \mathcal{K}, z \in E \right\}.$$

Every quasi-convex function is convex and close-to-convex and so is univalent. Also $f \in \mathcal{C}^*$ if and only if $zf' \in \mathcal{C}$.

Singh and Singh [10] discussed the class $\mathcal{C}_s^*$ as

$$\mathcal{C}_s^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{g'(z)}\right) > 0, g \in \mathcal{S}^*, z \in E \right\}.$$

For $-1 \leq B < A \leq 1$, Xiong and Liu [11] and Singh and Singh [10] studied the classes $\mathcal{C}^*(A, B)$ and $\mathcal{C}_s^*(A, B)$ respectively, which are the subclasses of quasi-convex functions defined as:

$$\mathcal{C}^*(A, B) = \left\{ f : f \in \mathcal{A}, \frac{(zf'(z))'}{h'(z)} \prec \frac{1 + Az}{1 + Bz}, h \in \mathcal{K}, z \in E \right\}$$

and

$$\mathcal{C}_s^*(A, B) = \left\{ f : f \in \mathcal{A}, \frac{(zf'(z))'}{g'(z)} \prec \frac{1 + Az}{1 + Bz}, g \in \mathcal{S}^*, z \in E \right\}.$$

Some other subclasses of quasi-convex functions were studied in [8, 9, 10].

For $-1 \leq D < C \leq 1$ and $0 \leq \alpha < 1$, Polatoglu et al. [7] defined the class $\mathcal{P}(C, D; \alpha)$ which consists of the functions $p(z)$ such that $p(z) \prec \frac{1 + [D + (C - D)(1 - \alpha)]z}{1 + Dz}$. For $\alpha = 0$, the class $\mathcal{P}(C, D; \alpha)$ agrees with $\mathcal{P}(C, D)$, which is a subclass of $\mathcal{A}$ introduced by Janowski [2].

Noor [6] introduced the class $\mathcal{K}_1$ which includes the functions $f \in \mathcal{A}$ such that $Re\left(\frac{f'(z)}{g'(z)}\right) > 0$ where $g \in \mathcal{C}^*$. Obviously, $\mathcal{C}^* \subset \mathcal{C} \subset \mathcal{K}_1$.

Throughout this paper, we assume that

$$-1 \leq D \leq B < A \leq C \leq 1, 0 \leq \alpha < 1, z \in E.$$

Motivated and stimulated by the above work, we now introduce the following subclasses of univalent functions:

Definition 1 Let $\mathcal{K}_1(A, B; C, D; \alpha)$ be the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\frac{f'(z)}{g'(z)} \prec \frac{1 + [D + (C - D)(1 - \alpha)]z}{1 + Dz},$$

where $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{C}^*(A,B)$.

The following observations are obvious:
(i) $\mathcal{K}_1(A,B;C,D;0)\equiv\mathcal{K}_1(A,B;C,D)$.
(ii) $\mathcal{K}_1(1,-1;C,D;0)\equiv\mathcal{K}_1(C,D)$.
(iii) $\mathcal{K}_1(1,-1;1,-1;0)\equiv\mathcal{K}_1$.

Definition 2 Let $\mathcal{K}_1'(A,B;C,D;\alpha)$ denote the class of functions $f\in\mathcal{A}$ and satisfying the condition

$$\frac{f'(z)}{h'(z)}\prec\frac{1+[D+(C-D)(1-\alpha)]z}{1+Dz},$$

where $h(z)=z+\sum_{n=2}^{\infty}d_nz^n\in\mathcal{C}_s^*(A,B)$.

We have the following observations:
(i) $\mathcal{K}_1'(A,B;C,D;0)\equiv\mathcal{K}_1'(A,B;C,D)$.
(ii) $\mathcal{K}_1'(1,-1;C,D;0)\equiv\mathcal{K}_1'(C,D)$.
(iii) $\mathcal{K}_1'(1,-1;1,-1;0)\equiv\mathcal{K}_1'$.

The paper is concerned with the study of the classes $\mathcal{K}_1(A,B;C,D;\alpha)$ and $\mathcal{K}_1'(A,B;C,D;\alpha)$. We obtain the coefficient estimates, distortion theorems, growth theorems and inclusion relations for the functions in these classes. Some earlier established results follow as special cases.

2. Preliminary Results

Lemma 1 [7] If $P(z)=\dfrac{1+[D+(C-D)(1-\alpha)]w(z)}{1+Dw(z)}=1+\sum_{n=1}^{\infty}p_nz^n$, then

$$|p_n|\le(C-D)(1-\alpha),n\ge1.$$

Lemma 2 [10] If $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{C}^*(A,B)$, then for $A-(n-1)B\ge(n-2),n\ge3$,

$$|b_n|\le\frac{1}{n}+\frac{(n-1)(A-B)}{2n}.$$

Lemma 3 [10] If $g(z)\in\mathcal{C}^*(A,B)$, then for $|z|=r<1$,
for $B\ne-1$,

$$\frac{A-B}{r(1+B)^2}log\left(\frac{1-Br}{1+r}\right)+\frac{1+A}{(1+B)(1+r)}\le|g'(z)|\le\frac{A-B}{r(1+B)^2}log\left(\frac{1-r}{1+Br}\right)+\frac{1+A}{(1+B)(1-r)},$$

and for $B=-1$,

$$-\frac{1+A}{2r(1+r)^2}+\frac{A}{r(1+r)}+\frac{1}{2r}(1-A)\le|g'(z)|\le\frac{1+A}{2r(1-r)^2}-\frac{A}{r(1-r)}+\frac{1}{2r}(A-1).$$

Lemma 4 [10] If $h(z)=z+\sum_{n=2}^{\infty}d_nz^n\in\mathcal{C}_s^*(A,B)$, then for $A-(n-1)B\ge(n-2),n\ge3$,

$$|d_n|\le1+\frac{(A-B)(n-1)(2n-1)}{6n}.$$

Lemma 5 [10] If $h(z)\in\mathcal{C}_s^*(A,B)$, then for $|z|=r<1$,
for $B\ne-1,\ B\ne0$,

$L_1 \leq |h'(z)| \leq L_2$, where

$$L_1 = \frac{(B-1)}{r(B+1)^3} log \left|\frac{1+r}{1-Br}\right| + \left[\frac{(B-1)}{B(B+1)^2} - \frac{A}{B}\right] \frac{1}{1+r} + \frac{1}{2}\left[1 + \frac{A}{B} - \frac{(B-1)}{B(B+1)}\right] \frac{(2+r)}{(1+r)^2},$$

$$L_2 = \frac{(B-1)}{r(B+1)^3} log \left|\frac{1+Br}{1-r}\right| + \left[\frac{(B-1)}{B(B+1)^2} - \frac{A}{B}\right] \frac{1}{1-r} + \frac{1}{2}\left[1 + \frac{A}{B} - \frac{(B-1)}{B(B+1)}\right] \frac{(2-r)}{(1-r)^2},$$

and for $B = -1$,

$$\frac{A}{1+r} - \frac{(3A+1)(2+r)}{2(1+r)^2} + \frac{2(A+1)(3+3r+r^2)}{3(1+r)^3}$$

$$\leq |h'(z)| \leq \frac{A}{1-r} - \frac{(3A+1)(2-r)}{2(1-r)^2} + \frac{2(A+1)(3-3r+r^2)}{3(1-r)^3}.$$

Lemma 6 [4] Let $-1 \leq D_2 \leq D_1 < C_1 \leq C_2 \leq 1$, then

$$\frac{1+C_1 z}{1+D_1 z} \prec \frac{1+C_2 z}{1+D_2 z}.$$

3. The class $\mathcal{K}_1(A,B;C,D;\alpha)$

Theorem 1 Let $f(z) \in \mathcal{K}_1(A,B;C,D;\alpha)$, then for $A-(n-1)B \geq (n-2), n \geq 2$,

$$|a_n| \leq \frac{1}{n} + \frac{(n-1)(A-B)}{2n} + \frac{(C-D)(1-\alpha)(n-1)}{n}\left[1 + \frac{(A-B)(n-2)}{4}\right]. \quad (2)$$

The bounds are sharp.

Proof. In Definition 1, using Principle of subordination, we have

$$f'(z) = g'(z)\left(\frac{1+Cw(z)}{1+Dw(z)}\right), w(z) \in \mathcal{U}. \quad (3)$$

On expanding (3), it yields

$$1 + 2a_2 z + 3a_3 z^2 + ... + na_n z^{n-1} + ...$$

$$= (1 + 2b_2 z + 3b_3 z^2 + ... + nb_n z^{n-1} + ...)(1 + p_1 z + p_2 z^2 + ... + p_{n-1} z^{n-1} + ...). \quad (4)$$

Equating the coefficients of z^{n-1} in (4), we have

$$na_n = nb_n + (n-1)p_1 b_{n-1} + (n-2)p_2 b_{n-2} ... + 2p_{n-2} b_2 + p_{n-1}. \quad (5)$$

Applying triangle inequality and Lemma 1 in (5), it gives

$$n|a_n| \leq n|b_n| + (C-D)(1-\alpha)\left[(n-1)|b_{n-1}| + (n-2)|b_{n-2}| ... + 2|b_2| + 1\right]. \quad (6)$$

Using Lemma 2 in (6), the result (2) is obvious.

For $n = 2$, equality sign in (2) holds for the functions f_n defined as

$$f'_n = \frac{1}{z}\left(\frac{1+[D+(C-D)(1-\alpha)]\delta_1 z^n}{1+D\delta_1 z^n}\right)\int_0^z \frac{1}{(1-\delta_2 z)^2}\left(\frac{1+A\delta_3 z^n}{1+B\delta_3 z^n}\right) dz, \quad (7)$$

where $|\delta_1| = |\delta_2| = |\delta_3| = 1$.

Remark 1 (i) On putting $\alpha = 0$ in Theorem 1, we can easily get the result for the class $\mathcal{K}_1(A,B;C,D)$.

(ii) For $A = 1, B = -1, \alpha = 0$, Theorem 1 yields the result for the class $\mathcal{K}_1(C,D)$.

(iii) For $A = 1, B = -1, C = 1, D = -1, \alpha = 0$, Theorem 1 agrees with the result due to Noor [6].

Theorem 2 If $f(z) \in \mathcal{K}_1(A,B;C,D;\alpha)$, then for $|z|=r, 0<r<1$, we have, for $B \neq -1$,

$$\left(\frac{1-[D+(C-D)(1-\alpha)]r}{1-Dr}\right)\left[\frac{A-B}{r(1+B)^2}log\left(\frac{1-Br}{1+r}\right)+\frac{1+A}{(1+B)(1+r)}\right] \leq |f'(z)|$$
$$\leq \left(\frac{1+[D+(C-D)(1-\alpha)]r}{1+Dr}\right)\left[\frac{A-B}{r(1+B)^2}log\left(\frac{1-r}{1+Br}\right)+\frac{1+A}{(1+B)(1-r)}\right]; \tag{8}$$

$$\int_0^r \left(\frac{1-[D+(C-D)(1-\alpha)]t}{1-Dt}\right)\left[\frac{A-B}{t(1+B)^2}log\left(\frac{1-Bt}{1+t}\right)+\frac{1+A}{(1+B)(1+t)}\right]dt \leq |f(z)|$$
$$\leq \int_0^r \left(\frac{1+[D+(C-D)(1-\alpha)]t}{1+Dt}\right)\left[\frac{A-B}{t(1+B)^2}log\left(\frac{1-t}{1+Bt}\right)+\frac{1+A}{(1+B)(1-t)}\right]dt, \tag{9}$$

and for $B=-1$,

$$\left(\frac{1-[D+(C-D)(1-\alpha)]r}{1-Dr}\right)\left[-\frac{1+A}{2r(1+r)^2}+\frac{A}{r(1+r)}+\frac{1}{2r}(1-A)\right] \leq |f'(z)|$$
$$\leq \left(\frac{1+[D+(C-D)(1-\alpha)]r}{1+Dr}\right)\left[\frac{1+A}{2r(1-r)^2}-\frac{A}{r(1-r)}+\frac{1}{2r}(A-1)\right]; \tag{10}$$

$$\int_0^r \left(\frac{1-[D+(C-D)(1-\alpha)]t}{1-Dt}\right)\left[-\frac{1+A}{2t(1+t)^2}+\frac{A}{t(1+t)}+\frac{1}{2t}(1-A)\right]dt \leq |f'(z)|$$
$$\leq \int_0^r \left(\frac{1+[D+(C-D)(1-\alpha)]t}{1+Dt}\right)\left[\frac{1+A}{2t(1-t)^2}-\frac{A}{t(1-t)}+\frac{1}{2t}(A-1)\right]dt. \tag{11}$$

Estimates are sharp.
Proof. From (3), we have

$$|f'(z)| = |g'(z)|\left|\frac{1+[D+(C-D)(1-\alpha)]w(z)}{1+Dw(z)}\right|, w(z) \in \mathcal{U}. \tag{12}$$

It can be easily proved that

$$\frac{1-[D+(C-D)(1-\alpha)]r}{1-Dr} \leq \left|\frac{1+[D+(C-D)(1-\alpha)]w(z)}{1+Dw(z)}\right| \leq \frac{1+[D+(C-D)(1-\alpha)]r}{1+Dr}. \tag{13}$$

As $g(z) \in \mathcal{C}^*(A,B)$, so using Lemma 3 and (13) in (12), the results (8) and (10) can be easily obtained. On integrating (8) and (10) from 0 to r, the results (9) and (11) are obvious.
Sharpness follows for the function f_n defined in (7).

Remark 2 (i) On putting $\alpha=0$ in Theorem 2, we can easily get the result for the class $\mathcal{K}_1(A,B;C,D)$.
(ii) For $A=1, B=-1, \alpha=0$, Theorem 2 agrees with the result for the class $\mathcal{K}_1(C,D)$.
(iii) For $A=1, B=-1, C=1, D=-1, \alpha=0$, the result due to Noor [6] can be easily obtained from Theorem 2.

Theorem 3 Let $-1 \leq D_2 = D_1 < C_1 \leq C_2 \leq 1$ and $0 \leq \alpha_2 \leq \alpha_1 < 1$, then

$$\mathcal{K}_1(A,B;C_1,D_1;\alpha_1) \subset \mathcal{K}_1(A,B;C_2,D_2;\alpha_2).$$

Proof As $f \in \mathcal{K}_1(A, B; C_1, D_1; \alpha_1)$, so

$$\frac{f'(z)}{g'(z)} \prec \frac{1 + [D_1 + (C_1 - D_1)(1 - \alpha_1)]z}{1 + D_1 z}.$$

As $-1 \leq D_2 = D_1 < C_1 \leq C_2 \leq 1$ and $0 \leq \alpha_2 \leq \alpha_1 < 1$, we have

$$-1 \leq D_1 + (1 - \alpha_1)(C_1 - D_1) \leq D_2 + (1 - \alpha_2)(C_2 - D_2) \leq 1.$$

So by Lemma 6, we obtain

$$\frac{f'(z)}{g'(z)} \prec \frac{1 + [D_2 + (C_2 - D_2)(1 - \alpha_2)]z}{1 + D_2 z},$$

which implies $f \in \mathcal{K}_1(A, B; C_2, D_2; \alpha_2)$.

Theorem 4 If $f \in \mathcal{K}_1(A, B; C, D; \alpha)$, then

$$|a_2| \leq \frac{1}{4}\left[2 + 3(A - B) + 2(C - D)(1 - \alpha)\right] \quad (14)$$

and

$$|a_3| \leq \frac{7}{9} + \frac{(C - D)(1 - \alpha)}{6}[4 + 3(A - B)] + \frac{1}{3}D^2 + \frac{(A - B)}{36}[58 + 27A - 35B]. \quad (15)$$

The estimates are sharp.

Proof. Using the principle of subordination in Definition 1, we obtain

$$\frac{f'(z)}{g'(z)} = \frac{1 + [D + (C - D)(1 - \alpha)]w(z)}{1 + Dw(z)}.$$

On expanding and comparing the coefficients, it leads to

$$a_2 = b_2 + \frac{(C - D)(1 - \alpha)}{2} c_1 \quad (16)$$

and

$$a_3 = b_3 + \frac{2}{3}(C - D)(1 - \alpha)b_2 c_1 + \frac{(C - D)(1 - \alpha)}{3}[c_2 - Dc_1^2] + \frac{4}{3}b_2^2 - \frac{1}{3}D^2 c_1^2. \quad (17)$$

For $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{C}^*(A, B)$,

$$|b_2| \leq \frac{1}{4}[2 + 3(A - B)] \quad (18)$$

and

$$|b_3| \leq \frac{1}{18}[8 + (A - B)\{11 - 4B\}]. \quad (19)$$

Also it was proved in [5], that for any complex number γ,

$$|c_2 - \gamma c_1^2| \leq max\{1, |\gamma|\}. \quad (20)$$

Using (18), (19) and (20) along with the inequality $|c_1| \leq 1$ in (16) and (17), the results (14) and (15) are obvious.
The results are sharp for the functions defined in (7).

4. The class $\mathcal{K}_1'(A,B;C,D;\alpha)$

Theorem 5 Let $f(z)\in\mathcal{K}_1'(A,B;C,D;\alpha)$, then for $A-(n-1)B\geq(n-2), n\geq 2$,

$$|a_n|\leq 1+\frac{(n-1)(2n-1)(A-B)}{6n}+\frac{(C-D)(1-\alpha)(n-1)}{2}\left[1+\frac{(n-2)(4n-3)(A-B)}{18n}\right]. \tag{21}$$

The results are sharp.

Proof. From Definition 2, using Lemma 1 and Lemma 4 and following the procedure of Theorem 1, the result (21) can be easily derived.

For $n=2$, equality sign in (14) hold for the functions $f_n(z)$ defined by

$$f_n'=\frac{1}{z}\left(\frac{1+[D+(C-D)(1-\alpha)]\delta_1 z^n}{1+D\delta_1 z^n}\right)\int_0^z\frac{1+\delta_2 z}{(1-\delta_2 z)^3}\left(\frac{1+A\delta_3 z^n}{1+B\delta_3 z^n}\right)dz, |\delta_1|=|\delta_2|=|\delta_3|=1. \tag{22}$$

Remark 3 (i) On putting $\alpha=0$ in Theorem 5, we can easily get the result for the class $\mathcal{K}_1'(A,B;C,D)$.
(ii) For $A=1, B=-1, \alpha=0$, Theorem 5 agrees with the result for the class $\mathcal{K}_1'(C,D)$.
(iii) For $A=1, B=-1, C=1, D=-1, \alpha=0$, the result for the class $\mathcal{K}_1'$ can be easily obtained from Theorem 5.

Theorem 6 If $f(z)\in\mathcal{K}_1'(A,B;C,D;\alpha)$, then for $|z|=r, 0<r<1$, we have, for $B\neq-1$, $B\neq 0$,

$$\left(\frac{1-[D+(C-D)(1-\alpha)]r}{1-Dr}\right)L_1\leq|f'(z)|\leq\left(\frac{1+[D+(C-D)(1-\alpha)]r}{1+Dr}\right)L_2;$$

$$\int_0^r\left(\frac{1-[D+(C-D)(1-\alpha)]t}{1-Dt}\right)L_1 dt\leq|f(z)|\leq\int_0^r\left(\frac{1+[D+(C-D)(1-\alpha)]t}{1+Dt}\right)L_2,$$

where

$$L_1=\frac{(B-1)}{r(B+1)^3}log\left|\frac{1+r}{1-Br}\right|+\left[\frac{(B-1)}{B(B+1)^2}-\frac{A}{B}\right]\frac{1}{1+r}+\frac{1}{2}\left[1+\frac{A}{B}-\frac{(B-1)}{B(B+1)}\right]\frac{(2+r)}{(1+r)^2},$$

$$L_2=\frac{(B-1)}{r(B+1)^3}log\left|\frac{1+Br}{1-r}\right|+\left[\frac{(B-1)}{B(B+1)^2}-\frac{A}{B}\right]\frac{1}{1-r}+\frac{1}{2}\left[1+\frac{A}{B}-\frac{(B-1)}{B(B+1)}\right]\frac{(2-r)}{(1-r)^2},$$

and for $B=-1$,

$$\left(\frac{1-[D+(C-D)(1-\alpha)]r}{1-Dr}\right)\left[\frac{A}{1+r}-\frac{(3A+1)(2+r)}{2(1+r)^2}+\frac{2(A+1)(3+3r+r^2)}{3(1+r)^3}\right]$$

$$\leq|f'(z)|\leq\left(\frac{1+[D+(C-D)(1-\alpha)]r}{1+Dr}\right)\left[\frac{A}{1-r}-\frac{(3A+1)(2-r)}{2(1-r)^2}+\frac{2(A+1)(3-3r+r^2)}{3(1-r)^3}\right];$$

$$\int_0^r\left(\frac{1-[D+(C-D)(1-\alpha)]t}{1-Dt}\right)\left[\frac{A}{1+t}-\frac{(3A+1)(2+t)}{2(1+t)^2}+\frac{2(A+1)(3+3t+t^2)}{3(1+t)^3}\right]dt$$

$$\leq|f(z)|\leq\int_0^r\left(\frac{1+[D+(C-D)(1-\alpha)]t}{1+Dt}\right)\left[\frac{A}{1-t}-\frac{(3A+1)(2-t)}{2(1-t)^2}+\frac{2(A+1)(3-3t+t^2)}{3(1-t)^3}\right]dt.$$

Estimates are sharp.

Proof. Using Lemma 5 and following the procedure of Theorem 2, the results of Theorem 6 can be easily established.

Sharpness follows for the function f_n defined in (22).

Remark 4 (i) On putting $\alpha = 0$ in Theorem 6, we can easily get the result for the class $\mathcal{K}_1'(A, B; C, D)$.
(ii) For $A = 1, B = -1, \alpha = 0$, Theorem 6 agrees with the result for the class $\mathcal{K}_1'(C, D)$.
(iii) For $A = 1, B = -1, C = 1, D = -1, \alpha = 0$, the result for the class $\mathcal{K}_1'$ can be easily obtained from Theorem 6.

Theorem 7 Let $-1 \leq D_2 = D_1 < C_1 \leq C_2 \leq 1$ and $0 \leq \alpha_2 \leq \alpha_1 < 1$, then

$$\mathcal{K}_1'(A, B; C_1, D_1; \alpha_1) \subset \mathcal{K}_1'(A, B; C_2, D_2; \alpha_2).$$

Proof. Using Lemma 6 and following the procedure of Theorem 3, the proof of Theorem 7 is obvious.

Theorem 8 If $f \in \mathcal{K}_1'(A, B; C, D; \alpha)$, then

$$|a_2| \leq \frac{1}{4}\left[2 + 3(A - B) + 2(C - D)(1 - \alpha)\right] \tag{23}$$

and

$$|a_3| \leq \frac{11}{6} + \frac{(C-D)(1-\alpha)}{6}[4+3(A-B)] + \frac{1}{3}D^2 + \frac{(A-B)}{108}[194+81A-105B]. \tag{24}$$

The bounds are sharp.

Proof. Following the procedure of Theorem 4 and using the results that, for $h(z) = z + \sum_{n=2}^{\infty} d_n z^n \in \mathcal{C}_s^*(A, B)$,

$$|d_2| \leq \frac{1}{4}[2 + 3(A - B)]$$

and

$$|d_3| \leq \frac{1}{54}[81 + (A - B)\{43 - 12B\}],$$

the proof is obvious.
The results are sharp for the functions defined in (22).

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